the complete y at $\gamma=1$ is due to the phase shift δ_3 alone. If δ_3 were proportional to $\bar{\eta}^3$ in a narrow region near $\bar{\eta}=0$ (while going over to $-0.11\bar{\eta}$ at higher energies), the divergence of y would be removed. However, as Fig. 2 shows, the spin-flip dispersion relations would still be violated by the α' phase shifts, because of the rapid variations of y near the resonance and the fact that a straight-line relationship with the correct coupling constant could not be obtained. Thus it can be concluded that the α' phase shifts are inconsistent with the spin-flip dispersion relations.

IV. CONCLUSIONS

We have applied the spin-flip dispersion relations[3] for the pion-nucleon scattering to the Minami phase shifts derived from the Fermi set and to a set of phase shifts (α') obtained by applying the Minami transformation[6] to the well-known Yang phase shifts. It has been shown that the Minami and the α' phase shifts both give a divergence (at $\gamma=1$) in the curve of y *vs* x, in definite disagreement with the straight-line behavior deduced from the dispersion relations. The divergence of y at $\gamma=1$ arises from the fact that δ_3 is proportional to $\bar{\eta}$ at low energies ($\delta_3=-0.11\bar{\eta}$).[8] In addition to the divergence, the curve of y *vs* x for the α' phase shifts has rapid variations in the region of the resonance. From these results, one can conclude that both the Minami and the α' phase shifts are incompatible with the dispersion relations.

The fact that the Yang, Minami, and α' phase shifts are all in very marked disagreement with the requirements of the spin-flip dispersion relations, while on the other hand the Fermi set is in agreement, makes it almost certain that the Fermi set is the only correct set and is also the unique solution for the pion-nucleon scattering at low energies up to $\sim$300 Mev.

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Decay of the Pi Meson

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A quantitative study of $\pi\rightarrow\mu+\nu$ decay is presented using the techniques of dispersion theory. The discussion is based on a model in which the decay occurs through pion disintegration into a nucleon-antinucleon pair, the latter annihilating via a Fermi interaction to produce the leptons. The weak vertex contains effectively both axial vector and pseudoscalar couplings even if one adopts the point of view of a universal axial vector and vector Fermi interaction. The pion-nucleon vertex which enters our model is also calculated using dispersion techniques. Under the assumption that this vertex is damped for large momentum transfers, we obtain a result for the pion lifetime largely independent of the detailed properties of the vertex and one which is in very close agreement with experiment. The precise prediction of our theory depends on the energy dependence of the complex phase shift for nucleon-antinucleon scattering in the ${}^1S_0$ isotopic triplet state.

I. INTRODUCTION

THE main interest in the problem of pion decay at the present time concerns the experimental absence of the modes $\pi\rightarrow e+\nu$ [1] and $\pi\rightarrow e+\nu+\gamma$.[2] Beyond this, however, one would also like to understand quantitatively the mechanism of the observed decay mode $\pi\rightarrow\mu+\nu$.

This process is customarily described in terms of virtual dissociation of the pion into a nucleon-antinucleon pair, the latter annihilating via the μ-capture Fermi interaction to produce the lepton pair. Only the axial vector and pseudoscalar Fermi couplings can contribute here. The former is of special relevance, since it and the vector coupling now appear to dominate in the other Fermi interactions: μ and β decay. Furthermore, a universal axial vector coupling would imply a suppression of $\pi\rightarrow e+\nu$ decay relative to $\pi\rightarrow\mu+\nu$ decay by a factor of $\sim10^{-4}$.

It is possible that the physical picture described above has to be extended to include also Fermi couplings of hyperon pairs with leptons, although at the present time there is no experimental evidence for β decay of hyperons. In any case, if only to sharpen the problem, we want to see to what extent the simple picture based on an axial vector μ-capture coupling can be reconciled with the known rate for $\pi\rightarrow\mu+\nu$ decay.

It is necessary here to make precise what is meant by our assumption that the coupling is axial vector. What we assume is that the Fermi interaction Lagrangian contains only nonderivative axial vector (and vector) covariants. In the μ-capture reaction, however, the nucleons involved are surrounded by clouds of pions, pairs, etc. This means that the S-matrix element will in general contain terms which simulate Fermi interactions with derivative nucleon couplings. When reduced to the

[1] H. L. Anderson and C. M. G. Lattes, Nuovo cimento **6**, 1356 (1957).
[2] Cassels, Rigby, Wetherell, and Wormald, Proc. Phys. Soc. (London) **A70**, 729 (1957).

standard form with no derivatives, the matrix element will contain a pseudoscalar covariant, in addition to the basic vector and axial vector terms. Moreover, the coupling "constants" will in fact be scalar functions of the momentum transfer. These matters are discussed more fully in another paper,[3] where it is made apparent that, in particular, the effective pseudoscalar coupling coefficient may be appreciable at the momentum transfer involved in μ capture. Likewise, for pion decay it plays a decisive role in the model to be discussed here.

As in reference 3, the discussion here is based on the use of dispersion-relation techniques. In the approximation ultimately adopted, the pion decay rate is expressed in terms of the (complex) phase shift for nucleon-antinucleon scattering in the ${}^1S_0$ isotopic triplet state. The essential assumption which has to be made here is that the nucleon-pion vertex function vanishes for infinite momentum transfer. If this is so, our final expression for the pion decay rate, although not free of approximations, is at least unambiguous.

The phase shift in question is not at present known with any experimental accuracy. It is not possible therefore to quote a definite theoretical result based on the present model. For a wide range of possibilities, however, our expression for the pion decay rate is not sensitive to the detailed properties of the phase shift and the calculated pion decay rate is in surprisingly close agreement with experiment. We may note here that in a perturbation-theoretic treatment[4,5] of pion decay one encounters (for axial vector coupling) a logarithmic divergence, so that the result is quite ambiguous. A similar calculation for pseudoscalar coupling leads to a quadratically divergent result which is naturally totally meaningless.

For definiteness of writing we assume throughout the validity of the two-component neutrino theory, though this is not really relevant to the discussion.

II. DISPERSION RELATIONS FOR π DECAY

The quantum field theory of unstable states is not a well understood subject and we shall not contribute to it here. But there seems to be little doubt how to proceed where we are willing to treat the weak interaction responsible for the decay in lowest order.

We begin by deriving a general formula for the decay of a negative pion into μ meson and neutrino. Loosely speaking, we are interested in the S-matrix element $\langle\mu\nu\text{ "out"}|\pi\rangle$. In spite of the fact that the state $|\pi\rangle$ is unstable, we apply the customary formalism of field theory[6] and write

$$\langle\mu\nu\text{ "out"}|\pi\rangle=i(2\pi)^4\delta(p_\mu+p_\nu-p_\pi)\times\langle\mu|\bar{f}_\nu(0)|\pi\rangle(1+\gamma_5)u(p_\nu),\quad(1)$$

where we have introduced the source of the neutrino field, f, according to

$$\gamma_\mu\frac{\partial}{\partial x_\mu}\psi_\nu=f_\nu.\quad(2)$$

The neutrino is described by the Dirac spinor $u(p_\nu)$. Now f is assumed to be proportional to the small coupling constant g linking μ and ν to the nucleon (or perhaps more generally, baryon) fields. Since we have exhibited the weak link explicitly, we now turn off the β coupling so that the state $|\pi\rangle$ really exists.

Inserting normalization factors for convenience, let us now write out the spinor coefficient of $(1+\gamma_5)u(p_\nu)$:

$$\begin{aligned}\mathfrak{M}&\equiv(p_{\mu0}/m_\mu)^{\frac{1}{2}}(2p_{\pi0})^{\frac{1}{2}}\langle\mu|\bar{f}_\nu|\pi\rangle\\&=i\left(\frac{p_{\mu0}}{m_\mu}\right)^{\frac{1}{2}}\int dx\,e^{ip_\pi\cdot x}\langle\mu|(\bar{f}_\nu(0)J(x))_+|0\rangle\\&=i\left(\frac{p_{\mu0}}{m_\mu}\right)^{\frac{1}{2}}\int dx\,e^{-i(p_\pi-p_\mu)\cdot x}\langle\mu|(\bar{f}_\nu(x)J(0))_+|0\rangle,\end{aligned}\quad(3)$$

where J is the source of the charged meson field, $(\mu^2-\Box)\varphi$, and the second line follows by making the usual contraction on the state $|\pi\rangle$.[6] The third form of (3) follows from translation invariance. We can use momentum conservation to replace $p_\pi-p_\mu$ by p_ν. Finally, using the identity

$$(f_\nu(x)J(0))_+=[\bar{f}_\nu(x),J(0)]\theta(x)+J(0)f_\nu(x),\quad(4)$$

where $\theta(x)$ is the step function (which vanishes for $x_0<0$ and is unity for $x_0>0$), we may write

$$\mathfrak{M}=i\left(\frac{p_{\mu0}}{m_\mu}\right)^{\frac{1}{2}}\int dx\,e^{-ip_\nu\cdot x}\theta(x)\langle\mu|[\bar{f}_\nu(x),J(0)]|0\rangle.\quad(5)$$

We have dropped the second term in Eq. (4) since it makes no contribution to physical pion decay. The remaining discussion is based on Eq. (5).

It follows from invariance principles that $\mathfrak{M}$ must have the form

$$\mathfrak{M}=F[(p_\mu+p_\nu)^2]\bar{u}(p_\mu)\gamma_5\gamma_\lambda(p_\mu+p_\nu)_\lambda.\quad(6)$$

Actually, since $\mathfrak{M}$ is ultimately going to be contracted with the spinor $(1+\gamma_5)u(p_\nu)$, the $\gamma\cdot p_\nu$ term above may be dropped; and further, we may set $\bar{u}(p_\mu)\gamma\cdot p_\mu=im_\mu\bar{u}(p_\mu)$. The reason then for displaying the factor $(p_\mu+p_\nu)_\lambda$ is that when we later make the explicit assumption about the axial vector character of the lepton coupling with nucleons, the factors will come out in a natural way.

We now show that $\mathfrak{M}$, or more precisely, F, satisfies a dispersion relation.[7] This is most easily seen by choosing a coordinate system in which the μ meson is at rest. By

[3] M. L. Goldberger and S. B. Treiman (to be published).

[4] M. Ruderman and R. Finkelstein, Phys. Rev. **76**, 1458 (1949).

[5] S. B. Treiman and H. W. Wyld, Phys. Rev. **101**, 1552 (1956).

[6] Lehmann, Symanzik, and Zimmermann, Nuovo cimento **1**, 205 (1955).

[7] See the Appendix for a more general derivation.

virtue of the masslessness of the neutrino, we may express $\mathfrak{M}$ in this frame as

$$\mathfrak{M}=i\int dx\exp[i\nu_0(x_0-\mathbf{n}\cdot\mathbf{x})]\theta(x_0)\times\langle\mu|[\bar{f}_\nu(x),J(0)]|0\rangle, \quad (7)$$

where $\mathbf{n}$ is a unit vector in the (irrelevant) direction of $\mathbf{p}_\nu$. Since $|x_0|\geqslant\mathbf{n}\cdot\mathbf{x}$ by virtue of the causality condition $[f_\nu(x),J(0)]=0$ for $\mathbf{x}^2-x_0^2>0$, and since x_0 is restricted by $\theta(x_0)$ to positive values, we see that $\mathfrak{M}(\nu_0)$ defines a function which is analytic in the upper half of the ν_0 plane. Thus we may write a dispersion relation for $\mathfrak{M}$, or better, for the function F defined in (6), in the form

$$F(-m_\mu^2-2m_\mu\nu_0)=\frac{1}{\pi}\int_{-\infty}^{\infty}d\nu_0'\frac{\mathrm{Im}F(-m_\mu^2-2m_\mu\nu_0')}{\nu_0'-\nu_0-i\epsilon}. \quad (8)$$

This may be written in terms of the invariant $\xi\equiv(p_\mu+p_\nu)^2$ as

$$F(\xi)=\frac{1}{\pi}\int_{-\infty}^{\infty}d\xi'\frac{\mathrm{Im}F(-\xi')}{\xi'+\xi-i\epsilon}. \quad (9)$$

We are of course making a definite assumption about the behavior of F at infinity; if we were to encounter a divergent integral we would in the usual way have to write a dispersion relation with a subtraction. Unless there were some value of ξ for which F is known, for the present purposes we would be defeated. This is because we require for the π decay rate the value of F at a single point, namely $\xi=(p_\mu+p_\nu)^2=p_\pi^2=-m_\pi^2$. Since this is just a number, we cannot tolerate the presence of any unknown constants. In the specific model which we shall adopt, no subtractions are necessary if the lepton-nucleon coupling is axial vector.

The imaginary or, better, absorptive part of F may be computed by going back to Eq. (5); we write $\mathfrak{M}=D+iA$ and identify A, the absorptive part, with the contribution from the term $\frac{1}{2}$ in $\theta(x_0)=\frac{1}{2}+\frac{1}{2}\epsilon(x_0)$. We then find, after inserting a sum over a complete set of states and carrying out the space-time integrations,

$$A=\pi(p_{\mu0}/m_\mu)^{\frac{1}{2}}\sum\nolimits_n\langle\mu|f_\nu|n\rangle\langle n|J|0\rangle\delta(p_n-p_\nu-p_\mu) \quad (10)$$

(the three-vector part of the δ function must, with our normalization, be regarded as a Kronecker δ function). In order to maintain the proper reality conditions at all stages of approximation, it is convenient to write the sum over states $|n\rangle$ as one-half the sum over "out" plus "in" states. We shall not write this explicitly but it will be assumed from now on that the sum has this meaning.

We evidently cannot hope to evaluate (10) completely, so we must resort to physical arguments to select the important, and tractable, intermediate states. We need consider only states of zero baryon number; and the least massive of these is that consisting of three pions (states containing two pions, or two K particles, cannot contribute if parity is conserved in the strong interactions). In fact, however, we want to explore the possibility that only the baryon-antibaryon states are important, in particular the state consisting of neutron and antiproton. We have no quantitative argument for throwing out other states of comparable or smaller mass. One may, for example, try to argue that pion-pion scattering is weak so that $\langle 3\pi|J|0\rangle$ might be expected to make a smaller contribution than $\langle n\bar{p}|J|0\rangle$. States with even more pions are so complicated that we can have no rational feelings about them. As far as baryon pairs other than $n-\bar{p}$ are concerned, their importance depends in part on whether they are directly coupled to the leptons. If so, they may be expected to make contributions comparable in magnitude to that coming from the $n-\bar{p}$ state and the amplitudes would interfere. Even if hyperon pairs are not directly coupled to leptons, they are indirectly coupled in the sense that through K meson transfer they may convert to nucleon pairs. In the latter case, no direct hyperon-lepton interaction, the hyperon pair states may be relatively unimportant.

We shall proceed then on the assumption that only the intermediate state $|n\bar{p}\rangle$ need be considered. This is the state which has been considered in earlier investigations, and retaining just this state provides us at least with a definite model which can to a certain extent be evaluated and compared with experiment.

The relevant term in (10) involves the product $\langle\mu|f_\nu|n\bar{p}\rangle\langle n\bar{p}|J|0\rangle$. The first factor involves the weak link; on general invariance grounds, it must have the form

$$\langle\mu|f_\nu|n\bar{p}\rangle=\left(\frac{m_\mu m^2}{p_{\mu0}n_0\bar{p}_0}\right)^{\frac{1}{2}}\bar{u}(\bar{p})\{ia\gamma_\lambda\gamma_5-b(n+\bar{p})_\lambda\gamma_5\}u(n)\times\bar{u}(p_\mu)i\gamma_\lambda\gamma_5; \quad (11)$$

where we have included only those terms which are relevant for pion decay and where the assumption that the basic Fermi interaction is axial vector is made apparent by the structure of the lepton factor. The coefficients a and b are, in general, functions of the momentum transfer $(n+\bar{p})^2$. The term containing the coefficient b simulates a pseudoscalar coupling, though it in fact has its origin in radiative corrections of the basic axial vector coupling. The only experimental information concerning the coefficients a and b can at present come from μ-capture measurements and would concern values of the argument $(n+\bar{p})^2$ of order m_μ^2.

A discussion of these coefficients is given in reference 3. It is shown there that a is very likely a slowly varying function of momentum transfer, at least for small momentum transfer. We shall make the perhaps drastic assumption that, even for the large momentum transfers involved here $[-(n+\bar{p})^2>4m^2]$, it retains effectively its μ-capture value: $a=a(m_\mu^2)=g_A$, where we identify g_A with the Gamow-Teller coupling constant of β decay.

As for the coefficient b, we take the result of reference 3: for small momentum transfer,

$$b[(n+\bar{p})^2]=-\sqrt{2}GF(-m_\pi^2)[(n+\bar{p})^2+m_\pi^2]^{-1}, \quad (12)$$

where G is the pion-nucleon coupling constant and $F(-m_\pi^2)$ is just the effective pion-decay coupling coefficient defined in Eq. (6) and evaluated at the momentum transfer of free pion decay. The relation (12) is based on the assumption that b satisfies a dispersion relation with no subtractions and no additive constant. This represents the major assumption of the present discussion. Again, we shall assume the validity of (12) even for the large momentum transfers involved here.

The superficially unwarranted assumptions concerning the ξ dependence of a and b for large ξ have, in fact, been investigated in reference 3. We have found that a more exact treatment does not modify our numerical conclusions qualitatively.

Next we turn to the matrix element $\langle n\bar{p}|J|0\rangle$. For definiteness assume that $|n\bar{p}\rangle$ is an "out" state. (The distinction between "in" and "out" was of no consequence in our discussion of $\langle\mu|f_\nu|n\bar{p}\rangle$ since we ultimately set $a=g_A=$constant.) This vertex has been studied by us in detail in connection with the problem of nucleon structure.[8] We content ourselves here with quoting the main results and trying to make them plausible. We note first that since J is a pseudoscalar, the intermediate state $|n\bar{p}\rangle$ must have zero angular momentum and odd parity. Imagine that we are in a coordinate system where $\mathbf{n}+\bar{\mathbf{p}}=0$. The state in question is evidently the $^1S_0$ isotopic triplet. It is evident from invariance considerations that the matrix element must have the form

$$(n_0\bar{p}_0/m^2)^{\frac{1}{2}}\langle n\bar{p}|J|0\rangle=iK[(n+\bar{p})^2]\bar{u}(n)\gamma_5 u(\bar{p}). \quad (13)$$

We shall assume, as appears quite reasonable, that the function $K(\xi)$ is analytic in the complex ξ plane, cut from $-(3m_\pi)^2$ to $-\infty$. Now the exact behavior of K for large ξ is not known. It has been shown,[9] however, that the proper vertex function, which is very closely related to K, does approach zero for $\xi\to\infty$. The precise relation is as follows:

$$\langle n\bar{p}|J|0\rangle=(\xi+m_\pi^2)\Delta_{Fc}(\xi)\bar{u}(n)i\gamma_5 u(\bar{p})f(\xi),$$

where $\Delta_{Fc}(\xi)$ is the exact pion propagation function. The function $f(\xi)\to 0$ as $-\xi\to\infty$; thus K will show the same behavior provided the product $\xi\Delta_{Fc}(\xi)$ is finite in the limit. This limit is in fact Z_3^{-1}, where Z_3 is the usual meson wave-function renormalization constant. If Z_3 is not zero then we may conclude that our function K approaches zero as $-\xi\to\infty$.

Now in general, for real ξ, $K(\xi)$ is a complex function and we may write the self-evident formula

$$K(\xi)=\frac{e^{i\varphi(-\xi)}}{\cos[\varphi(-\xi)]}\,\mathrm{Re}K(\xi),$$

where the argument of φ has been written as $-\xi$ for later convenience. We now assume that $K(\xi)$ can be extended to a function analytic in the cut ξ plane and, further, that it satisfies a dispersion relation with only one subtraction, namely,

$$K(\xi)=K(-m_\pi^2)-\left(\frac{\xi+m_\pi^2}{\pi}\right)\times\int_{(3m_\pi)^2}^{\infty}\frac{\mathrm{Im}K(-\xi')}{(\xi'-m_\pi^2)(\xi'+\xi-i\epsilon)}d\xi'$$

$$=K(-m_\pi^2)-\left(\frac{\xi+m_\pi^2}{\pi}\right)\times\int_{(3m_\pi)^2}^{\infty}\frac{\tan[\varphi(\xi')]}{(\xi'-m_\pi^2)(\xi'+\xi-i\epsilon)}\mathrm{Re}K(-\xi')d\xi'. \quad (14)$$

The "$-i\epsilon$" has been inserted to conform with the "out" boundary condition on the state $|n\bar{p}\rangle$.

To proceed further we must of course identify the phase φ. If we assume that all matrix elements connecting the one-pion state to states other than $|n\bar{p}\rangle$ are negligible, it turns out that

$$\tan[\varphi(\xi)]=\frac{\mathrm{Re}(e^{i\delta}\sin\delta)}{1-\mathrm{Im}(\epsilon^{i\delta}\sin\delta)}\theta(\tfrac{1}{4}\xi-m^2), \quad (15)$$

where δ is the complex phase shift for $n-\bar{p}$ scattering in the $^1S_0$ state and is a function of the center-of-mass wave number $(\frac{1}{4}\xi-m^2)^{\frac{1}{2}}$. This result may be understood in the following way. Note first that if δ is real then $\varphi=\delta$; and one has a familiar-*looking* result, namely, that the matrix element is a real number times $e^{i\delta}$, where δ is the phase shift for the final state. Actually this form would be expected only if the interaction responsible for the process $\pi\to n+\bar{p}$ were weak. Now the matrix element $\langle n\bar{p}|J|0\rangle$, aside from irrelevant kinematic factors, is just equal to $-iS_{n\bar{p},\pi}$, where $S_{n\bar{p},\pi}$ is the S-matrix element for the rather unusual process of a π^- meson converting into a neutron and antiproton. From (10) it is evident that we are concerned with the above matrix element only for such weird π mesons that the process is in fact allowed by energy-momentum conservation; i.e., we have $\delta(p_n-p_\nu-p_\mu)=\delta(n+\bar{p}-p_\pi)$. Since the conservation laws are satisfied we may utilize the formal properties of the S-matrix in the usual way. In particular, we write S in terms of the real reaction matrix Q. Only one angular momentum and parity combination is involved here, namely 0^-; and the matrices are labeled by the particle symbols, π, 3π, $\cdots n\bar{p}$, $\cdots$. Next we write $Q=Q_0+Q'$, where Q_0 is defined as having no matrix elements connecting to the one-pion state. We then write

$$S_{n\bar{p},\pi}=\left(\frac{1+iQ_0+iQ'}{1-iQ_0-iQ'}\right)_{n\bar{p},\pi}=\frac{i}{4}[(1+S)Q']_{n\bar{p},\pi}, \quad (16)$$

where the second equality may be verified by expanding the first form of S, appropriately symmetrized, in powers of Q' and then regrouping. If, finally, we assume

[8] Federbush, Goldberger, and Treiman (to be published).

[9] Lehmann, Symanzik, and Zimmermann, Nuovo cimento 2, 425 (1955).

that all the matrix elements of Q' are negligible except that connecting π and $n\bar{p}$, we have

$$S_{n\bar{p},\pi}\sim i\langle n\bar{p}|J|0\rangle=\frac{i}{4}(1+S)_{n\bar{p},n\bar{p}}Q'_{n\bar{p},\pi}. \quad (17)$$

Now we know that $Q'_{n\bar{p},\pi}$ is real and $\frac{1}{2}(1+S)_{n\bar{p},n\bar{p}}$ is simply $e^{i\delta}\cos\delta$, where δ is the complex phase shift for $n-\bar{p}$ scattering in the $^1S_0$ state. The important point to notice is that we do not have to assume that $Q'_{n\bar{p},\pi}$ is weak to obtain this result, so that (17) is a generalization of the usual result and is valid provided that the elements of Q' connecting π to states other than $n\bar{p}$ are weak. Since $\tan\varphi=\mathrm{Im}K/\mathrm{Re}K$, we have

$$\tan\varphi=\frac{\mathrm{Im}(e^{i\delta}\cos\delta)}{\mathrm{Re}(e^{i\delta}\sin\delta)},$$

which is equivalent to the result of Eq. (15).

We now return to the dispersion relation (14), where of course the lower limit of the integral is now $(2m)^2$. It may be shown that the solution of (14) is given by

$$K(\xi)=K(-m_\pi^2)\exp\left\{-\left(\frac{\xi+m_\pi^2}{\pi}\right)\times\int_{4m^2}^{\infty}d\xi'\frac{\varphi(\xi')}{(\xi'-m_\pi^2)(\xi'+\xi-i\epsilon)}\right\}. \quad (18)$$

Note that since only $\tan\varphi$ is defined experimentally, φ itself is undefined to within additive multiples of π. But on physical grounds we suppose that $\tan\varphi\rightarrow 0$ for $\xi'\rightarrow 4m^2$ and we take $\varphi(4m^2)=0$. If, as would appear reasonable, there is always some absorption at all wave numbers (δ always complex), then clearly for all wave numbers $|\varphi|<\pi/2$. We shall assume this to be true. Moreover, if K is to vanish for infinite momentum transfer—as must be assumed in the present discussion—it is necessary that φ approach a positive limit as $\xi'\rightarrow\infty$.

We are now ready to put the pieces together to finally effect the evaluation of (10). Substituting (11) for $\langle\mu|f_\nu|n\bar{p}\rangle$ and (13) for $\langle n\bar{p}|J|0\rangle$, and carrying out the sum over spins and integration over phase volume implied in (10), we find

$$A(\xi)=-\frac{1}{4\pi}(ma-\tfrac{1}{2}\xi b)\,\mathrm{Re}K(\xi)\times\left(\frac{\xi+4m^2}{\xi}\right)^{\frac{1}{2}}\bar{u}(p_\mu)\gamma_\lambda\gamma_5(p_\mu+p_\nu)_\lambda. \quad (19)$$

From this and Eq. (12), we thus have

$$\mathrm{Im}F(\xi)=-\frac{1}{4\pi}\left[mg_A+\frac{\sqrt{2}}{2}GF(-m_\pi^2)\frac{\xi}{\xi+m_\pi^2}\right]\times\mathrm{Re}K(\xi)\left(\frac{\xi+4m^2}{\xi}\right)^{\frac{1}{2}}. \quad (20)$$

For the dispersion relation (9), which we rewrite

$$F(-m_\pi^2)=\frac{1}{\pi}\int_{4m^2}^{\infty}d\xi\frac{\mathrm{Im}F(-\xi)}{\xi-m_\pi^2}, \quad (21)$$

we require

$$\mathrm{Re}K(-\xi)=\sqrt{2}G\cos\varphi(\xi)\exp\left\{\left(\frac{\xi-m_\pi^2}{\pi}\right)\times\int_{4m^2}^{\infty}dy\frac{\varphi(y)}{(y-\xi)(y-m_\pi^2)}\right\}, \quad (22)$$

where we have used the fact that $K(-m_\pi^2)=\sqrt{2}G$. A principal-value integration is implied here.

It is convenient now to introduce as variable the center-of-mass wave number k for nucleon-antinucleon scattering. Let

$$\xi=4(k^2+m^2),\quad y=4(k'^2+m^2). \quad (23)$$

Finally, for algebraic convenience, let us set the pion mass equal to zero in the above expressions (this introduces no appreciable error). Collecting all our results, we obtain

$$F(0)=-\frac{m}{2\pi^2}\sqrt{2}Gg_A\frac{J}{1+(G^2/4\pi)(2J/\pi)}; \quad (24)$$

$$J=\int_0^\infty dk\frac{k^2}{(k^2+m^2)^{\frac{3}{2}}}\cos\varphi(k)\exp\left\{\frac{2}{\pi}\int_0^\infty dk'k'\varphi(k')\times\left(\frac{1}{k'^2-k^2}-\frac{1}{k'^2+m^2}\right)\right\}. \quad (25)$$

III. NUMERICAL ESTIMATES

Let us first write down the expression for the pion decay rate ω. In the standard way, one finds

$$\omega=\frac{1}{2\pi^4}m_\pi\left(\frac{m_\mu}{m_\pi}\right)^2\left(\frac{m}{m_\pi}\right)^2\left(1-\frac{m_\mu^2}{m_\pi^2}\right)^2\times\left(\frac{G^2}{4\pi}\right)(g_Am_\pi^2)^2\left(\frac{J}{1+G^2J/2\pi^2}\right)^2. \quad (26)$$

If we take $G^2/4\pi=15$, and if we adopt for g_A the value of the Gamow-Teller coupling constant in β decay (there is as yet no experimental justification for the latter, but aside from small corrections, it is implied by the notion of a universal Fermi interaction), we then find from the known pion lifetime that

$$\left(\frac{J}{1+G^2J/2\pi^2}\right)_{\mathrm{exp}}\simeq 0.13. \quad (27)$$

On the theoretical side, provided only that $J\gg\frac{1}{10}$, so that we can neglect the unit term in the denominator of Eq. (27), we obtain a result independent of J (i.e., independent of the details of the nucleon-antinucleon scattering); namely, we obtain the value 0.10. This is rather surprisingly good agreement with experiment. One sees here the decisive damping effect of the denominator in Eq. (27).

We remark that the expression for $F(0)$ which one obtains in perturbation theory is precisely given by Eq. (24) if one neglects the damping term in the denominator and if one sets $\varphi=0$ in Eq. (25). In this case the integral over k in Eq. (25) of course diverges logarithmically.

In the present discussion, what we require for good agreement with experiment is of course that the integral of Eq. (25) shall be finite, i.e., that the pion-nucleon vertex function vanish for infinitely large momentum transfer; and that the value of J be large compared to $(G^2/2\pi^2)^{-1}\approx\frac{1}{10}$. We cannot argue in detail the correctness of our assumptions. No one can really be confident about a conjectured behavior of a complex phase shift for very large energies. It is our general feeling that even in the face of the opening up of ever newer absorptive channels with increasing energy, the real part of the ${}^1S_0$ phase shift will remain finite, so that damping of the nucleon vertex will persist. The argument is that the wave function will ultimately stay out of the region of annihilation.

If, as we are hoping, the behavior of φ for large wave number k is such as to guarantee fairly rapid convergence, the main numerical contributions to J would come from small values of k. In this low-energy region a two-parameter scattering length approximation for $n-\bar{p}$ ${}^1S_0$ scattering seems reasonable, and we now turn to this simplification. As stated, the approximation presumably makes little sense at large wave numbers, but the final result may be numerically reasonable. This will be especially so if the integral J is in any case large, since then the pion decay rate does not depend on J.

The ${}^1S_0$ $n-\bar{p}$ scattering amplitude f is related to the complex phase shift δ by

$$f=(e^{i\delta}\sin\delta)/k. \tag{28}$$

We now introduce two positive constants, a and b, and we suppose that

$$\tan\delta=k(a+ib). \tag{29}$$

Then, from (15),

$$\tan\varphi=ka/(1+kb). \tag{30}$$

The ${}^1S_0$ $n-\bar{p}$ scattering and absorption cross sections, σ_s and σ_a, are related to the parameters according to

$$\sigma_s=4\pi\frac{a^2+b^2}{(1+kb)^2+k^2a^2}, \tag{31}$$

$$\sigma_a=4\pi\left(\frac{b}{k}\right)\frac{1}{(1+kb)^2+k^2a^2}. \tag{32}$$

Let us now define

$$I(k)=\frac{2}{\pi}\int_0^\infty dk'\,k'\varphi(k')\left(\frac{1}{k'^2-k^2}-\frac{1}{k'^2+m^2}\right). \tag{33}$$

Even with our simplified expression (30) we have not been able to carry out this integration in closed form for the general case. We instead consider two limiting possibilities (in the following discussion we set $m=1$, which means that the parameters a and b are expressed in units of the nucleon mass).

1. $a\ll b$

In this limit we can set $\tan\varphi\approx\varphi$ (the approximation is in fact not unreasonable even for a as large as b), and we find

$$\exp I(k)=C(a,b)\exp\left\{-\frac{2a}{\pi b}\left(\frac{b^2k^2}{b^2k^2-1}\right)\ln(bk)\right\}, \tag{34}$$

where

$$C(a,b)=\exp\left\{\frac{2a}{\pi}\left(\frac{1}{b^2+1}\right)\left[\tfrac{1}{2}\pi+b\ln b\right]\right\}. \tag{35}$$

As for $\cos\varphi$ we need make no approximations and we have

$$\cos\varphi=\frac{1+kb}{[(1+kb)^2+k^2a^2]^{\frac{1}{2}}}. \tag{36}$$

As a numerical example, if we take $a=b=3$ we find $J\approx1.7$.

2. $a\gg b$

We carry out this approximation to lowest order in b/a and we further suppose that $a>1$. We then find

$$\exp I(k)=(a+1)\exp\left\{-\frac{2}{\pi}\frac{b}{a}\left(\frac{a^2}{a^2-1}\right)\ln a\right\}(k^2a^2+1)^{-\frac{1}{2}}$$
$$\times\exp\left\{\frac{2}{\pi}\frac{b}{a}\left(\frac{k^2a^2}{k^2a^2+1}\right)\ln(ka)\right\}. \tag{37}$$

In the extreme limit $a\gg b$ where we can completely ignore b/a, we can carry out in closed form the subsequent integration for J [see Eq. (24)]. In this limit

$$J=(a-1)^{-1}\{1-(a^2-1)^{-\frac{1}{2}}\tan^{-1}(a^2-1)^{\frac{1}{2}}\}. \tag{38}$$

Thus for $b\ll a$ and $a\gg1$, J is inversely proportional to a. As an example, for $a=3$ we find $J=0.28$.

It is evident from this brief discussion that for a large range of possibilities concerning the ${}^1S_0$ phase shift, the integral J is appreciably larger than $(G^2/2\pi^2)^{-1}$, in which case our result for the pion decay rate does not depend much on J. The actual low-energy behavior of the phase shift should become known in time.

APPENDIX

In the derivation of the dispersion relation for $F(p_\pi^2)$ which we have sketched in the text, explicit use was made of the vanishing of the neutrino mass. In fact this last condition is not required, as can easily be shown by an extension of the discussion given in the text. Rather than doing so, we indicate instead an alternate derivation in which the assumption is made at the outset that the leptons emerge from a point, with axial vector coupling.

We begin with Eq. (5), which we rewrite for ease of reference:

$$\mathfrak{M}=i\left(\frac{p_{\mu 0}}{m_\mu}\right)^{\frac{1}{2}}\int dx\, e^{-ip_\nu\cdot x}\theta(x)\langle\mu|[\bar{f}_\nu(x),J(0)]|0\rangle. \quad \text{(A-1)}$$

We now introduce the explicit assumption that f_ν is given by

$$f_\nu(x)=f_\beta\bar{\psi}_\mu i\gamma_\lambda\gamma_5\cdot\bar{\psi}_p i\gamma_\lambda\gamma_5\psi_n(x).$$

The constant f_β is the product of the unrenormalized Fermi coupling constant and wave-function renormalization constants. We now replace $\bar{\psi}_\mu$ by a free-field operator, since we work to lowest order in the weak coupling. Then Eq. (A-1) may be written

$$\mathfrak{M}=V_\lambda(p_\pi)\bar{u}(p_\mu)i\gamma_\lambda\gamma_5, \quad \text{(A-2)}$$

where

$$V_\lambda(p_\pi)=if_\beta\int dx\, e^{-ip_\pi\cdot x}\langle 0|[B_\lambda(x),J(0)]|0\rangle\theta(x), \quad \text{(A-3)}$$

and p_π stands for $p_\mu+p_\nu$. The effective vector operator $B_\lambda(x)$ is defined by

$$B_\lambda(x)=\bar{\psi}_p(x)i\gamma_\lambda\gamma_5\psi_n(x). \quad \text{(A-4)}$$

Since there are no vectors other than p_π which enter into (A-3), it is clear that $V_\lambda(p_\pi)$ may be written

$$V_\lambda(p_\pi)=(p_\pi)_\lambda W(p_\pi^2), \quad \text{(A-5)}$$

and

$$W(p_\pi^2)=\frac{if_\beta}{p_\pi^2}\int dx\, e^{-ip_\pi\cdot x} \times\langle 0|[p_\pi\cdot B(x),J(0)]|0\rangle\theta(x). \quad \text{(A-6)}$$

Since we have to do here with the Fourier transform of the vacuum expectation value of a completely retarded commutator, it follows in the standard way that $W(p_\pi^2)$ may be written in the form

$$W(p_\pi^2)=\int_0^\infty d\sigma^2\frac{\rho(\sigma^2)}{\sigma^2+p_\pi^2-i\zeta\epsilon(p_{\pi 0})}, \quad \text{(A-7)}$$

where $\zeta\rightarrow 0_+$ and $\rho(\sigma^2)$ is evidently proportional to the imaginary part of $W(-\sigma^2)$.

It is clear from our definitions that except for trivial factors, $W(p_\pi^2)$ is identical to $F(p_\pi^2)$ defined in Eq. (6). Having made the identification, we can of course evaluate $F(p_\pi^2)$ by any method we choose; and the procedure followed in the text seems to be the most convenient for the present purposes.