Connection between Spin and Statistics

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The proof of the connection between spin and statistics for interacting fields is divided into two parts: commutation relations involving components of a single field, and commutation relations between different fields. The first problem is treated in this paper: the connection between spin and statistics is shown to follow from a few simple postulates. The explicit discussion is limited to the cases of spin zero and of spin one-half.

1. INTRODUCTION

THE theorem on the connection between spin and statistics states that particles with integral spin obey Bose statistics and particles with half-integral spin obey Fermi statistics. Fields corresponding to particles of integral spin are to be quantized with minus commutation relations whereas plus commutation (anticommutation) relations have to be employed in the quantization of fields with half-integral spin. The connection between spin and statistics was proved by Pauli[1] for noninteracting fields on the basis of a few simple postulates.[2] In the presence of interaction the theorem splits into two parts:

Commutation relations between two operators of the same field.—Minus commutation relations for fields of integral spin (Bose fields), plus commutation relations for fields of half-integral spin (Fermi fields).

Commutation relations between different fields.—Minus commutation relations between different Bose fields and between one Bose and one Fermi field, plus commutation relations between different Fermi fields. We shall refer to this choice of commutation relations between different fields as the "normal case."

There is a close relation between the TCP theorem[3] and the connection between spin and statistics. The derivations and discussions by Lüders, Bell, and Pauli use, among other postulates, the usual connection between spin and statistics; the commutation relations of a field with itself are taken for granted, those between different fields are explicitly postulated.[4] On the other hand, Schwinger[5] postulated the TCP theorem (or actually the validity of TC and P separately) and inferred the connection between spin and statistics. Earlier work by Belinfante and Pauli[6] had shown that the connection between spin and statistics can be deduced for charged fields if one postulates invariance under C for their interaction with the electromagnetic field; this interaction is in any case invariant under T and P. The recent very satisfactory proof of the TCP theorem by Jost[7] still has to use the usual connection between spin and statistics; the commutation character not only of a field with itself but also that between different fields has to be assumed.

This situation is rather unsatisfactory; the theorem of the connection between spin and statistics and the TCP theorem support each other mutually but no independent proof of either of these theorems has been given. It is not clear whether Pauli's arguments[1] do not lose their strength when interactions are present (and this is, of course, the only case when the TCP theorem is a nontrivial statement). In the present paper a new derivation of the connection between spin and statistics for operators of the same field will be given. The proof is valid in the presence of interactions and does not postulate the TCP theorem or an equivalent of it. We analyze fields which either are Hermitian or, if they are non-Hermitian, admit a gauge transformation of the first kind:

$$\varphi(x)\rightarrow\varphi(x)e^{i\alpha},\quad \varphi^*(x)\rightarrow\varphi^*(x)e^{-i\alpha}; \tag{1}$$

similarly for spinor fields, etc. This gauge transformation may, of course, involve simultaneously several fields. The operator which generates it may represent the electric charge or other conserved quantities like baryon number, strangeness (in strong and electromagnetic interactions), or lepton charge (if this quantity is conserved). We believe that two Hermitian fields can be reasonably combined into a single non-Hermitian field only under the assumption of some such gauge invariance.[8] Otherwise such a combination

[1] W. Pauli, Phys. Rev. **58**, 716 (1940).

[2] The postulates will be listed at the places where they are needed for our own analysis of the problem.

[3] G. Lüders, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **28**, No. 5 (1954) and Ann. phys. **2**, 1 (1957); J. S. Bell, Proc. Roy. Soc. (London) **A231**, 79 (1955); W. Pauli, in *Niels Bohr and the Development of Physics*, edited by W. Pauli (McGraw-Hill Book Company, Inc., New York, and Pergamon Press, Inc., London, 1955).

[4] Actually only anticommutativity of different spinor fields was postulated. Apparently none of the authors was at that time aware of the fact that an equally nontrivial problem is given by the commutation relations between two Bose fields or one Bose and one Fermi field. A discussion of this point is given at the end of this section.

[5] J. Schwinger, Phys. Rev. **82**, 914 (1951); **91**, 713 (1953).

[6] F. J. Belinfante, Physica **6**, 870 (1939); W. Pauli and F. J. Belinfante, Physica **7**, 177 (1940).

[7] R. Jost, Helv. Phys. Acta **30**, 409 (1957).

[8] To derive the equality of masses (and lifetimes) of particles and antiparticles from C invariance or from the TCP theorem

is artificial and one actually has two different Hermitian fields.

The derivation will be given for spin zero and one-half. The extension to spin one is evident. Higher spins will not be treated. The postulates on which the derivation is based will be given at those places where they are needed for the proof. Formal complications arise in the presence of a quantized electromagnetic field. Special care is needed to apply our arguments to that case; the discussion will not be given here.

No detailed analysis appears to have been given so far of the problem of the commutation relations between different fields. It has been known for some time that the above-specified normal case is not always the only possible one; interactions can be constructed for which the character of the commutation relations between different fields is to a certain extent arbitrary. A trivial example is that of noninteracting fields; nontrivial cases have been given by various authors.[9] The problem will be discussed in general in a forthcoming paper by one of the authors (G. L.), under the assumption that the theory is specified by a local interaction and that the resulting differential equations are also local. Here we shall only summarize his results. It can first be shown that the above-mentioned normal case provides a possible choice of commutation relations. In the presence of a certain invariance property of the theory, however, other choices are also possible. They can all be obtained from the normal case by means of one or more generalized Klein transformations.[10] This freedom in the choice of the commutation relations can be shown not to affect the validity of the *TCP* theorem. It is in fact possible to satisfy this theorem in all cases by a simple redefinition of the *TCP* transformation.[11]

2. HERMITIAN SPIN-ZERO FIELD

Let $\varphi(x)$ be a Hermitian spin-zero field in the Heisenberg representation.

We first postulate:

I. The theory is invariant with respect to the proper inhomogeneous Lorentz group (which includes four-dimensional translations but does not contain any reflections).

It follows that the expectation value $\langle\varphi(x)\varphi(y)\rangle_0$ with respect to the physical vacuum is an invariant function of the difference four-vector

$$\xi_\mu = x_\mu - y_\mu. \tag{2}$$

One then has

$$\langle\varphi(x)\varphi(y)\rangle_0 = f(\xi), \tag{3}$$

where $f(\xi)$, for spacelike ξ, depends only upon the invariant $\xi_\mu\xi_\mu$ but for timelike ξ depends also upon whether this vector points into the future or past light cone.[12] Use is made of the special consequence of Postulate I that the physical vacuum is a Lorentz-invariant (and nondegenerate) state. One concludes from Eq. (3) that

$$\langle[\varphi(x),\varphi(y)]\rangle_0 = 0 \quad (\xi \text{ spacelike}). \tag{4}$$

We now postulate:

II. Two operators of the same field at points separated by a spacelike interval either commute or anticommute (Locality).[13]

Relying upon this postulate, we do not discuss the problem as to whether quite different types of relations between field operators might be possible. We therefore have only to show that the assumption

$$\langle\{\varphi(x),\varphi(y)\}\rangle_0 = 0 \quad (\xi \text{ spacelike}) \tag{5}$$

leads to contradictions with postulates which shall be given later. From Eqs. (4) and (5) one concludes that

$$\langle\varphi(x)\varphi(y)\rangle_0 = 0 \quad (\xi \text{ spacelike}). \tag{6}$$

If further one postulates:

III. The vacuum is the state of lowest energy,

one finds by the method of analytic continuation as used by Hall and Wightman[14] that Eq. (6) holds, not only for ξ spacelike but for all ξ. We assume as usual that

IV. The metric of the Hilbert space is positive definite.

This postulate allows one to conclude from Eq. (6) that

$$\varphi(x)\Omega = 0, \tag{7}$$

where Ω is the physical vacuum. We finally postulate:

V. The vacuum is not identically annihilated by a field.

Since Eq. (7) is in contradiction with this postulate, the assumption (5) is untenable.

Postulate V is probably less familiar than the others and may require a word of explanation. If the field $\varphi(x)$ represents a stable particle, so that incoming and outgoing fields can be associated with $\varphi(x)$, Eq. (7) certainly cannot hold since there are known to exist

one also has to postulate the existence of gauge invariance, i.e., of a generalized charge operator. This requirement is, however, usually not stated explicitly.

[9] K. Nishijima, Progr. Theoret. Phys. Japan **5**, 187 (1950); S. Oneda and H. Umezawa, Progr. Theoret. Phys. Japan **9**, 685 (1953); T. Kinoshita, Phys. Rev. **96**, 199 (1954); Umezawa, Podolanski, and Oneda, Proc. Phys. Soc. (London) **A68**, 503 (1955); R. Spitzer, Phys. Rev. **105**, 1919 (1957).

[10] O. Klein, J. phys. **9**, 1 (1938).

[11] The above results throw some light on questions raised in a recent paper by T. Kinoshita and A. Sirlin [Phys. Rev. **108**, 844 (1957), footnote 22].

[12] More explicit expressions for $f(\xi)$ under the additional assumption of Postulate III were given by G. Källén, Helv. Phys. Acta **25**, 417 (1952); H. Lehmann, Nuovo cimento **11**, 342 (1954); M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

[13] We do not wish to formulate at this point the much weaker requirement of locality which is needed in Jost's proof of the *TCP* theorem.

[14] D. Hall and A. S. Wightman, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **31**, No. 5 (1957).

nonvanishing matrix elements of $\varphi(x)$ between the vacuum and the one-particle states. But even in a more general case it seems unlikely that a Hilbert space can be constructed if Postulate V is not satisfied.

The whole analysis holds in particular in the absence of interactions, the case originally studied by Pauli[1]. His deduction rests, for integral spin, on Postulate I and Postulate II. Postulate IV is tacitly assumed; the role of this postulate was studied later by the same author.[15] Postulate V is evident for free fields because of the well-known procedure of constructing the Hilbert space in this case. Postulate III, on the other hand, does not seem to play any role in Pauli's proof for the case of integral spin.

3. HERMITIAN SPIN ONE-HALF FIELD

Let the spin one-half field $\psi_\alpha(x)$ be Hermitian, or rather be a Majorana field,

$$\bar{\psi}_\alpha(x)=\psi_\beta(x)C_{\beta\alpha}, \tag{8}$$

where C is the 4-by-4 charge conjugation matrix.[16] We shall analyze the vacuum expectation value

$$\langle\psi_\beta{}^*(x)\psi_\beta(y)\rangle_0=C_{\alpha\delta}\gamma_{4\delta\beta}\langle\psi_\alpha(x)\psi_\beta(y)\rangle_0. \tag{9}$$

This expression has the properties of positive definiteness which will be needed below. In the following, we write

$$C_{\alpha\delta}\gamma_{4\delta\beta}=\eta_{\alpha\beta}. \tag{10}$$

The matrix η is symmetric:

$$\eta_{\alpha\beta}=\eta_{\beta\alpha}. \tag{11}$$

Since expression (9) is the fourth component of a four-vector, it follows from *Postulate I* that

$$\eta_{\alpha\beta}\langle\psi_\alpha(x)\psi_\beta(y)\rangle_0=\xi_4 g(\xi), \tag{12}$$

where $g(\xi)$ is an invariant function of the difference vector ξ. Making use also of Eq. (11), one sees that

$$\eta_{\alpha\beta}\langle\{\psi_\alpha(x),\psi_\beta(y)\}\rangle_0=0 \quad (\xi \text{ spacelike}). \tag{13}$$

Because of *Postulate II* we have only to show that the assumption

$$\eta_{\alpha\beta}\langle[\psi_\alpha(x),\psi_\beta(y)]\rangle_0=0 \quad (\xi \text{ spacelike}) \tag{14}$$

leads to contradictions with the other postulates. In analogy to Eq. (6), one finds that in this case

$$\eta_{\alpha\beta}\langle\psi_\alpha(x)\psi_\beta(y)\rangle_0=0 \quad (\xi \text{ spacelike}). \tag{15}$$

From *Postulate III* it follows that Eq. (15) holds without restriction on ξ. Remembering (9), one finds, from *Postulate IV*, that

$$\psi_\alpha(x)\Omega=0, \tag{16}$$

which is in contradiction with *Postulate V*.

The case of no interaction is again a specialization. Whereas our analysis proceeds along parallel lines for spin zero and spin one-half, Pauli's proof proceeds rather differently in the two cases. For half-integral spin his deduction rests very heavily on Postulate III. In our own analysis, on the other hand, this postulate does not play a more important role here than it did in the case of integral spin. It should also be pointed out that Pauli's argument requires a modification in the case of Majorana fields, since the c-number expression for the energy vanishes identically in that case.

4. NON-HERMITIAN FIELDS

As explained in the Introduction, we regard non-Hermitian fields as reasonable concepts only in the presence of some gauge invariance, i.e., if the theory is, for spin-zero fields, invariant under the transformation (1). From this postulate of gauge invariance it follows that

$$\langle\varphi(x)\varphi(y)\rangle_0=\langle\varphi^*(x)\varphi^*(y)\rangle_0=0. \tag{17}$$

Since our technique is restricted to vacuum expectation values of the products of two field operators we can, using *Postulate II*, only try to show that the assumption

$$\langle\{\varphi^*(x),\varphi(y)\}\rangle_0=0 \quad (\xi \text{ spacelike}) \tag{18}$$

leads to contradictions with our postulates. Applying *Postulate I*, one concludes from Eq. (18) that

$$\langle\varphi^*(x)\varphi(y)+\varphi(x)\varphi^*(y)\rangle_0=0 \quad (\xi \text{ spacelike}); \tag{19}$$

compare the derivation of Eq. (4) from Eq. (3). From the *Postulate of Gauge Invariance* [i.e., Eq. (17)], one then obtains

$$\langle(\varphi(x)\pm\varphi^*(x))(\varphi(y)\pm\varphi^*(y))\rangle_0=0 \quad (\xi \text{ spacelike}); \tag{20}$$

the equation is valid both with two plus signs and with two minus signs. *Postulate III* makes this equation generally valid. Since both $\varphi(x)+\varphi^*(x)$ and $i(\varphi(x)-\varphi^*(x))$ are Hermitian fields, *Postulate IV* leads to

$$\varphi(x)\Omega=\varphi^*(x)\Omega=0, \tag{21}$$

which is in contradiction with *Postulate V*.

The conclusions for spin one-half run quite parallel. Since, however, some formal care is needed we reproduce the arguments in detail. The analog of Eq. (17) is

$$\langle\psi_\alpha(x)\psi_\beta(y)\rangle_0=\langle\bar{\psi}_\alpha(x)\bar{\psi}_\beta(y)\rangle_0=0. \tag{22}$$

Equation (18) is to be replaced by

$$\langle[\psi_\alpha{}^*(x),\psi_\alpha(y)]\rangle_0=0 \quad (\xi \text{ spacelike}). \tag{23}$$

This equation is to be disproved. Since the left-hand side transforms like the fourth component of a vector

[15] W. Pauli, Progr. Theoret. Phys. Japan **5**, 526 (1950). Our assumption that $\varphi(x)$ is a Hermitian field does, of course, not imply Postulate IV. If use were made of a nonpositive-definite scalar product, Hermiticity would be defined with respect to this scalar product (self-adjointness in Pauli's terminology).

[16] Various definitions of this matrix occur in the literature. We use the one given by W. Pauli in reference 3. Summation over repeated spinor indices is to be understood in our formulas.

one finds, instead of Eq. (19),

$$\langle\psi_\alpha^*(x)\psi_\alpha(y)+\psi_\alpha(x)\psi_\alpha^*(y)\rangle_0=0 \quad (\xi \text{ spacelike}); \quad (24)$$

compare the transition from Eq. (13) to Eq. (14). Equation (20) is to be replaced by[17]

$$\eta_{\alpha\beta}\langle(\psi_\alpha(x)\pm\zeta_{\alpha\gamma}\psi_\gamma^*(x))(\psi_\beta(y)\pm\zeta_{\beta\delta}\psi_\delta^*(y))\rangle_0=0 \quad (\xi \text{ spacelike}), \quad (25),$$

where the (symmetric) matric ζ is the inverse of the matrix η defined in Eq. (10) and $\zeta_{\alpha\beta}^*=\eta_{\alpha\beta}$. Since $\psi_\alpha(x)+\zeta_{\alpha\gamma}\psi_\gamma^*(x)$ and $i(\psi_\alpha(x)-\zeta_{\alpha\gamma}\psi_\gamma^*(x))$ are Majorana fields, one is led to

$$\psi_\alpha(x)\Omega=\bar{\psi}_\alpha(x)\Omega=0, \quad (26)$$

which replaces Eq. (21) and contradicts Postulate V.

[17] The formally simpler expressions $\langle(\psi_\alpha(x)\pm\psi_\alpha^*(x))(\psi_\alpha(y)\pm\psi_\alpha^*(y))\rangle_0$ would also vanish for spacelike ξ. Such expressions, however, do not have simple transformation properties under the proper Lorentz group and so do not permit application of the methods of Hall and Wightman.

The analogs of Eqs. (4) and (13) for non-Hermitian fields, i.e.,

$$\langle[\varphi^*(x),\varphi(y)]\rangle_0=\langle\{\psi_\alpha^*(x),\psi_\alpha(y)\}\rangle_0=0 \quad (\xi \text{ spacelike}), \quad (27)$$

or the stronger relation

$$\langle\{\bar{\psi}_\alpha(x),\psi_\beta(y)\}\rangle_0=0 \quad (\xi \text{ spacelike}), \quad (28)$$

cannot be derived from Postulate I. To obtain these equations one has to invoke Postulate II.

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Nucleon Size Contributions to the Hyperfine Structure of Hydrogen and Helium*

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The effect of the electromagnetic structure of the proton and neutron upon the hyperfine splitting of S states in hydrogen and helium is calculated nonrelativistically. Convenient formulas are given, assuming that the charge and moment distributions of the proton and neutron are Gaussian. A summary of calculations of nuclear structure effects in deuterium is given and the present experimental situation is reviewed. If there is no relativistic or interaction moment contribution to the magnetic moment of the deuteron, the percent D state is 3.9, which is, under the same assumptions, consistent with the hyperfine splitting of deuterium.

I. INTRODUCTION

Nuclear structure contributions to the hyperfine structure (hfs) of hydrogen and helium have been a subject of continual interest. First Bohr,[1] and then Low[2] more carefully, calculated nonrelativistically such effects in deuterium, and subsequently there was a calculation by Salpeter and Newcomb[3] of relativistic corrections, while tritium and He^3 have been investigated by other workers.[4,5] None of these calculations includes the effect of nucleon structure, which is calculated in this paper. For the proton there have been a number of calculations summarized in the work of Zemach.[6] He has exhibited a general form for the hfs of hydrogen, which, at least as far as nucleon structure is concerned, is given in terms of the first statistical moment of a distribution which characterizes the proton structure, and furthermore can be determined by other experiments such as electron-proton scattering. Zemach rigorously establishes by field-theoretic arguments the validity of the nonrelativistic calculation.

In this investigation the nonrelativistic method of Zemach is extended, without field-theoretic justification, to H^2, H^3, He^{3+}, and He^3. This procedure is equivalent to neglecting relativistic and interaction moment contributions. In Sec. II the nucleon size contribution to H^2 and H^3 is derived assuming for convenience of calculation that the nucleons can be characterized by Gaussian charge and magnetic moment distributions. Although this is apparently not true experimentally for the proton,[7] the difference between a Gaussian and the true

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[1] A. Bohr, Phys. Rev. **73**, 1109 (1948).
[2] F. Low, Phys. Rev. **77**, 361 (1950).
[3] E. E. Salpeter and W. Newcomb, Phys. Rev. **87**, 150 (1952).
[4] E. N. Adams, II, Phys. Rev. **81**, 1 (1951); A. M. Sessler and H. M. Foley, Phys. Rev. **94**, 761 (1954).
[5] A. M. Sessler and H. M. Foley, Phys. Rev. **98**, 6 (1955).
[6] A. C. Zemach, Phys. Rev. **104**, 1771 (1956).
[7] F. Bumiller and R. Hofstadter, Bull. Am. Phys. Soc. Ser. II, **3**, 50 (1958).