# Hall and Transverse Magnetoresistance Effects for Warped Bands and Mixed Scattering*

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Evaluation of the transport integrals for warped bands using the method of McClure was done for relaxation times determined by mixed scattering from acoustic phonons and ionized impurities. Hall and transverse magnetoresistance coefficients were calculated for parameters characteristic of the degenerate valence bands in germanium and silicon, as obtained from cyclotron resonance data. For germanium, results are consistent with observation in regard to a density-of-states ratio for fast and slow holes of approximately 4% and in the occurrence of fine structure, especially a minimum in the Hall coefficient between 1000 and 2000 gauss at 80°K, depending on the impurity content. With parameters representative of silicon, substantially different behavior is predicted, in line with experiment.

## I. INTRODUCTION

THE magnetic field dependence of galvanomagnetic effects in semiconductors is quite strikingly affected by the charge-carrier scattering mechanisms and by the band structure. The effects of different energy dependencies of the relaxation time on the Hall coefficient and magnetoresistance are illustrated by Wilson,[1] and the influence of re-entrant energy surfaces on Hall mobility was discussed by Shockley.[2] Johnson and Whitesell,[3] Madelung,[4] Appel,[5] and others have examined cases of mixed scattering by acoustic phonons and ionized impurities, while Beer, Armstrong, and Greenberg[6] have considered the effect of degenerate band structures on the Hall coefficient, and have also applied the treatment to Corbino magnetoresistance and thermomagnetic effects. In all of these cases, calculations were done only for a model with spherical energy surfaces. The restriction was necessary inasmuch as closed solutions of the Boltzmann equation, exact in magnetic field strength, exist only when the electron energy is a quadratic function of the wave number.[7] Consequently, studies of the magnetic-field dependence of transport effects in semiconductors have been confined largely to ellipsoidal models, as for $n$-type germanium,[8,9] or, in the case of degenerate bands in $p$-type germanium, to a spherical model where the fundamental characteristics of each band were determined from experiment.[10]

Consideration of anisotropic energy surfaces has, for the most part, been confined to the weak magnetic field region where the Jones-Zener[11] series solution of the Boltzmann equation in powers of $H$ is applicable. Such a procedure has been effectively applied to determine in first approximation the effect of warped surfaces in germanium and silicon[12] on state density, conductivity, and Hall effect for the case of a single scattering process. Characterization of the warped energy surfaces was done by means of cyclotron resonance information. More recently the coefficients have been evaluated for transverse and longitudinal magnetoresistance.[13]

For examining the transport coefficients as a function of magnetic field, however, power series in H must be avoided since such representations do not converge for high-mobility charge carriers even at moderate magnetic-field strengths. A different approach is therefore necessary. Such a method has been provided by McClure[14] who obtained the conductivity tensor in the form of Fourier series expansions in harmonics of the frequency of the carrier around the hodograph in the magnetic field. This development applies quite generally to arbitrary values of magnetic field and shapes of energy surfaces, provided the relaxation time is a function of energy alone. To determine the Fourier components of the harmonics for the warped surfaces characteristic of germanium and silicon would be quite involved. Straightforward calculations are of course possible in limiting cases—for example, when the energy surfaces are spheres (the series reducing to a single term) and also for the case of a cube. Goldberg, Adams, and Davis[15] have analyzed this latter case and have shown that with $H$ in the [100] direction such a surface leads to a ratio of Hall to drift mobility of 0.5 for a relaxation time independent of energy. This is to be compared to a value of 1.0 for spherical surfaces and

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[1] A. H. Wilson, *The Theory of Metals* (Cambridge University Press, New York, 1953), pp. 239, 240.
[2] W. Shockley, *Electrons and Holes in Semiconductors* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1950), p. 338.
[3] V. A. Johnson and W. J. Whitesell, Phys. Rev. **89**, 941 (1953).
[4] O. Madelung, Z. Naturforsch. **9a**, 667 (1954).
[5] J. Appel, Z. Naturforsch. **9a**, 167 (1954).
[6] Beer, Armstrong, and Greenberg, Phys. Rev. **107**, 1506 (1957).
[7] A. H. Wilson, reference 1, p. 224.
[8] B. Abeles and S. Meiboom, Phys. Rev. **95**, 31 (1954).
[9] M. Shibuya, Phys. Rev. **95**, 1385 (1954).
[10] Willardson, Harman, and Beer, Phys. Rev. **96**, 1512 (1954).
[11] H. Jones and C. Zener, Proc. Roy. Soc. (London) **A145**, 268 (1934).
[12] B. Lax and J. G. Mavroides, Phys. Rev. **100**, 1650 (1955); these authors will hereafter be designated by LM.
[13] J. G. Mavroides and B. Lax, Phys. Rev. **107**, 1530 (1957); **108**, 1648(E) (1957).
[14] J. W. McClure, Phys. Rev. **101**, 1642 (1956).
[15] Goldberg, Adams, and Davis, Phys. Rev. **105**, 865 (1957); these authors will hereafter be designated by GAD.

0.71 as obtained by LM-DZL[16] for the heavy-mass band in germanium. In regard to the low value of $\mu^H/\mu_\sigma$, GAD point out that the amplitudes of the harmonics of the velocity would be less for the warped surface than for a cube, and they discuss a treatment including only the third and fifth harmonics, with amplitudes chosen to as to give the LM value for $\mu^H/\mu_\sigma$.

Our analysis also involves the third and fifth harmonics. Furthermore, in addition to the amplitudes of the velocity harmonics, there are other coefficients whose values depend on the shape of the energy surfaces. These include the parameter connecting the cyclotron frequencies with magnetic field, and a coefficient associated with the density of states.

A further modification in our development is to replace the constant-$\tau$ treatment by the energy-dependent relation approximating scattering by acoustic phonons and ionized impurities. The light-mass band is treated by the spherical model, for which a greater contribution to the scattering by ionized impurities is assumed.

## II. DEVELOPMENT OF EQUATIONS

### A. Constant Relaxation Time

McClure's theory yields for the conductivity coefficients[17] $S_{xx}$ and $S_{xy}$ the following expressions:

$$S_{xx}=pe\mu^s\alpha[1/(1+\omega^2\tau^2)+b_3/(1+9\omega^2\tau^2)+b_5/(1+25\omega^2\tau^2)+\cdots], \quad (1)$$

$$S_{xy}/H=p(e/c)(\mu^s)^2\alpha a[1/(1+\omega^2\tau^2)-3b_3/(1+9\omega^2\tau^2)+5b_5/(1+25\omega^2\tau^2)-\cdots+\cdots], \quad (2)$$

where $\mu^s$ is the conductivity mobility at zero magnetic field for a spherical energy surface. In the present case of constant $\tau$, it is equivalent to $e\tau/m^*$. The constants $\alpha$, $a$, $b_3$, and $b_5$, are indicative of the band structure. They are known for spherical and cubical energy surfaces and are readily related to the LM parameters. The parameter $a$ is associated with the cyclotron frequency $\omega$ as follows:

$$\omega=aeH/m^*c. \quad (3)$$

At weak magnetic fields, the LM representations give

$$S_{xx}=pe\mu^s(\mathfrak{A}_{xx}/\mathfrak{A}_d), \qquad H\to 0 \quad (4)$$

$$S_{xy}/H=p(e/c)(\mu^s)^2(\mathfrak{A}_{xy}/\mathfrak{A}_d), \quad H\to 0 \quad (5)$$

where the anisotropic factors $\mathfrak{A}_d$, $\mathfrak{A}_{xx}$, and $\mathfrak{A}_{xy}$ represent the $\gamma$ series in LM Eqs. (10), (11), and (12), respectively. The quantity $p/\mathfrak{A}_d$ represents the carrier density in a spherical band. The zero-magnetic-field conductivity mobility $\mu$ in the warped band is given by $\mu^s\mathfrak{A}_{xx}/\mathfrak{A}_d$.

Determination of $\alpha$, $a$, $b_3$, and $b_5$ for the heavy-mass band is subject to the conditions that the values of these parameters be between those for spherical and cubical surfaces, that at low magnetic fields they agree with the LM results, and that at strong fields the Hall coefficient yields $(pec)^{-1}$, with a proportionality factor of unity.[18]

TABLE I. Values of parameters determing the conductivity coefficients for several types of energy surfaces.

| Parameter | Value for type of surface indicated: Sphere | Cube,[b] $H$ along [100] | Warped band in Ge[c] | Our choice |
|---|---|---|---|---|
| *Basic* | | | | |
| $\alpha$ | 1 | $8/\pi^2$ | ... | 0.96 |
| $a$ | 1 | $\pi/4$ | ... | 0.935 |
| $b_3$ | 0 | 1/9 | ... | 0.085 |
| $b_5$ | 0 | 1/25 | ... | 0.035 |
| *Derived*[a] | | | | |
| $\mu/\mu^s$ | 1 | 1 | 1.075 | 1.075 |
| $\mu^H\mu/(\mu^s)^2$ | 1 | 1/2 | 0.823 | 0.826 |
| $\mu^H/\mu$ | 1 | 1/2 | 0.713 | 0.714 |
| $\mu_\infty\mu^s/(\mu_\infty^H)^2$ | 1 | 4/3 | ... | 1.110 |
| $\mu_\infty/\mu_\infty^H$ | 1 | 1 | ... | 1.005 |

[a] See Eqs. (6)–(10). For comparison with the LM results, an energy-independent relaxation time was used here.
[b] From GAD. The values of $\alpha$, $a$, and $\mu/\mu^s$ should be increased somewhat—with a corresponding decrease in $\mu_\infty\mu^s/(\mu_\infty^H)^2$—due to a density-of-states factor which is not readily evaluated. The fundamental ratios $\mu^H/\mu$ and $\mu_\infty/\mu_\infty^H$ are unchanged.
[c] From LM-DZL.

It is convenient to write the equations in the limiting cases of weak and strong magnetic fields so that only the parameters characteristic of the band shape are present. This can be done in terms of mobility ratios by using Eqs. (1), (4) and (2), (5) with the result that:

$$\mu/\mu^s\equiv S_{xx}[1/pe\mu^s]=\alpha(1+b_3+b_5+\cdots)=\mathfrak{A}_{xx}/\mathfrak{A}_d, \quad H\to 0 \quad (6)$$

$$\mu^H\mu/(\mu^s)^2\equiv[S_{xy}/H][c/pe(\mu^s)^2]=\alpha a(1-3b_3+5b_5-\cdots+\cdots)=\mathfrak{A}_{xy}/\mathfrak{A}_d, \quad H\to 0. \quad (7)$$

The quantity $\mu^H$ is the Hall mobility in the warped band, defined in terms of the transport properties of the band, namely the limiting Hall coefficient at zero magnetic field and the conductivity $\sigma_0$, as follows[19]:

$$\mu^H\equiv R_0\sigma_0c. \quad (8)$$

One can also define mobilities in the warped band in the strong-field limit: $\mu_\infty\equiv\sigma_\infty/pe$ and $\mu_\infty^H\equiv R_\infty\sigma_\infty c$, so that Eqs. (1) and (2) may be written

$$\mu_\infty\mu^s/(\mu_\infty^H)^2\equiv S_{xx}^\infty[\mu^sH^2/pec^2]=(\alpha/a^2)\times[1+b_3/9+b_5/25+\cdots], \quad H\to\infty \quad (9)$$

$$\mu_\infty/\mu_\infty^H\equiv S_{xy}^\infty[H/pec]=(\alpha/a)\times[1-b_3/3+b_5/5-\cdots+\cdots], \quad H\to\infty. \quad (10)$$

The basic parameters in the preceding equations are listed in Table I for spherical and cubical surfaces along

[16] This notation refers to the equations of LM using the revised values of the parameters given in Dexter, Zeiger, and Lax, Phys. Rev. **104**, 637 (1956).
[17] These are equivalent to the $S_{11}$ and $S_{12}$, respectively, of GAD.
[18] For a general derivation of this relationship, consult J. A. Swanson, Phys. Rev. **99**, 1799 (1955); Lifshitz, Azbel, and Kaganov, J. Exptl. Theoret. Phys. U.S.S.R. **31**, 63 (1956) [translation: Soviet Physics JETP **4**, 41 (1957)].
[19] See reference 2, p. 209. The Gaussian system of units is being used throughout this section.

with the predictions from cyclotron resonance experiments on germanium. The values used in subsequent calculations are shown in the last column.

### B. Case of Mixed Scattering by Lattice Vibrations and Ionized Impurities

The common expression for the dependence of the relaxation time on energy applicable in this case may be written[20]

$$\tau=\frac{3\sqrt{\pi}}{4}\frac{m^*}{e}\mu_L{}^s\frac{x^{\frac{3}{2}}}{\beta+x^2},\quad x\equiv\epsilon/kT, \tag{11}$$

where $\epsilon$ is the energy and where $\mu_L{}^s$ is the lattice mobility for a spherical band. The factor $\beta$ is a slowly varying function of energy which is usually taken as an appropriately chosen constant throughout an integration.[6] Its magnitude is an indication of the degree of impurity scattering. For pure lattice scattering, $\beta$ is zero. The effective mass $m^*$ is related to the band parameters $A$ and $B$, as defined by LM, as follows:

$$m^*=m_0/(A\pm B),\quad (+\text{for light mass}, \; -\text{for heavy mass}). \tag{12}$$

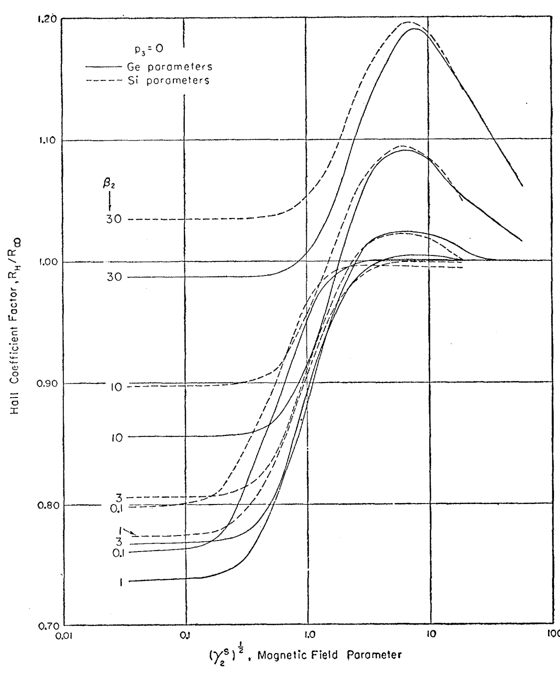


FIG. 1. Hall coefficient factor for a single warped band. The solid lines are for parameters representative of the heavy-mass band in germanium as determined from cyclotron resonance. The dashed lines are applied to silicon. The parameter $\beta_2$ measures the degree of impurity scattering.

[20] Information on this expression can be found in reference 3 and in the literature cited in reference 6. A detailed presentation is available in the chapter by Harvey Brooks, *Advances in Electronics and Electron Physics* (Academic Press, Inc., New York, 1955), Vol. 7.

For our case, McClure's expressions may be written as follows[21]:

$$S_{xx}=\frac{2}{3}\frac{pe^2}{m^*}\alpha\left\langle x\tau\sum_{N\text{ odd}}\frac{b_N}{[1+(N\omega\tau)^2]}\right\rangle, \tag{13}$$

$$S_{xy}=\frac{2}{3}\frac{pe^2}{m^*}\alpha\left\langle x\omega\tau^2\sum_{N\text{ odd}}\frac{b_N(-1)^nN}{[1+(N\omega\tau)^2]}\right\rangle,\quad n\equiv\tfrac{1}{2}(N-1), \tag{14}$$

where the average over energy is defined by

$$\langle q\rangle\equiv\frac{2}{\sqrt{\pi}}\int_0^\infty x^{\frac{1}{2}}e^{-x}q\,dx. \tag{15}$$

Using the expression for $\tau$ from Eq. (11), we obtain

$$S_{xx}=pe\mu_L{}^s\alpha\sum_{N\text{ odd}}b_NK(\beta,a^2N^2\gamma^s), \tag{16}$$

$$S_{xy}=\tfrac{1}{2}(\pi)^{\frac{1}{2}}pe\mu_L{}^s\alpha a(\gamma^s)^{\frac{1}{2}}\sum_{N\text{ odd}}b_N(-1)^nNL(\beta,a^2N^2\gamma^s), \tag{17}$$

where

$$\gamma^s\equiv(9\pi/16)(\mu_L{}^sH/c)^2,\quad\text{Gaussian units.} \tag{18}$$

In the common case where mobility is expressed in $\text{cm}^2$/volt-sec and $H$ is in gauss, Eqs. (16) and (17) apply with $\gamma^s$ defined as follows:

$$\gamma^s\equiv(9\pi/16)(\mu_L{}^sH/10^8)^2,\quad\text{laboratory units.} \tag{19}$$

The integrals $K(\beta,\gamma)$ and $L(\beta,\gamma)$ are tabulated in reference 6.

The Hall coefficient and transverse magnetoresistance are given by

$$R_H=S_{xy}/[H(S_{xx}{}^2+S_{xy}{}^2)], \tag{20}$$

$$\Delta\rho/\rho_H=1-[S_{xx}{}^2+S_{xy}{}^2]/S_{xx}S_{xx}{}^0, \tag{21}$$

where the transport coefficients are summed over each band, with the $S_{xy}$ positive for holes and negative for electrons.

It is convenient to introduce a dimensionless quantity, the Hall coefficient factor, which relates the Hall coefficient and carrier densities. This factor may be written $R_H/R_\infty\equiv R_Hpe$; and, as has been pointed out by numerous investigators, $R_{H\to0}/R_\infty\equiv\mu^H/\mu$.

## III. CALCULATED RESULTS

With a spherical energy surface for the light-mass band (subscript 3) and a warped surface characterized by the constants determined in Sec. A for the heavy-mass band (subscript 2), the magnetic-field dependence

[21] See GAD Eqs. (7.5) and (7.6), recalling our different definition of $x$.

of the Hall coefficient factor may be written

$$\frac{R_H}{R_\infty}=\frac{3\pi}{8}\left(1+\frac{p_3}{p_2}\right)\times\frac{\mathcal{L}(\beta_2,\gamma_2^s)+(p_3/p_2)(\mu_{L\,,3}^s/\mu_{L\,,2}^s)^2L(\beta_3,\gamma_3^s)}{[\mathcal{K}(\beta_2,\gamma_2^s)+(p_3/p_2)(\mu_{L\,,3}^s/\mu_{L\,,2}^s)K(\beta_3,\gamma_3^s)]^2+\frac{1}{4}\pi\gamma_2^s[\mathcal{L}(\beta_2,\gamma_2^s)+(p_3/p_2)(\mu_{L\,,3}^s/\mu_{L\,,2}^s)^2L(\beta_3,\gamma_3^s)]^2}, \tag{22}$$

where

$$\mathcal{L}(\beta_2,\gamma_2^s)\equiv a\alpha[L(\beta_2,a^2\gamma_2^s)-3b_3L(\beta_2,9a^2\gamma_2^s)+5b_5L(\beta_2,25a^2\gamma_2^s)], \tag{23}$$

$$\mathcal{K}(\beta_2,\gamma_2^s)\equiv \alpha[K(\beta_2,a^2\gamma_2^s)+b_3K(\beta_2,9a^2\gamma_2^s)+b_5K(\beta_2,25a^2\gamma_2^s)], \tag{24}$$

with the parameters $a$, $\alpha$, $b_3$, and $b_5$ for germanium given in Table I.

The transverse magnetoresistance coefficient is given by

$$\frac{\Delta\rho}{\rho_H\gamma_2^s}=\frac{1}{\gamma_2^s}\left[1-\frac{[\mathcal{K}(\beta_2,\gamma_2^s)+(p_3/p_2)(\mu_{L\,,3}^s/\mu_{L\,,2}^s)K(\beta_3,\gamma_3^s)]^2+\frac{1}{4}\pi\gamma_2^s[\mathcal{L}(\beta_2,\gamma_2^s)+(p_3/p_2)(\mu_{L\,,3}^s/\mu_{L\,,2}^s)^2L(\beta_3,\gamma_3^s)]^2}{[\mathcal{K}(\beta_2,\gamma_2^s)+(p_3/p_2)(\mu_{L\,,3}^s/\mu_{L\,,2}^s)K(\beta_3,\gamma_3^s)][\mathcal{K}(\beta_2,0)+(p_3/p_2)(\mu_{L\,,3}^s/\mu_{L\,,2}^s)K(\beta_3,0)]}\right]. \tag{25}$$

Evaluations of the transport integrals, $K$ and $L$, were done from tabulated values,[6] using suitable interpolation procedures. The effect of the warping on the Hall coefficient factor for a single band is shown in Fig. 1, the solid lines being representative of germanium. It is seen that the magnitude of the maximum is conspicuously dependent on the degree of impurity scattering, given by the factor $\beta\equiv 6(\mu_L^s/\mu_I^s)$.

The effect of the fast holes is to raise $R_H/R_\infty$ substantially at weak fields, depending on the amount of impurity scattering. Results are shown in Fig. 2 for ratios $p_3/p_2$ of 0.03, 0.04, and 0.045. Calculations were done for values of $\beta_2$ and $\beta_3$ which appeared in the tables of the transport integrals, and which seem representative of germanium, as indicated by the ratio of heavy and light masses. The salient features of the curves agree with the experimental data, for example, the occurrence of minima and maxima and their shift with impurity scattering. In fact, in the example with least impurity scattering, the fine structure has disappeared. Some of these considerations may account for the fact that certain investigators have observed a minimum[22] in Hall coefficient as a function of magnetic field at 80°K, while others have not.[15] In addition, it is understandable that no minima have been observed at dry ice temperatures.

A comparison of some unpublished measurements of Reid and Willardson at 80°K with the theoretical curves for 4% fast holes is given in Fig. 3. A value $\mu_{L\,,2}^s$ of 55 000 cm²/volt-sec was chosen in order to place the minima of the theoretical curves at the magnetic field values indicated. Such a mobility is also consistent with the magnetoresistance data to be discussed later. Agreement of theoretical predictions and experimental values is semiquantitative, and it appears likely that values of the mixed scattering parameter $\beta$ exist which can bring the curves into a good coincidence.

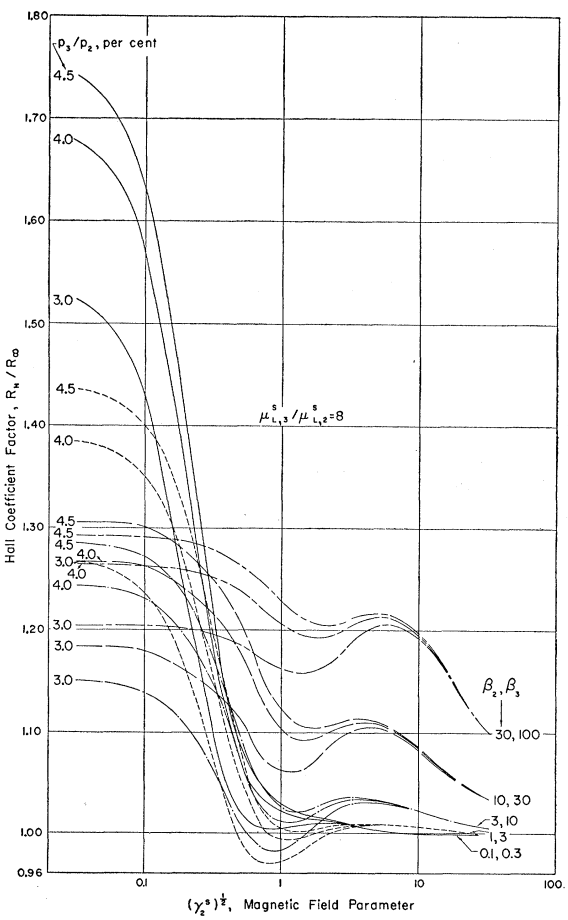


FIG. 2. Hall coefficient factor for degenerate valence bands representative of germanium. Parameters are density ratios of light and heavy holes, and impurity scattering factors.

[22] Peterson, Swanson, and Tucker, Bull. Am. Phys. Soc. Ser. II, 1, 117 (1956).

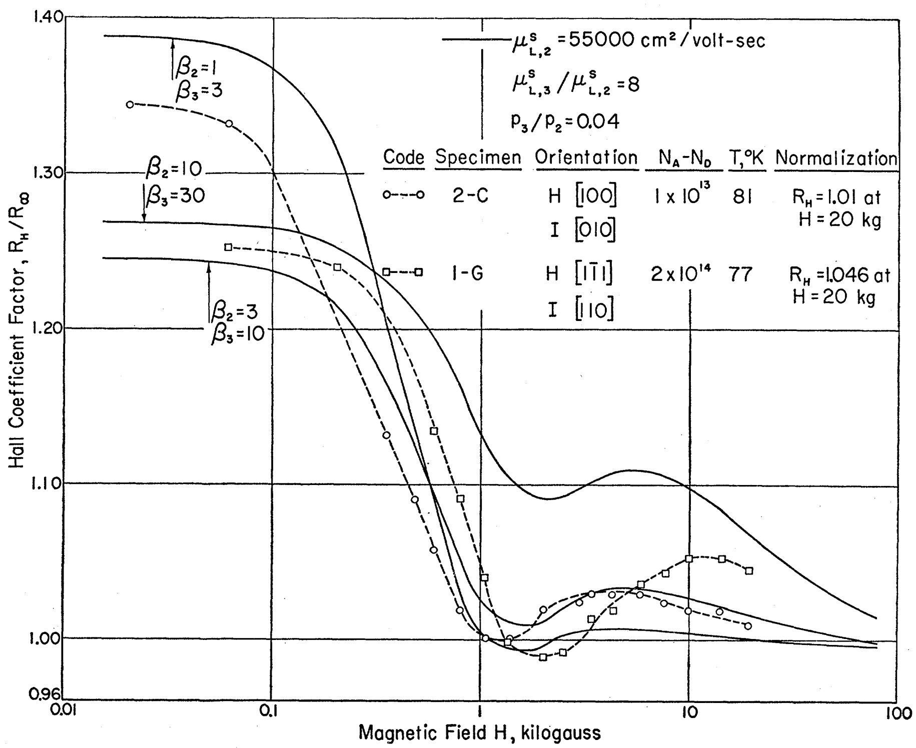


FIG. 3. Comparison of theoretical Hall coefficient factor (solid lines) with experimental data on germanium (dashed lines) for liquid nitrogen temperatures.

Even with the theoretical curves shown, there is good agreement in slope in the region of rapid change with field, and also in the shape of the minimum and the maximum in the case of specimen 2-C. Since our measurements did not reach the infinite field limit, it was necessary to choose a normalization value for the experimental curve. This was done on the basis of a deviation of $R_H$ at 20 kilogauss from $R_\infty$ which appeared reasonable for the particular sample and temperature of measurement.

Figure 4 shows a comparison of results at 196°K. At this temperature, a mobility $\mu_{L}{}^{s}{}_{,2}$ of 7000 cm²/volt-sec gives a good fit. From the experimental evidence, it is not possible to postulate the existence of a minimum. Such a behavior is in agreement with the calculated curve for the lower value of $\beta$. The manner of presenting the experimental data is not meant to imply a conclusion that $R_0/R_\infty$ is larger for the less pure specimen, since the difference at weak fields is only 4%. Furthermore, the normalization is somewhat arbitrary, inasmuch as the strong magnetic field limit could not be attained. It would not be unreasonable to expect that for specimen 1-G, $R_H$ should be normalized to 0.985 at 10 kilogauss, and for 2-C to 1.025 at 20 kilogauss. Such a procedure would place the curve for specimen 1-G slightly below that for 2-C at weak fields. Nevertheless, indications are that $R_0/R_\infty$ for specimen 1-G is larger than expected on the basis of simple theory. A possible reason might be the orientation, the effects of which were not considered in the calculations of the theoretical curves. Support for this suggestion is given in the results at 80°K where the maxima and minima of specimen 1-G are relatively prominent.

TABLE II. Values of parameters used for silicon.

| Parameter | Value from cyclotron resonance[b] | Our choice |
|---|---|---|
| *Basic* | | |
| $\alpha$ | | 0.96[c] |
| $a$ | | 0.935[c] |
| $b_3$ | | 0.072 |
| $b_5$ | | 0.030 |
| *Derived*[a] | | |
| $\mu/\mu^s$ | 1.049 | 1.058 |
| $\mu^H\mu/(\mu^s)^2$ | 0.847 | 0.838 |
| $\mu^H/\mu$ | 0.769 | 0.749 |
| $\mu_\infty\mu^s/(\mu_\infty{}^H)^2$ | | 1.108 |
| $\mu_\infty/\mu_\infty{}^H$ | | 1.008 |

[a] See Eqs. (6)–(10). For comparison with LM results, an energy-independent relaxation time was used here.
[b] From LM-DZL.
[c] For simplicity in computation, the germanium values were used.

The magnetoresistance calculations are presented in Fig. 5 for the mobilities corresponding to the two temperatures. Since the ordinates here are quite sensitive to the lattice mobility ($\gamma\sim\mu_L{}^2$), it is gratifying to find that those mobility values chosen on the basis of the Hall measurements are consistent with those predicted from magnetoresistance. At the high fields, the experimental data show evidence of the failure of the resistivity to saturate as predicted by most theories, in

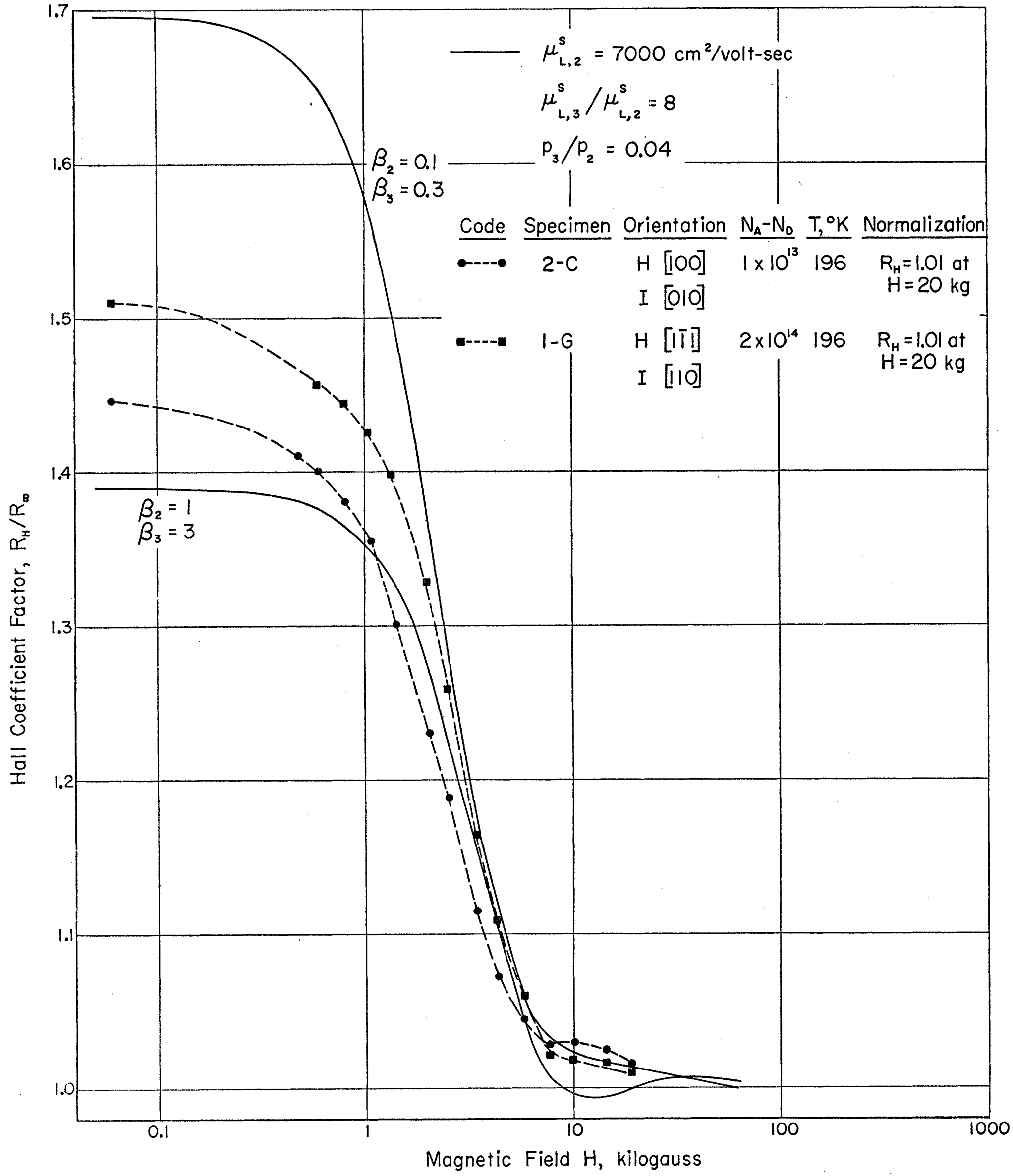


FIG. 4. Comparison of theoretical Hall coefficient factor (solid lines) with experimental data on germanium (dashed lines) for a temperature of 196°K.

agreement with results reported by several investigators.[23]

There is additional information available from experimental data for use in checking the consistency of our calculations, namely, the $R_0\sigma_0$ values. Knowing these, we can use relationships developed in the preceding section to determine $\beta$, given a particular $\mu_L{}^s$. Details are given in the Appendix. For an assumed ratio $\beta_3/\beta_2$ of 3, we obtain values of $\beta_2$ of 1.5 and 4 for specimens 2-C and 1-G, respectively, using the $\mu_L{}^s$ determined from the Hall data in Fig. 3. These results are in good agreement with predictions based on the relative location of the theoretical and experimental curves. Similarly, one obtains values of 0.6 and 1.1 for $\beta_2$ at 196°K.

## IV. RESULTS FOR SILICON

A limited number of calculations were made with parameters approximating those predicted by cyclotron resonance experiments on silicon. To save computational effort, the germanium values for $\alpha$ and $a$ were used (see Table II). The Hall characteristics of this band are shown by the dashed lines in Fig. 1. It is noted that these results indicate smaller effects due to the warping than were found in the germanium. Further discussion of this point is made later.

The light-mass band was again assumed spherical, with the values of $p_3/p_2=0.16$ and $m_3/m_2=3$, as sug-

[23] See reference 15, for example.

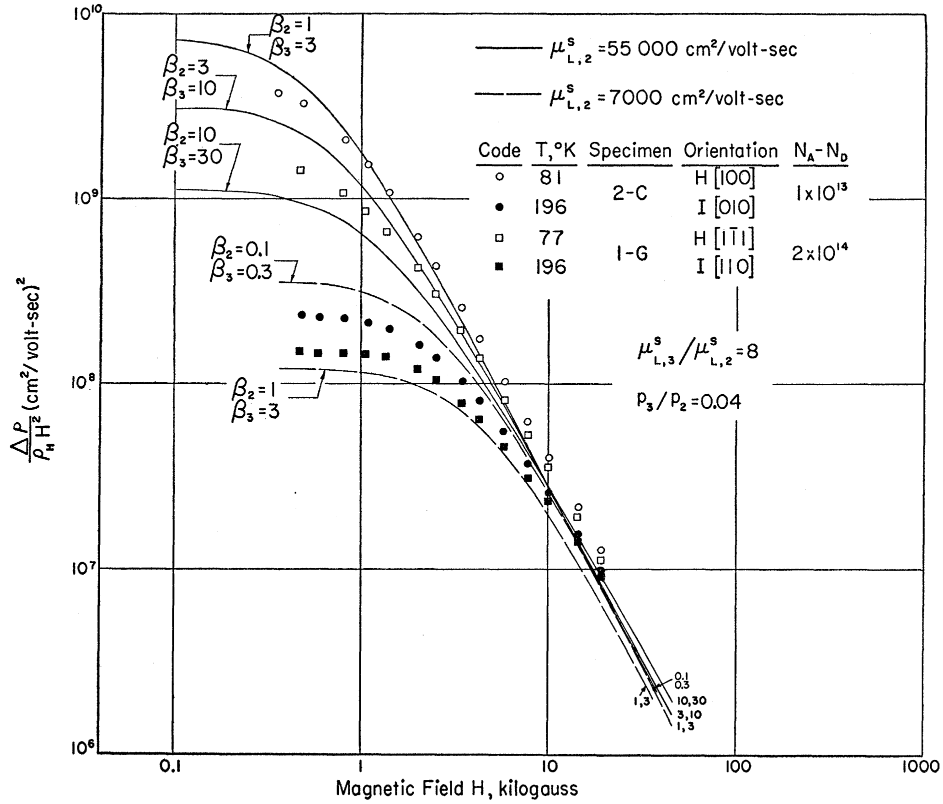


FIG. 5. Comparison of theoretical transverse magnetoresistance coefficients (solid lines and broken lines) with experimental data on germanium (isolated points) at liquid nitrogen and dry ice temperatures.

gested by the results of LM-DZL. On the basis of the mass ratios, $\mu_{L^s,3}/\mu_{L^s,2}$ was chosen[24] as 3. Calculations were done with values of the impurity scattering parameters $\beta_2$ and $\beta_3$ as were used for germanium[25] and also for $\beta_2=\beta_3$— this being the next entry in the tabulations of the transport integrals. It is expected that $\beta_3/\beta_2$ would be less for the silicon.

No curves are shown of the Hall coefficients for $\beta_3=\beta_2$ since, contrary to experiment, they were all monotonically decreasing with $\gamma$, being different from most of those for $\beta_3/\beta_2\sim3$. Interpolation for $1<\beta_3/\beta_2<3$ was, therefore, not possible.

Results for the Hall coefficient ratio are presented in Fig. 6. Experimental data, representing some unpublished measurements of Bate and Reid of these laboratories, are also shown. The theory accounts semiquantitatively for the observed behavior, although the agreement is not so good as in the case of germanium. It would appear that parameters representative of a greater degree of warping are necessary. This postulate finds support in the magnetoresistance data in Fig. 7, where it is seen that the theoretical curves are low. The experimental points could, of course, be lowered by increasing $\mu_{L^s,2}$, but this does not appear justified since it would shift the Hall curves to larger values of $\gamma_2{}^s$. It would also increase the $\beta_2$'s obtained from the $R_0\sigma_0$ data. Since these values for specimens 12-*t* and 14-*b* are reasonable, namely, 0.8 and 1.9 (see Appendix), substantial changes there are not warranted.

The theoretical magnetoresistance curves could also be raised by increasing $p_3/p_2$ or $\mu_{L^s,3}/\mu_{L^s,2}$. Either of these procedures would, however, decrease the amplitudes of the Hall maxima, which are already small. Another consideration, which we did not examine, is orientation effects. Our silicon specimens were cut from pulled crystals with no attempts at special orientations. Some experimental data on orientation are available from measurements of Pearson and Herring[26] and more recently by Long.[27]

Problems in connection with the anisotropy parameters, as determined from magnetoresistance, were also encountered by Long at the higher temperatures. His comments concerning nonparabolic behavior of the light-mass band and increased contributions from the split-off band appear pertinent.

[24] This result is obtained for equal relaxation times of light and heavy holes, a relationship followed quite closely in germanium and applicable to a lesser extent to silicon. For further information, consult p. 152 of the article by Brooks listed in reference 20.

[25] To make use of some of the germanium calculations, mobility ratios of $a\sqrt{10}$ or 2.96 were used.

[26] G. L. Pearson and C. Herring, Physica **20**, 975 (1954).

[27] D. Long, Phys. Rev. **107**, 672 (1957); also D. Long and J. Myers, Phys. Rev. **109**, 1098 (1958).

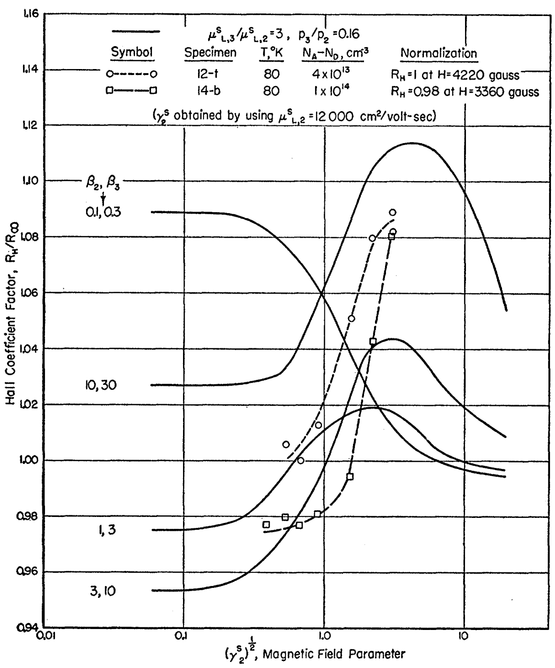


FIG. 6. Comparison of theoretical Hall coefficient factor (solid lines) with experimental data on silicon (dashed and broken lines) for a temperature of 80°K.

## V. DISCUSSION

The theory gives a semiquantitative fit to the Hall and magnetoresistance data of extrinsic $p$-type germanium including the prediction of fine structure for the Hall measurements at 80°K. Significantly, this occurs with strict use of parameters as determined by cyclotron resonance experiments. In the case of germanium, this means a density ratio for fast and slow holes of about 4%. In previous calculations it was necessary to use ratios of 2%,[10,28] when warped bands were not taken into account and impurity scattering was assumed negligible at temperatures well above 80°K. Other treatments applied to data at 80°K called for a mobility ratio of fast and slow holes different from the cyclotron resonance predictions.[29]

Lattice mobility values, as determined from the magnetic field strengths at the Hall minima, are consistent with those required to fit the magnetoresistance data. An independent check was obtained through the experimental determinations of $R_0\sigma_0$. A consideration of the relationship between $\mu_L{}^s$ and $R_0\sigma_0$ reveals that differences between these quantities may be appreciably augmented by multiband contributions to the transport process. Therefore, a knowledge only of the temperature dependence of $R_0\sigma_0$ does not permit firm conclusions on the behavior of the mobility unless additional information is available on the band structure.

[28] Mochan, Obraztsov, and Krylov, J. Tech. Phys. U.S.S.R. **27**, 242 (1957) [translation: Soviet Phys.—Tech. Phys. **2**, 213 (1957)].

[29] See discussion in GAD.

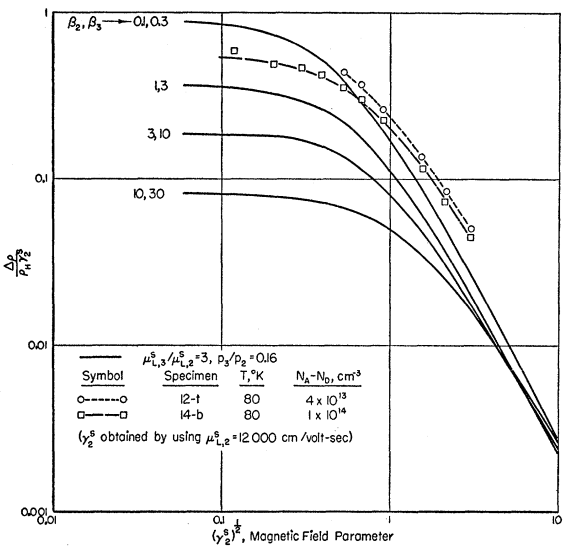


FIG. 7. Comparison of theoretical transverse magnetoresistance coefficients (solid lines) with experimental data on silicon (dashed and broken lines) at 80°K.

Results indicate the importance of considering mixed scattering processes as well as warped bands, especially as regards the fine structure, which in our measurements included a minimum followed by a secondary maximum. This maximum renders difficult the precise determination of the strong-field limit. The data also indicate that when one is concerned with accuracies of a few percent, ionized impurity scattering must be considered throughout the extrinsic range.

The importance of the light-mass band in modifying the characteristics from those due to the warped band alone is striking. In silicon, due to the lower mobility ratio, the effect is not so pronounced as in germanium, and the primary maxima are still evident, except in cases where the impurity scattering is unusually small.

Probably the most obvious area for refinement is in the treatment of the scattering. The ratios $\beta_3/\beta_2$ were chosen on the basis of impurity scattering mobilities in the two bands varying as the inverse square root of the effective masses, and then using the nearest entry in the tabulations of the transport integrals. In germanium this gives a ratio of approximately 3—in silicon it should be substantially less. This is probably still another reason why less satisfactory agreement was obtained for silicon than for germanium. However, isolated calculations were done for different values of $\beta_3/\beta_2$, and these support the belief that our choice is adequate, at least for germanium. The latitude in $\beta_3/\beta_2$ is effectively constrained by the fact that small ratios accent the effect of the spherical light-mass band in washing out the fine structure, while large values significantly reduce the magnetoresistance. A rigorous theoretical examination of $\beta_3/\beta_2$ would be especially desirable, as would a more sophisticated treatment of the thermal scattering, which we approximated by the acoustic phonon process.[30] Such a simple mechanism does not account for the observed temperature dependence of the lattice mobility. Therefore, the particular values obtained for $\mu_L{}^s$ should not be regarded strictly. A more involved analysis, taking into account scattering also by optical modes, could be expected to change somewhat the $\mu_L{}^s$ values, and also those for the $\beta$'s.

Further refinements might be possible in the series for the transport coefficients of the warped band, including the addition of more harmonics and redetermination of the other parameters. The value of this complication is perhaps marginal for germanium but may be significant in the case of silicon, where it appears that a greater anisotropy than was used here is necessary. Further support for this suggestion is found in the fact that at 77°K, the ratio of longitudinal and transverse magnetoresistances is larger in silicon than in germanium by a factor of at least 5.[27,31]

It would be interesting if Hall and magnetoresistance data were available as a function of temperature and impurity concentration for the same crystal orientation. This would eliminate some of the uncertainties in the interpretation of the present experimental data. It would be of especial interest if material of higher purity were studied, since our treatment predicts the prominence of the fine structure to be sensitive to the degree of impurity scattering. For sufficiently pure silicon, the Hall coefficient curves at 77°K should exhibit the germanium characteristic.

Finally, we wish to point out that studies of the fine structure in the magnetic-field dependence of the Hall coefficient can provide useful information on new semiconductors. For example, similarities in the behavior of $p$-type indium antimonide and $p$-type germanium have been established by several investigators. Recent measurements in our laboratories have indicated that the behavior of diamond is reminiscent of silicon rather than germanium. In aluminum antimonide, the Hall coefficient at 80°K was found to exhibit the large maximum characteristic of silicon.

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## VII. APPENDIX

### A. Theoretical Relationship between $\mu_L{}^s$ and $R_0\sigma_0$

In terms of carrier densities and actual mobilities in the heavy- and light-mass bands, one can write

$$R_0\sigma_0=[R_0/R_\infty][(p_2\mu_2+p_3\mu_3)/(p_2+p_3)]. \quad \text{(A-1)}$$

With the relationships given in Sec. II, the above equation can be put in the form

$$R_0\sigma_0=\mu_L{}^s{}_{,2}[R_0/R_\infty][(\mathcal{A}_{xx}/\mathcal{A}_d)K(\beta_2) + (p_3/p_2)(\mu_L{}^s{}_{,3}/\mu_L{}^s{}_{,2})K(\beta_3)][p_2/(p_2+p_3)]. \quad \text{(A-2)}$$

### B. Application to Experiment

For the parameters listed in Tables I and II, Eq. (A-2) becomes

$$R_0\sigma_0=0.962\mu_L{}^s{}_{,2}[R_0/R_\infty][1.075K(\beta_2) +0.32K(\beta_3)], \quad \text{(for Ge)}, \quad \text{(A-3)}$$

$$R_0\sigma_0=0.862\mu_L{}^s{}_{,2}[R_0/R_\infty][1.058K(\beta_2) +0.48K(\beta_3)], \quad \text{(for Si)}. \quad \text{(A-4)}$$

In using the above equations, the values of $\mu_L{}^s{}_{,2}$ were those obtained from fitting the curves of theoretical and experimental Hall coefficients as functions of magnetic fields. Approximate values of $R_0/R_\infty$ were obtained from the experimental data. Determinations were then made of the impurity scattering parameter, $\beta_2$, with 3 as an assumed value of the ratio $\beta_3/\beta_2$.

Results are listed in Table III.

TABLE III. Values of $\beta_2$ as determined from $R_0\sigma_0$ data.

| Specimen | $T$, °K | $\rho_0$ [a] ohm-cm | $R_0\sigma_0$ [a] cm²/volt-sec | $R_0/R_\infty$ [b] | $\mu_L{}^s{}_{,2}$ | $\beta_2$ |
|---|---|---|---|---|---|---|
| Ge 2-C | 81 | 10.80 | 54 600 | 1.35 | 55 000 | 1.5 |
| | 196 | 74.10 | 9420 | 1.45 | 7000 | 0.6 |
| Ge 1-G | 77 | 1.03 | 36 000 | 1.25 | 55 000 | 4.0 |
| | 196 | 5.75 | 8500 | 1.51 | 7000 | 1.1 |
| Si 12-$t$ | 80 | 15.00 | 10 250 | 1.00 | 12 000 | 0.8 |
| Si 14-$b$ | 80 | 7.17 | 7680 | 0.98 | 12 000 | 1.9 |

[a] From experiment.
[b] From Figs. 3, 4, or 6.

[30] See, for example, F. J. Blatt, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press, Inc., New York, 1957), Vol. 4, p. 340. Further literature pertinent to the problem of mixing scattering is cited in Sec. IV of reference 6.

[31] G. L. Pearson and H. Suhl, Phys. Rev. **83**, 768 (1951).