Comparison of Σ^+p and Σ^-n Systems*

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Assuming that the nuclear interactions between Σ^-n and Σ^+p are charge-symmetric, the effect of the Coulomb potential in the Σ^+p system on the binding energy and wave function of a possible bound state of the Σ^+p system is calculated. Using a phenomenological square-well potential the binding energies, ϵ_+ and ϵ_-, of the Σ^+p and Σ^-n systems are determined. In particular, it is found that the limiting condition, $\epsilon_+ \to 0$, implies that $\bar{\epsilon}_-$ is 0.38, 0.47, 0.62 Mev when the square-well range parameter, b, equals 1.4×10^{-13}, 1.0×10^{-13}, 0.6×10^{-13} cm respectively. If the Σ^-n interaction determines ϵ_- to lie in the small but finite region $0<\epsilon_-<\bar{\epsilon}_-$, then the Σ^-n system is bound and the Σ^+p system unbound. Conversely, if Σ^+p is bound, then $\epsilon_->\bar{\epsilon}_-$. A brief discussion of the experimental situation with regard to these hyperon compounds is given.

I. INTRODUCTION

A STUDY of the interactions of hyperons with nucleons is of fundamental importance in understanding the nature of these new baryons. Since the hyperons have such short lifetimes, $\sim10^{-10}$ sec, the accumulation of data on the scattering of hyperons by nucleons is necessarily an exceedingly slow process. The existence and study of bound states of Λ^0 hyperons with nucleons has furnished a considerable amount of information about the Λ^0-nucleon interaction. It is clear that a similar investigation of Σ-nucleon bound systems, if they exist, would be of interest.

The classification scheme of Gell-Mann[1] and Nishijima[2] implies that the only possible "stable" compounds of Σ hyperons are a Σ^+ hyperon with one or more protons and a Σ^- hyperon with one or more neutrons.[3] A compound containing for example a Σ^+ hyperon and a neutron would disintegrate rapidly, $\sim10^{-21}$ sec, via the exothermic reaction $\Sigma^++n\to\Lambda^0+p$. Several authors[4,5] have discussed the binding energies of the Σ^-n and Σ^+p systems from the field-thoretic point of view. Making several plausible assumptions and approximations, they conclude that the Σ^-n system and possibly also the Σ^+p system may be bound with very small binding energies.

Experimentally, one event[6] has been found which was interpreted as the decay from rest of a Σ^+p compound. While profile[6] measurements and energy-momentum balance considerations agree very well with this interpretation, there still remains a probability of the order of 2% that this event was produced by a K^- meson rather than by a Σ^+p compound.[7]

A Σ^-n compound can only decay via the reaction

$$\Sigma^-n\to n+n+\pi^-+(117-\epsilon_-)\text{ Mev}. \quad (1)$$

By analogy to the decay characteristics of the lightest Λ^0 hyperfragments, a Σ^+p compound is expected to decay predominantly via the emission of a π^+ or π^0 meson, that is,

$$\Sigma^+p\to p+n+\pi^++(110-\epsilon_+)\text{ Mev} \quad (2)$$

$$\Sigma^+p\to p+p+\pi^0+(116-\epsilon_+)\text{ Mev}. \quad (3)$$

The event reported in reference 6 is of type (2). From the experimental point of view, reaction (2) provides the most readily identifiable Σ-compound event. Among the known particles, only a K^- meson or a Σ^+p compound can give rise to an event in which a slowly moving particle disintegrates into a proton and an energetic π^+ meson ($\sim$100 Mev). Hence, if such an event were seen in a situation wherein the K^- meson hypothesis can be ruled out *a priori*, the existence of a Σ^+p bound state would be conclusively demonstrated. Such an unambiguous possible source of Σ^+p compounds does exist in the following sequence of events. A K^- meson comes to rest, is captured inside a nucleus and produces a Σ^+ hyperon. The Σ^+ hyperon then picks up a proton on the way out of the nucleus to form a Σ^+p compound. Perhaps about 1000 Σ^+ hyperons produced by K^- mesons captured at rest in nuclear emulsion have been examined by the world's emulsion groups without the detection of a Σ^+p compound via reaction (2). This indicates that if the Σ^+p compound is stable, the probability[8] of its formation and subsequent decay via reaction (2) is much less than 1%.

Reaction (3) cannot easily be distinguished from some types of nonmesonic decay or of π^0 mesonic decay

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[1] M. Gell-Mann, Phys. Rev. **92**, 833 (1953); and M. Gell-Mann and A. Pais, in *Proceedings of the International Conference on High-Energy Physics* (Pergamon Press, London, 1955).

[2] T. Nakano and K. Nishijima, Progr. Theoret. Phys. (Japan) **10**, 581 (1953) and K. Nishijima, Progr. Theoret. Phys. (Japan) **13**, 285 (1955).

[3] W. Holladay, quoted by R. G. Sachs, Phys. Rev. **99**, 1573 (1955).

[4] D. B. Lichtenberg and M. H. Ross, Phys. Rev. **107**, 1714 (1957).

[5] F. Ferrari and L. Fonda, Nuovo cimento **6**, 1027 (1957).

[6] Baldo-Ceolin, Fry, Greening, Huzita, and Limentani, Nuovo cimento **6**, 144 (1957).

[7] W. F. Fry, *Proceedings of the Padua-Venice Conference on Elementary Particles, 1957* (Suppl. Nuovo cimento, to be published).

[8] More precisely, it is the probability of decay via reaction (2) into a proton that has sufficient energy to make a visible track that is the relevant quantity. In nuclear emulsion this corresponds to a proton kinetic energy greater than about 0.3 Mev.

of a Λ^0 hyperfragment with $Z=2$. Reaction (1) cannot easily be distinguished from the decay of an ordinary Σ^- hyperon into one neutron and a π^- meson, except in those special cases in which the π^- meson is followed to the end of its range so that its energy can be determined with great precision. An additional complication in the case of reaction (1) is that analogously to the behavior of an ordinary Σ^- hyperon, a Σ^-n compound that comes to rest will be absorbed by the nucleus that captures it into an atomic orbit long before it has a chance to decay via reaction (1). Hence the method of measuring the π^- energy with great precision to distinguish a Σ^-n compound from a Σ^- hyperon can only be applied to decays in flight, but in such cases there is a considerable uncertainty in the Σ^- or Σ^-n velocity at the point of decay which will in most cases render the separation of Σ^-n decays from Σ^- decays impossible.

This discussion illustrates that in order to detect a Σ^-n compound, or a Σ^+p compound that decays via reaction (3), a great deal of painstaking measurement on the track of the suspect particle must be carried out in order to determine its charge and mass. Swami[9] has made a start in this direction on a limited number of events superficially classified as Σ^- hyperons, with inconclusive results as to the existence of a Σ^-n compound.

A priori, there is a finite probability that the Σ^+p system be unbound and the Σ^-n system be bound. Consequently, a careful comparison of these two systems from a phenomenological point of view seems warranted. If charge symmetry is assumed for the Σ-hyperon–nucleon interaction, as will be done in this note, the wave functions of the Σ^+p and Σ^-n systems differ only through the presence of a repulsive Coulomb potential in the Σ^+p system. The purpose of this note is to determine theoretically the effect of this repulsive Coulomb potential on the binding energy and wave function of the Σ^+p system. In particular, the extent of the energy region in which the Σ^-n system is bound and the Σ^+p system is unbound is determined.

Section II describes the mathematical problem and the physical approximations made. Section III contains the solution to the problem in the limit that the Σ^+p binding energy approaches zero. Section IV gives the Σ^+p wave function and binding energy for a particularly simple special case that corresponds to the Σ^+p binding energy having the value 0.014 Mev. Section V contains some further discussion and conclusions.

II. MATHEMATICAL PROBLEM

We assume that the Σ-nucleon interaction can be represented by an attractive, static, central potential, which is the same for Σ^-n as for Σ^+p except for the Coulomb potential. For definiteness the Σ^-n potential, $V_-(r)$, is taken to be a square well:

$$\begin{aligned} V_-(r) &= -V_0, \quad r\leq b \\ &= 0, \qquad r>b. \end{aligned} \tag{4}$$

The parameters V_0 and b are related to the binding energy ϵ_- of the lowest state of the Σ^-n system by the well-known formula[10]

$$\kappa_- \cot\kappa_- b = -\gamma_-, \tag{5}$$

where

$$\kappa_-^2 = K_-^2 - \gamma_-^2, \qquad \gamma_-^2 = (2\mu_-/\hbar^2)\epsilon_-,$$
$$K_-^2 = (2\mu_-/\hbar^2)V_0, \qquad \mu_- = M_{\Sigma^-}M_n/(M_{\Sigma^-}+M_n).$$

For the Σ^+p system we assume a potential $V_+(r)$ of the form

$$\begin{aligned} V_+(r) &= -V_0 + e^2/b, \quad r\leq b \\ &= e^2/r, \qquad r>b. \end{aligned} \tag{6}$$

To avoid the unphysical singularity of the Coulomb potential at the origin, and to approximate crudely the effect of the smeared-out charge distributions of the proton and Σ^+ hyperon, the $1/r$ increase of the Coulomb potential has been cut off at the boundary of the square well. With these assumptions the problem is to determine the relationship between ϵ_+, the binding energy of the lowest state of the Σ^+p system, and ϵ_- for various values of b. Of particular interest is the value of ϵ_- in the limit $\epsilon_+ \to 0$. Analogous to the nucleon-nucleon interaction, b is expected to be small $\sim 10^{-13}$, and V_0 large, $\gg e^2/b$. As a result the major contribution of the Coulomb potential arises from its effect on the asymptotic form of the Σ^+p wave function in the region $r>b$. Hence the differences between ϵ_+ and ϵ_- should be insensitive to the precise method of rounding off the Coulomb potential in the region $r<b$.

To determine the dependence of ϵ_+ on V_0 and b, one must solve the Schrödinger equation

$$-(\hbar^2/2\mu_+)\nabla^2\psi + V_+(r)\psi = -\epsilon_+\psi, \tag{7}$$

where $V_+(r)$ is given by Eq. (6), and $\mu_+ = M_{\Sigma^+}M_p/(M_{\Sigma^+}+M_p)$.[11] Before proceeding in this manner, we can obtain an order of magnitude estimate of $\epsilon_- - \epsilon_+$ using first-order perturbation theory. The normalized ground-state Σ^-n wave function is simply[10]

$$\begin{aligned} r\psi_-(r) &= \left[\left(\frac{1}{4\pi}\right)\left(\frac{2\gamma}{1+\gamma b}\right)\right]^{\frac{1}{2}} \sin\kappa r, \qquad r\leq b \\ &= \left(\frac{\kappa}{K}\right)\left[\left(\frac{1}{4\pi}\right)\left(\frac{2\gamma}{1+\gamma b}\right)\right]^{\frac{1}{2}} e^{-\gamma(r-b)}, \quad r>b. \end{aligned} \tag{8}$$

[9] M. S. Swami, doctoral dissertation, University of Wisconsin, 1957 (unpublished).

[10] See, for example, R. G. Sachs, *Nuclear Theory* (Addison-Wesley Publishing Company, Cambridge, 1953), Chap. 3.

[11] Hereafter we shall not distinguish between μ_+ and μ_-, $\mu_- \cong \mu_+ = 524.49$ Mev $= \mu$. Similarly $K_-^2 \cong K_+^2 = K^2$. This simplification introduces a negligible error since μ_+ and μ_- differ by 0.34%.

(The subscript $(-)$ has been suppressed on the symbols γ and κ.) Then the first-order change in energy due to the additional potential

$$V_+(r)-V_-(r)=e^2/b, \quad r\leq b \qquad =e^2/r, \quad r>b, \tag{9}$$

is

$$\Delta E=4\pi\int_0^\infty \psi_-^2(r)[V_+(r)-V_-(r)]r^2dr$$

$$=\frac{2\gamma}{1+\gamma b}\left(\frac{e^2}{b}\right)\left[\frac{b}{2}-\frac{\sin 2\kappa b}{4\kappa}\right]$$

$$+\frac{\kappa^2}{K^2}\left(\frac{2\gamma}{1+\gamma b}\right)e^{2\gamma b}e^2[-\mathrm{Ei}(-2\gamma b)], \tag{10}$$

where $-\mathrm{Ei}(-x)$ is the exponential integral function.[12] In the energy region ϵ_- near zero, $\kappa b\approx\pi/2$, and γb and γ/κ are small compared to 1. Then

$$\Delta E\cong\gamma e^2+2\gamma e^2[-\mathrm{Ei}(-2\gamma b)]. \tag{11}$$

If we equate ΔE of Eq. (11) to ϵ_-, and solve for the value of γ_- or ϵ_- that satisfies the resulting equation, we obtain the value of the $\Sigma^- n$ binding energy, $\bar{\epsilon}_-$, such that ϵ_+ has the limiting value of zero binding energy. For the values $b=1.4\times10^{-13}$, 1.0×10^{-13}, and 0.6×10^{-13} cm, this procedure yields $\bar{\epsilon}_-=0.43$, 0.54, and 0.76 Mev respectively. As will be seen in the next section, these perturbation theory results differ from the exact results for $\epsilon_--\epsilon_+$ in the limit $\epsilon_+\to0$ by $\lesssim20\%$.

III. EXACT SOLUTION IN THE LIMIT $\epsilon_+\to0$

The solution of the Schrödinger equation for the $\Sigma^+ p$ system is somewhat different from the standard Coulomb field problems discussed in books on elementary quantum mechanics. In this case we wish to determine the lowest energy eigenvalue for a bound state, where the total potential consists of the superposition of an attractive short-range potential and a repulsive, long-range Coulomb potential. Since the Coulomb potential drastically modifies the form of the wave function in the external region, $r>b$, the matching of the logarithmic derivative of the internal and external solutions at the point $r=b$ introduces complications.

For an $l=0$, bound state solution, Eq. (7) can be written in the form

$$(r\psi_+)''+\left(K^2-\frac{2}{Db}-\gamma_+^2\right)(r\psi_+)=0, \quad r\leq b, \tag{12}$$

$$(r\psi_+)''-\left(\frac{2}{Dr}+\gamma_+^2\right)(r\psi_+)=0, \quad r>b, \tag{13}$$

where

$$K^2=(2\mu/\hbar^2)V_0, \quad \gamma_+^2=(2\mu/\hbar^2)\epsilon_+,$$

$$1/D=\mu e^2/\hbar^2=1.940\times10^{11}\ \mathrm{cm}^{-1}.$$

For $r\leq b$,

$$\psi_+(r)=\frac{C_1\sin\kappa_+ r}{r}, \tag{14}$$

where

$$\kappa_+^2=K^2-\gamma_+^2-2/Db, \tag{15}$$

and C_1 is a normalization constant. The logarithmic derivative of $\psi_+(r)$ at $r=b_<$, i.e., the limit as r approaches b from within, is simply

$$\left.\frac{1}{\psi_+}\frac{d\psi_+}{dr}\right|_{r=b_<}=\kappa_+\cot\kappa_+ b-\frac{1}{b}. \tag{16}$$

In the external region, let

$$\psi_+(r)=e^{-\gamma_+ r}g(r). \tag{17}$$

Equation (13) becomes

$$\rho\frac{d^2g}{d\rho^2}+(2-\rho)\frac{dg}{d\rho}-\left[1+\frac{1}{\gamma_+ D}\right]g=0, \tag{18}$$

where $\rho=2\gamma_+ r$. Equation (18) is the second-order differential equation for the confluent hypergeometric function,[13] with the particular parameters $c=2$, and

$$a=1+(1/\gamma_+ D). \tag{19}$$

Since we seek a bound-state solution, we require the asymptotic condition that

$$\lim_{r\to\infty}\psi_+(r)=0. \tag{20}$$

Morse and Feshbach[13] discuss at length the properties of the solutions of Eq. (18). It is easily seen from their discussion that the unique solution of Eq. (18) that satisfies the boundary condition (20) is

$$\psi_+(r)=c_2e^{-\gamma_+ r}U_2(a|2|\rho), \tag{21}$$

where

$$U_2(a|2|\rho)=\frac{e^{ia\pi}\rho^{-a}}{\Gamma(a)}\int_0^\infty e^{-u}u^{a-1}\left(1+\frac{u}{\rho}\right)^{1-a}du. \tag{22}$$

Asymptotically,

$$e^{-\gamma_+ r}U_2(a|2|\rho)\underset{r\to\infty}{\longrightarrow}e^{ia\pi}(2\gamma_+ r)^{-a}e^{-\gamma_+ r}\underset{r\to\infty}{\longrightarrow}0. \tag{23}$$

The other independent solution of Eq. (18), $U_1(a|2|\rho)$, has the property

$$e^{-\gamma_+ r}U_1(a|2|\rho)\underset{r\to\infty}{\longrightarrow}(2\gamma_+ r)^{a-2}e^{+\gamma_+ r}, \tag{24}$$

which violates the boundary condition of Eq. (20).

[12] Jahnke-Emde, *Tables of Functions* (Dover Publications, New York, 1945), p. 1.

[13] See for example P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Chap. 5, p. 551.

The problem that remains is to match the logarithmic derivatives of ψ_+ in the two regions $r\geq b$ and $r\leq b$ at the point $r=b$. From Eq. (21) it follows that

$$\left.\frac{1}{\psi_+}\frac{d\psi_+}{dr}\right|_{r=b>}=-\gamma_+\left.+\left(\frac{1}{U_2}\frac{dU_2}{dr}\right)\right|_{r=b>}. \quad (25)$$

The dependence of γ_+ or ϵ_+ on the parameters V_0 and b is contained implicitly in the equation obtained by equating the right-hand sides of Eqs. (25) and (16). The evaluation of $(dU_2/dr)/U_2$ at $r=b_>$ is not simple. It has been carried out only in the limit as $\epsilon_+, \gamma_+\to 0$. This limiting value for ϵ_+ is the most interesting, since it determines the relationship between V_0 and b such that the Σ^+p compound has zero binding energy. By applying this same relationship between V_0 and b, for various values of b, to the Σ^-n system, the corresponding, nonvanishing binding energies of the Σ^-n system are obtained. An outline of the evaluation of the logarithmic derivative of U_2 is given in Appendix A. The result is, in the limit $\gamma_+\to 0$,

$$\left.\frac{1}{\psi_+}\frac{d\psi_+}{dr}\right|_{r=b>}=-\frac{1}{b}-\frac{2}{D}\frac{I_1}{I_2}, \quad (26)$$

where

$$I_1=\int_0^\infty e^{-(u+2b/Du)}\frac{du}{u}, \quad (27)$$

$$I_2=\int_0^\infty e^{-(u+2b/Du)}du. \quad (28)$$

Then

$$\left.\frac{1}{\psi_+}\frac{d\psi_+}{dr}\right|_{r=b<}=\left.\frac{1}{\psi_+}\frac{d\psi_+}{dr}\right|_{r=b>} \quad (29)$$

implies that

$$\kappa_+\cot\kappa_+b=-(2/D)(I_1/I_2). \quad (30)$$

Since $(2b/D)\ll 1$ for $b\sim 10^{-13}$ cm, the integrals I_1 and I_2 can be evaluated by expanding I_1 and I_2 as power series in $2b/D$, which we shall call δ.[14] Keeping all terms of lower order than δ^2, we obtain

$$\frac{I_1}{I_2}\cong 2\frac{\delta-(0.5772+\ln\sqrt{\delta})(1+\delta)}{1-\delta+2\delta(0.5772+\ln\sqrt{\delta})}. \quad (31)$$

Since we have assumed that V_0 and b are the same for the Σ^+p and Σ^-n systems, it is easy to expand $\kappa_+\cot\kappa_+b$ in terms of γ_-, b, and D, ignoring terms of order $(\gamma_-/\kappa_-)^2\ll 1$. In the limit $\gamma_+\to 0$, we obtain

$$\kappa_+\cot\kappa_+b\cong-\gamma_-(1+\tfrac{1}{2}\gamma_-b)-1/D. \quad (32)$$

Hence, equating Eqs. (32) and (30),

$$-\gamma_-(1+\tfrac{1}{2}\gamma_-b)-\frac{1}{D}=-\frac{2}{D}\left(\frac{I_1}{I_2}\right). \quad (33)$$

[14] See for example, W. Gröbner and N. Hofreiter, *Integraltafel* (Springer-Verlag, Berlin, 1950), Part II, p. 166.

TABLE I. Values of the binding energy $\bar{\epsilon}_-$, and $\bar{\gamma}_-=(2\mu\bar{\epsilon}_-/\hbar^2)^{\frac{1}{2}}$, of the Σ^-n system, for several values of the square-well range parameter b, in the limit that the binding energy, ϵ_+, of the Σ^+p system is zero.

b (10^{-13} cm)	$\bar{\gamma}_-$ (10^{12} cm^{-1})	$\bar{\epsilon}_-$ (Mev)
1.40	1.015	0.384
1.00	1.122	0.468
0.60	1.292	0.622

Equation (33) can be numerically solved for γ_- as a function of b. The results are given in Table I. The values of b were chosen to cover an *a priori* reasonable set of values for the range of the Σ^-n effective potential. The exact calculation reduces the difference between $\bar{\epsilon}_-$ and ϵ_+ as compared to the first-order perturbation result. If the Σ^-n interaction can bind the Σ^-n system with a binding energy *less* than the values of $\bar{\epsilon}_-$ listed in Table I for a given value of b, then the Σ^+p system will be unbound. We postpone further discussion of these results until Sec. V.

IV. ILLUSTRATIVE EXAMPLE OF Σ^+p BOUND STATE WAVE FUNCTION

In this section we consider the solution of Eq. (18), for the Σ^+p wave function, in a particularly simple special case; namely for $a=2$, or $(1/\gamma_+D)=1$. This choice of γ_+ corresponds to the Σ^+p system being bound with a binding energy, $\epsilon_+=0.014$ Mev. Since the parameter a is an integer, the general solution of Eq. (18) can be found by quadratures. Again letting

$$\psi_+(r)=e^{-\gamma_+r}g(r),$$

Eq. (18) with $a=2$ becomes

$$\rho g''(\rho)+(2-\rho)g'(\rho)-2g=0. \quad (34)$$

One solution of Eq. (34) is

$$g_1(\rho)=e^\rho=e^{2\gamma_+r}. \quad (35)$$

The second independent solution is then easily found to be

$$g_2(\rho)=Ae^\rho\int_\rho^\infty\frac{e^{-\rho}}{\rho^2}d\rho=A\left\{+\frac{1}{\rho}-e^\rho\int_\mu^\infty\frac{e^{-\rho}}{\rho}d\rho\right\}. \quad (36)$$

The solution that satisfies the boundary condition $\psi_+\to 0$ at infinity, is

$$\psi_+=e^{-\gamma_+r}g_2(\rho). \quad (37)$$

Using this solution in the external region, one can easily evaluate $[(d\psi_+/dr)\psi_+]$ at $r=b$. Equating internal and external logarithmic derivatives, one obtains a relationship between V_0 and b for this particular choice of $\gamma_+(=1/D=1.94\times 10^{11}$ cm$^{-1})$, namely

$$-\frac{1}{b}+\kappa_+\cot\kappa_+b=-\frac{1}{b}\frac{1}{[1+\delta e^\delta\,\mathrm{Ei}(-\delta)]}+\gamma_+, \quad (38)$$

where $\delta=2b/D$.

TABLE II. Values of the $\Sigma^- n$ binding energy ϵ_-, for several values of the square-well range parameter b, in the special case $\epsilon_+ = 0.014$ Mev.

b (10^{-13} cm)	ϵ_- (Mev)	$\epsilon_- - \epsilon_+$ (Mev)
1.4	0.415	0.401
1.0	0.500	0.486
0.6	0.656	0.642

Assuming charge symmetry, and neglecting terms $\sim\gamma_+ b$ and (γ_-^2/κ_-^2), the left-hand side of Eq. (38) can be expressed in terms of γ_- and b, and one obtains

$$\gamma_-(1+\tfrac{1}{2}\gamma_- b) = -\frac{2}{D}\left(\frac{e^\delta \operatorname{Ei}(-\delta)}{1+\delta e^\delta[\operatorname{Ei}(-\delta)]}\right). \quad (39)$$

Table II lists the values of ϵ_- obtained from the solution of Eq. (39) for several values of the square-well range parameter b.

As seen from a comparison of Table II and Table I, ϵ_- increased by ~ 0.032 Mev as ϵ_+ increased from 0 to 0.014 Mev.

It is instructive to compare the asymptotic properties of the wave function ψ_+ of Eq. (37) with that of ψ_-. Whereas ψ_- has the simple form

$$\psi_- = Ce^{-\gamma_- r}/r, \quad r > b, \quad (40)$$

ψ_+ has the form

$$\psi_+ \to A\left[\frac{1}{2\gamma_+ r} + \ln(3.562\gamma_+ r)\right], \quad b < r \ll \frac{1}{2\gamma_+} \quad (41)$$

and

$$\psi_+ \to Ae^{-\gamma_+ r}\frac{1}{(2\gamma_+ r)^2}, \quad r \gg \frac{1}{2\gamma_+}. \quad (42)$$

Special solutions of the type given by Eq. (37) for the $\Sigma^+ p$ bound state wave function may be of use in developing theories for the probability of $\Sigma^+ p$ formation and for the probability of $\Sigma^+ p$ decaying *via* reaction (2) into a proton that has sufficient energy to make a visible track in nuclear emulsion.[8]

V. DISCUSSION AND CONCLUSION

We have seen that if the $\Sigma^- n$ interaction binds the $\Sigma^- n$ system with a binding energy ϵ_- less than the values $\tilde{\epsilon}_-$ listed in Table I, then the $\Sigma^+ p$ system will be unbound. The values of $\tilde{\epsilon}_-$ increase as the choice of the range parameter b of the attractive $\Sigma^- n$ potential decreases. Clearly our choice of a square-well potential is only a device to simplify the calculations, but the order of magnitude of the effect of the Coulomb potential should not be altered significantly by a more complicated and more realistic choice for the $\Sigma^- n$ potential, provided its effective range of interaction lies in the region $(0.6\text{–}1.4)\times 10^{-13}$ cm. The region in which $\Sigma^- n$ is bound and $\Sigma^+ p$ is unbound is quite small, but since the most plausible estimates of the $\Sigma^- n$ binding energy from field theoretic calculations fall approximately in this region,[4,5] the possibility that only the $\Sigma^- n$ is bound cannot be overlooked. On the other hand, if confirmation is forthcoming for the existence of a bound $\Sigma^+ p$ compound,[6,7] then the values of $\tilde{\epsilon}_-$ listed in Table I can be used as lower bounds to the $\Sigma^- n$ binding energy.

Another parameter with which one can estimate the size of the region in which only $\Sigma^- n$ is bound is the value of the depth of the potential, V_0. That is, one can ask the question, by what percentage must V_0 be increased, for each value of b, so as to make ϵ_- increase from 0 to $\tilde{\epsilon}_-$? From Eq. (5) it is easy to show that

$$(Kb)^2 \cong (\tfrac{1}{2}\pi)^2\left[1+\frac{8}{\pi^2}(\gamma_- b)\right], \quad (43)$$

neglecting terms $\sim(\gamma_- b)^2 \ll 1$. It then follows that the percentage change in V_0 necessary to increase ϵ_- from 0 to $\tilde{\epsilon}_-$ is 12, 10, and 7% for $b = 1.4\times 10^{-13}$, 1.0×10^{-13}, and 0.6×10^{-13} cm respectively. This result again illustrates the small but non-negligible region involved. It is hoped that a continued and rigorous search for $\Sigma^+ p$ and $\Sigma^- n$ compounds will resolve these questions in the near future.

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APPENDIX A

Evaluation of $[(dU_2/dr)/U_2]_{r=b>}$ *in the limit* $\gamma_+ \to 0$.

$$U_2(a|2|\rho) = \frac{e^{ia\pi}\rho^{-a}}{\Gamma(a)}\int_0^\infty e^{-u}u^{a-1}\left(1+\frac{u}{\rho}\right)^{1-a} du, \quad (22)$$

where $a = 1+(1/\gamma_+ D)$, $\rho = 2\gamma_+ r = 2r/D(a-1)$, and $r \geq b$. Equivalently,

$$U_2(a|2|2\gamma_+ r) = \frac{e^{ia\pi}}{\Gamma(a-1)}\left(\frac{D}{2r}\right)\int_0^\infty e^{-u} \times\left(1+\frac{1}{(Du/2r)(a-1)}\right)^{1-a} du. \quad (44)$$

As $\gamma_+ \to 0$, $(a-1) = 1/\gamma_+ D \to \infty$. Hence, for

$$Du/2r \geq \eta > 0,$$

$$\left(1+\frac{1}{(Du/2r)(a-1)}\right)^{1-a} \to e^{-2r/Du} \text{ as } a \to \infty. \quad (45)$$

Choose a parameter N equal to $D\eta(a-1)/2r$, such that

$$1 \ll N \ll (D/2r)(a-1). \quad (46)$$

Then by breaking up $\int_0^\infty$ in Eq. (44) into two parts, $\int_0^\eta$ and $\int_\eta^\infty$, one can show that

$$\left|\left[U_2(a|2|2\gamma_+ r)\Big/\left(\frac{e^{ia\pi}}{\Gamma(a-1)}\frac{D}{2r}\right)\right] - \int_0^\infty \exp\left[-\left(u+\frac{2r}{Du}\right)\right]du\right| < \eta e^{-2r/D\eta}(1+\eta D/2r). \quad (47)$$

Since in the limit $a \to \infty$, η can be made arbitrarily small without violating Eq. (46), Eq. (47) implies that

$$U_2(a|2|2\gamma_+ r) \xrightarrow[\gamma_+ \to 0]{} \frac{e^{ia\pi}}{\Gamma(a-1)}\frac{D}{2r} \times \int_0^\infty e^{-(u+2r/Du)}du. \quad (48)$$

In an analogous way, it is easy to show that

$$\frac{dU_2}{dr}(a|2|2\gamma_+ r) \xrightarrow[\gamma_+ \to 0]{} -\frac{1}{r}U_2 - \frac{e^{ia\pi}}{\Gamma(a-1)}\frac{1}{r}\int_0^\infty e^{-(u+2r/Du)}\frac{du}{u}. \quad (49)$$

The result of Eq. (26) follows directly, since the quotient of Eqs. (49) and (48) yields

$$\left(\frac{1}{U_2}\frac{dU_2}{dr}\right)\Big|_{r=b_>} \xrightarrow[\gamma_+ \to 0]{} -\frac{1}{b} - \frac{2}{D}\frac{I_1}{I_2}, \quad (50)$$

where I_1 and I_2 are defined in Eqs. (27) and (28).

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Enumeration of the True Observables in Gauge-Invariant Theories

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By making use ot invariance arguments, we shall show that the total number of true observables which can serve as "coordinates" in a gauge-invariant theory is $N-2n$ where N is the number of field variables and n is the number of arbitrary functions needed to specify an element of the gauge group.

INTRODUCTION

IT has long been known that, in electrodynamics, the Euler-Lagrange equations of motion for the four-potentials are not of the Cauchy-Kowalewsky type. As a consequence, only certain functionals of the four-potentials have the property that their values at any time are uniquely determined by the equations of motion. Indeed, only the two independent components of the transverse part of the four-potentials and functionals thereof have this latter property. The same situation obtains in the general theory of relativity where, again, only two functionals of the ten $g_{\mu\nu}$ have the above-mentioned property. In what follows, we shall refer to dynamic variables whose motion is uniquely determined by the equations of motion as "true" observables. A unique state of the system is then specified by giving, at some instant of time, values of the true observables and their first time derivatives. In a sense, these true observables are the physically meaningful "coordinates" of the system.

The existence of fewer independent true observables than field variables is due to the particular invariance properties of the theory. In both electrodynamics and general relativity, we have invariance under what we call a gauge group, that is, a group whose elements are specified by one or more arbitrary space-time functions. In electrodynamics, the gauge group is just that group which is usually referred to as the gauge group and requires one arbitrary function to specify an element. In general relativity, the gauge group is that of all continuous coordinate transformations, and hence an element of the group is determined by four arbitrary functions.

The particular manner in which a gauge group acts to restrict the number of independent true observables in a theory has been worked out in the canonical form of the theory.[1] We shall not go into the details here except to say that the existence of the gauge group leads to a number of relations, called constraints, between the canonical variables. As a consequence of the existence of constraints, only certain variables, namely those which have vanishing Poisson brackets with the constraints, are true observables. The constraints themselves, of course, have this property; however, we do not treat them as true observables, since they always have the value zero throughout the motion.

One of the chief drawbacks to the canonical formalism is the difficulty of discovering, except in the simplest of cases, what are the true observables. Such information is not only necessary for the classical theory, but is essential to its quantization. With the advent of the

[1] J. L. Anderson and P. G. Bergmann, Phys. Rev. **83**, 1018 (1951).