Pion Scattering and Dispersion Relations*

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Inaccuracies in the experimental data which might give rise to the discrepancy in the dispersion relations for the forward scattering of negative pions (up to a few hundred Mev) are examined. The discrepancy cannot be removed without contradicting one of the several apparently accurate experimental results. A relation to check charge independence is suggested.

1. INTRODUCTION

THE dispersion relations for the forward scattering of pions by protons (which were developed by Goldberger *et al.*[1]) have been compared with the experimental results up to pion kinetic energies of 400 Mev (lab) by Puppi and Stanghellini.[2] The dispersion relations express the real part of the forward scattering amplitude in terms of the renormalized coupling constant f_1^2 and an integral over the total cross sections. The differential cross section $|f(0,\omega)|^2$ for forward scattering is deduced by assuming that only partial waves having small orbital angular momentum l are scattered. Using the optical theorem to evaluate $\mathrm{Im}f(0,\omega)$, the real part $D=\mathrm{Re}f(0,\omega)$ is deduced directly from the experimental results; it is not necessary to make a phase-shift analysis.

The experimental results for π^+ and π^- scattering are plotted in Figs. 1–3. The π^- data include the 307-Mev and 330-Mev values reported by S. M. Korenchenko and V. G. Zinov at the Padua Conference (1957).[3] The theoretical curves in Figs. 1 and 2 are those of Puppi and Stanghellini; Fig. 3 shows the curves of G. Salzman and H. Schnitzer.[4]

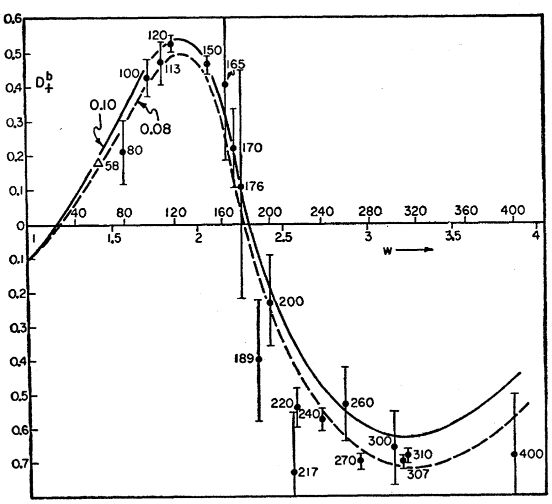

FIG. 1. The π^+-p scattering dispersion curves for $f^2=0.08$ and 0.10 calculated by Puppi and Stanghellini, together with the experimental values.

For π^+ scattering there is good agreement between the observed values and the dispersion relations with a coupling constant $f_1^2=0.09\pm0.01$. Figures 2 and 3 show that theory and experiment do not agree for π^- scattering. The dispersion relations for positive and negative pions are not independent; charge symmetry is assumed in their derivation. It follows that the π^+ and π^- data should be fitted using the same value of f_1^2. Figures 2

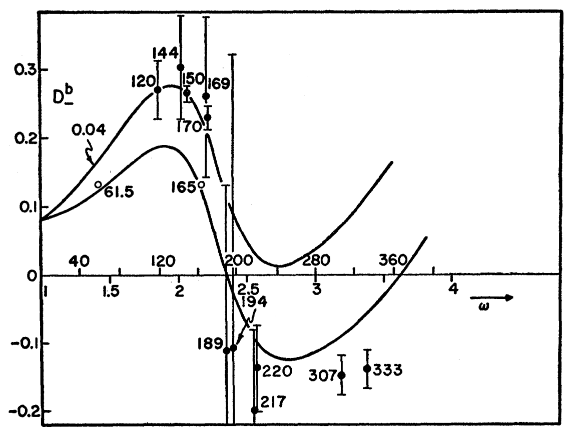

FIG. 2. The π^--p curves for $f^2=0.04$ and 0.08 as evaluated by Puppi and Stanghellini. The latest experimental values are shown. The errors for the values at 150 Mev and 170 Mev should be increased by about 40% to allow for the cross-section normalization errors and for error correlations.

and 3 show that the π^- data tend to favor small values of f_1^2, at least in the region 120–170 Mev. To obtain reasonable agreement with the π^+ data, we shall only consider coupling constant values $f_1^2\geq0.08$. This also is compatible with the coupling constant values deduced from the effective-range theory of pion-nucleon scattering,[5] and from the spin-flip dispersion relation [6]

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[1] Goldberger, Miyazawa, and Oehme, Phys. Rev. **99**, 986 (1955).

[2] G. Puppi and A. Stanghellini, Nuovo cimento **5**, 1305 (1957). (They only discussed π^-+p up to about 200 Mev.)

[3] I am indebted to Professors G. Puppi and E. Lomon fo information about these results.

[4] G. Salzman and H. Schnitzer (to be published). I am obliged to Dr. Salzman for permission to quote these results.

[5] For a discussion see reference 2 and J. Orear, Nuovo cimento 4, 856 (1956).

[6] W. C. Davidon and M. L. Goldberger, Phys. Rev. **104**, 1119

The curves of Salzman and Schnitzer are derived from the cross-section data given by Anderson and Piccioni.[7] The low-energy data of Anderson are in good agreement with his phase-shift analysis. Puppi and Stanghellini's curves are based on earlier cross-section data, and they give greater disagreement with the observed forward scattering amplitude in the 120–170 Mev region than do the curves of Salzman and Schnitzer. Here we shall use the same cross-section data as Salzman and Schnitzer.

In the region[8] $\omega=120$ to 170 Mev, the values of Puppi and Stanghellini for $D_-^B\equiv[\mathrm{Re}f_-(\omega)](k_B/k)$ are in error by $\delta D_-^B\simeq 0.10$, using $f_1^2=0.08$. In this region the values of Salzman and Schnitzer are better, but they are still in error by $\delta D_-^B=0.05$ to 0.06. In the region $\omega=250$ to 350 Mev, the curve of Salzman and Schnitzer is a little worse than Puppi and Stanghellini's; both are in appreciable disagreement with the observations. In both these energy regions the magnitude of the theoretical value of D_-^B is consistently less than the experimental results.

We shall examine the extent to which any of the following is adequate or necessary to explain the discrepancy in the π^- data:

(i) inaccuracies in the total cross-section values, and lack of data for the very high-energy cross sections;

(ii) errors in the scattering lengths;

(iii) electromagnetic interactions and the mass difference of charged and neutral pions;

(iv) virtual strange particle reactions;

(v) a failure of charge independence in the pion-nucleon interaction;

(vi) a breakdown of dispersion relations, implying a violation of local causality.

2. TOTAL CROSS SECTION

The forward scattering amplitudes for π^+ and π^- elastic scattering on protons satisfy

$$\mathrm{Re}f_\pm(\omega)-\frac{1}{2}\left(1+\frac{\omega}{\mu}\right)\mathrm{Re}f_\pm(\mu)-\frac{1}{2}\left(1-\frac{\omega}{\mu}\right)\mathrm{Re}f_\mp(\mu)$$
$$=-\frac{2f_1^2}{\mu^2}\left[\frac{k^2}{(\mu^2/2M)\mp\omega}\right]+\frac{k^2}{4\pi^2}\int_\mu^\infty\frac{d\omega'}{k'}$$
$$\times\left\{\frac{\sigma_+(\omega')}{\omega'\mp\omega}+\frac{\sigma_-(\omega')}{\omega'\pm\omega}\right\},\quad(1)$$

where $f_\pm(\omega)$ are the scattering amplitudes in the lab system for pion lab energy $\omega=(k^2+\mu^2)^{\frac{1}{2}}$, f_1^2 is the renormalized coupling constant, M is the nucleon mass, and $\sigma_\pm(\omega)$ are the total cross sections for $\pi^\pm$ on hydrogen.

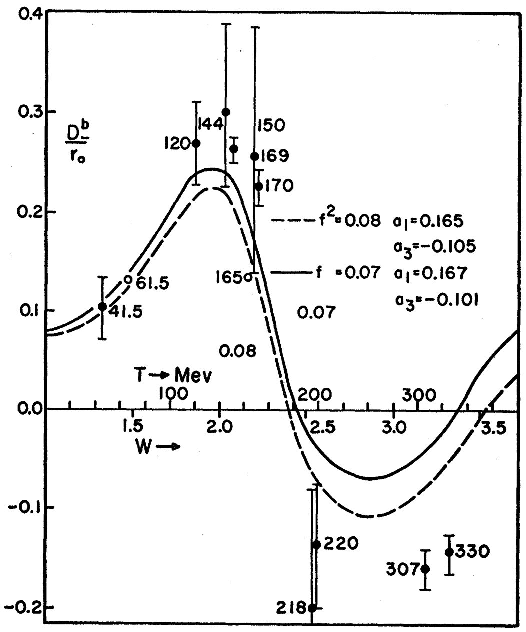

FIG. 3. The π^--p curves for $f^2=0.07$ and 0.08 calculated by G. Salzman and H. Schnitzer using the cross-section values given by Anderson and Piccioni (1957).

The corresponding center-of-mass variables are given by[9]

$$\frac{f_B(\omega_B)}{k_B}=\frac{f(\omega)}{k},\quad k_B=k\left(1+\frac{2\omega}{M}+\frac{\mu^2}{M^2}\right)^{-\frac{1}{2}}.$$

It is convenient to use nuclear units with $\hbar=c=\mu=1$; the units of area and energy are 20 mb and 140 Mev.

The effect of errors in the total cross sections $\sigma_\pm(\omega')$ will be different according as the errors fall into two types: (a) errors $\delta\sigma(\omega')$ at energies ω' which are appreciably higher than the energy ω at which we calculate $\mathrm{Re}f(\omega)$; (b) errors $\delta\sigma(\omega')$ at energies lower than or close to ω.

Type (a) Errors

If the error occurs for $\omega'>\Omega$ where Ω is much larger than ω, it produces a change in $\mathrm{Re}f(\omega)$:

$$\delta\,\mathrm{Re}f_\pm(\omega)\simeq\frac{k^2}{4\pi^2}\int_\Omega^\infty\frac{d\omega'}{k'\omega'}\{\delta\sigma_+(\omega')+\delta\sigma_-(\omega')\}.\quad(2)$$

Choosing $\Omega=1$ Bev, we see that an error $(\delta\sigma_+(\omega')+\delta\sigma_-(\omega'))=40$ mb for *all* $\omega'>\Omega$ gives $\delta D_\pm^B\simeq 0.016$ at a lab energy of 120 to 150 Mev. This large error [it is greater than 50% of the measured values of $(\sigma_++\sigma_-)$ in the 1-Bev to 2-Bev region] is unlikely.[10]

(1956); W. Gilbert and G. R. Screaton, Phys. Rev. **104**, 1758 (1956).

[7] H. L. Anderson, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics* (Interscience Publishers, Inc., New York, 1956), p. I-43; O. Piccioni, *ibid.*, p. IV-7.

[8] We should strictly say $\omega-\mu=120$–170 Mev.

[9] For example, $k/k_B=1.27$ at $\omega=140$ Mev, and 1.38 at 280 Mev.

[10] Cool, Piccioni, and Clark, Phys. Rev. **103**, 1082 (1956).

TABLE I. Possible errors in the total cross section $\sigma_-(\omega')$ and their effects on D_-^B at 150 Mev.

Energy range	Error $\delta\sigma_-(\omega')$	δD_-^B for $\omega=150$ Mev
1 Bev $>\omega'>420$ Mev	$+10$ mb	$\sim+0.01$
420 Mev$>\omega'>280$ Mev	$+20$ mb	$\sim+0.015$
280 Mev$>\omega'>210$ Mev	$+10$ mb	$\sim+0.01$

Physically it does not seem reasonable that $\sigma_\pm(\omega')$ should increase considerably at energies above 2 Bev; indeed since the shorter-range K-meson and hyperon forces become important at high energies, we might expect $\sigma_\pm(\omega')$ to decrease. It follows that errors in, or lack of knowledge of $\sigma_\pm(\omega')$ for $\omega'>1$ Bev cannot explain the discrepancy in $\mathrm{Re}f_-(\omega)$ around 150 Mev. It should also be noted that the above error $(\delta\sigma_++\delta\sigma_-)$ gives $\delta D_-^B\simeq 0.04$ in the 300- to 330-Mev region; this gives worse agreement with the observed values.

For errors $\delta\sigma_-(\omega')$ in (210 Mev$<\omega'<1$ Bev), we estimate the changes δD_-^B at $\omega\sim 150$ Mev by using Eq. (1). Typical values are shown in Table I. The magnitude of the errors $\delta\sigma_-$ is in each case appreciably larger than the experimental errors which are quoted for the measurements of $\sigma_-(\omega')$ in these energy regions.[10,11]

Similar errors in $\sigma_+(\omega')$ have less effect on $\mathrm{Re}f_-(\omega)$ ($\omega=150$ Mev), except for the range (1 Bev$>\omega'>420$ Mev) where $\delta\sigma_+(\omega')$ and $\delta\sigma_-(\omega')$ have almost the same effect.

The effect of errors of type (a) may be summed up by noting that a systematic underestimation of both $\sigma_+(\omega')$ and $\sigma_-(\omega')$ by the amounts indicated above for all $\omega'>210$ Mev would give a sufficiently large value of δD_-^B at $\omega=150$ Mev. The same corrections would give a large positive change in $\mathrm{Re}f_-(\omega)$ at $\omega=320$ Mev (δD_-^B would be expected to be of the order 0.07) which would increase the discrepancy in this region.[12] It would in any case be surprising if there were such large systematic errors (in one direction) in the total cross sections.

Type (b) Errors

Errors in the cross sections at energies ω' in the neighborhood of ω can have a large effect on $\mathrm{Re}f(\omega)$. Consider the contribution to $\mathrm{Re}f_-(\omega)$ from the integral over $\sigma_-(\omega')$; it can be written

$$\frac{k^2}{4\pi^2}\int_\mu^\infty \frac{d\omega'}{k'}\left\{\frac{\sigma_-(\omega')-\sigma_-(\omega)}{\omega'-\omega}+\frac{\sigma_-(\omega)}{\omega'-\omega}\right\}.$$

The contribution to this integral from any range of ω' which is symmetric about ω (i.e., $\omega-\omega_0\leq\omega'\leq\omega+\omega_0$) is

$$\frac{k^2}{4\pi^2}\int_{\omega-\omega_0}^{\omega+\omega_0}\frac{d\omega'}{k'}\left[\frac{\sigma_-(\omega')-\sigma_-(\omega)}{\omega'-\omega}\right]. \tag{3}$$

Thus in the neighborhood of ω' the dispersion integral is a measure of the derivative of $\sigma(\omega')$. [This explains the general form of the curves in Fig. 1 up to 300 Mev. $\mathrm{Re}f_+(\omega)$ rises where the slope of $\sigma_+(\omega)$ increases, up to about 120 Mev; it drops to zero at (approximately) the resonance, and it goes negative on the downward slope of $\sigma_+(\omega)$ beyond the resonance.]

The effect of errors $\delta\sigma_-(\omega')$ at energies $\omega'<210$ Mev has been studied by Zaidi and Lomon.[13] They show how the discrepancy in $\mathrm{Re}f_-(\omega)$ for $\omega\sim 120$ to 170 Mev can be considerably reduced by increasing the slope of $\sigma_-(\omega')$ in the region $\omega'\sim 100$ to 170 Mev (see Fig. 1 of their paper). Unfortunately their large increase[14] in the slope of $\sigma_-(\omega')$ over the region 110 Mev$<\omega'<150$ Mev cannot be justified; it implies an unduly large increase in the total cross section $\sigma_{\frac{1}{2}}(\omega')$ for the isotopic spin $T=\frac{1}{2}$ process at energies appreciably below the resonance. In this region it is much safer to keep as close as possible to Anderson's cross-section data[7] which agree with a reasonable phase shift analysis; this gives the curve of Salzman and Schnitzer (Fig. 3).

Cross Section near Resonance

In the range 150 Mev$<\omega'<200$ Mev Zaidi and Lomon use values of the total cross section $\sigma_-(\omega')$ which are significantly greater than the experimental values. It should be emphasized that an increase in the value of $\sigma_-(\omega')$ near resonance would considerably improve agreement with the π^- dispersion relation. At the resonance ($\omega'\simeq 180$ Mev) Anderson's curve gives $\sigma_-(\omega')=66$ mb. Consider the fairly extreme case in which the cross section is increased over the region 150 Mev to 200 Mev in such a way that the peak becomes sharper and the maximum is increased to 73 mb. We would then expect the theoretical value D_-^B to be increased by 0.015 to 0.025 at $\omega=150$ Mev; at $\omega=210$ Mev there would be a decrease in D_-^B of about the same size. Both corrections are in the right direction.

This increase in the peak of the $\sigma_-(\omega')$ curve would have another desirable effect. For $\omega=170$ Mev there would be little change in the theoretical value $\mathrm{Re}f_-(\omega)$, but the observed value would have to be reduced. The observation of D_-^B at 170 Mev (Figs. 2 and 3; the point is due to Ashkin *et al.*[15]) is based on a total cross section

$$\sigma_-(170\text{ Mev})=(62.7\pm1.9)\text{ mb}. \tag{4}$$

D_-^B is deduced from the differential cross section $|f(\theta,\omega)|^2$ using

$$D_-^B=\frac{k_B}{k}\left[|f(0,\omega)|^2-\left(\frac{k}{4\pi}\sigma_-(\omega)\right)^2\right]^{\frac{1}{2}}. \tag{5}$$

[11] S. J. Lindenbaum and L. C. L. Yuan, Phys. Rev. **100**, 306 (1955).

[12] To reduce the error in $f_-(\omega)$ at 320 Mev, the change in slope of $\sigma_-(\omega')$ should be made more negative, not more positive.

[13] M. H. Zaidi and E. L. Lomon, Phys. Rev. **108**, 1352 (1957). I am indebted to the authors for a preprint copy.

[14] The slope of the $\sigma_-(\omega')$ curve of Zaidi and Lomon is about 1.0 mb/Mev between $\omega'=110$ and 150 Mev. Anderson's values for σ_- give no more than 0.75 mb/Mev, while σ_+ has slope 2.2 mb/Mev in the same region.

[15] Ashkin, Blaser, Feiner, and Stern, Phys. Rev. **101**, 1149 (1956).

If the calibration has to be altered so that the differential cross section $|f(\theta,\omega)|^2$ is increased by (say) 10% for all θ, the observed value of D_-^B is reduced from 0.22 to 0.19. Again the correction is in the right direction.

In spite of the low value[16] of σ_- in (4), there is some indication[17] from a comparison of the total cross sections σ^+, σ_- that the σ_- curve might have a sharper and higher peak than Anderson's values[7] show. It would be valuable to have further careful determinations of $\sigma_-(\omega)$ in the region of its maximum.

3. SCATTERING LENGTHS

Equation (1) can be written

$$\mathrm{Re}f_\pm(\omega)-\mathrm{Re}f_\pm(\mu)$$
$$=\mp\left(\frac{\omega}{\mu}-1\right)\tfrac{1}{2}\{\mathrm{Re}f_-(\mu)-\mathrm{Re}f_+(\mu)\}$$
$$-\frac{2f_1^2}{\mu^2}\left[\frac{k^2}{(\mu^2/2M)\mp\omega}\right]+\frac{k^2}{4\pi^2}\int_\mu^\infty\frac{d\omega'}{k'}$$
$$\times\left\{\frac{\sigma_+(\omega')}{\omega'\mp\omega}+\frac{\sigma_-(\omega')}{\omega'\pm\omega}\right\}. \quad (6)$$

At low energies $\mathrm{Re}f_\pm(\omega)$ is particularly sensitive to the scattering lengths $\mathrm{Re}f_\pm(\mu)$; changing them changes the starting point of the curves $\mathrm{Re}f_\pm(\omega)$ at $\omega=\mu$, and it also changes the first term on the right of (6) which is linear in $(\omega-\mu)$.

Letting $\omega\to\infty$ in either Eq. (6) and making the plausible assumption[18] that $\mathrm{Re}f_\pm(\omega)/\omega\to 0$ as $\omega\to\infty$, gives the sum rule[19]

$$\frac{1}{\mu}\{\mathrm{Re}f_-(\mu)-\mathrm{Re}f_+(\mu)\}$$
$$=\frac{4f_1^2}{\mu^2}+\frac{1}{2\pi^2}\int_\mu^\infty\frac{d\omega'}{k'}\{\sigma_-(\omega')-\sigma_+(\omega')\}. \quad (7)$$

If $\mathrm{Re}f_\pm(\mu)$ and f_1^2 are related in this way, changes in the scattering lengths will not be important for large ω. [Using $\mathrm{Re}f_-(\mu)-\mathrm{Re}f_+(\mu)=0.205$, Eq. (7) gives $f_1^2=0.084$ when the integral over the cross sections is taken to 2 Bev; the contribution to the integral from higher energies appears to be small.[20] Because of the poor convergence of the integral, Eq. (7) is not a good determination of f_1^2.]

The values of the scattering lengths as determined by the well-known methods are

$$\mathrm{Re}f_-(\mu)=+0.089,\quad \mathrm{Re}f_+(\mu)=-0.116;$$

these are based on the center-of-mass system values[21]

$$a_1=+0.167\pm0.01,\quad a_3=-0.101\pm0.01. \quad (8)$$

Changes in the scattering lengths could be used to improve the agreement in Fig. 3 in the 120- to 170-Mev region. This requires keeping $\mathrm{Re}f_+(\mu)$ fixed and increasing substantially $\mathrm{Re}f_-(\mu)$. To avoid producing much more serious disagreement in the 300-Mev region it is necessary at the same time to increase f_1^2 above 0.08. Examples of such changes are given in Table II. These alterations would have little effect on the agreement between theoretical and experimental values of $\mathrm{Re}f_+(\omega)$ (see Fig. 1). They would make it possible to use $f_1^2\simeq0.09$ for both the π^+ and π^- data.

TABLE II. The effect on D_-^B of related changes in the π^- scattering length and the coupling constant f_1^2.

Parameter change	Change in D_-^B $\omega=150$ Mev	$\omega=300$ Mev
$\delta\,\mathrm{Re}f_-(\mu)=+0.022$, $\delta f_1^2=+0.007$	+0.010	+0.006
$\delta\,\mathrm{Re}f_-(\mu)=+0.035$, $\delta f_1^2=+0.011$	+0.016	+0.009

Changes +0.022 and +0.035 in $\mathrm{Re}f_-(\mu)$ correspond to increasing a_1 from 0.167 to 0.196 and 0.212, respectively. The values (8) are deduced from elastic π^- scattering between 40 Mev and 60 Mev[22]; similar results come from data in the 15-Mev to 30-Mev range.[23] Increases in a_1 of the above magnitude would be compatible with the low-energy π^- scattering only if the effective range for the phase shift α_1 were noticeably larger than is generally believed.

The values (8) are also related to charge-exchange scattering at threshold. They are consistent with Panofsky ratio $R=1.55$ and π^-/π^+ threshold ratio $r_0=1.87$; this does not include any Coulomb correction which might increase r_0, or a Noyes correction (which would be noticeable if the effective range of α_1 were large). These somewhat uncertain corrections[20] might increase a_1 by 10%. The latest and largest value of the Panofsky ratio[24] $R=1.85\pm0.09$ would, without any corrections give $a_1=0.187$ (assuming as always that $a_3=-0.101$).

[16] Ashkin *et al.* also give $\sigma_+=194.9\pm5.5$ mb at 170 Mev; in relation to this, σ_- is low.

[17] See, for example, the data in reference 4.

[18] It is plausible for a scatterer of finite radius because $\mathrm{Re}f_B(\omega_B)/k_B\to 0(1)$ implies that, if only elastic scattering occurs, for large ω_B an appreciable fraction of *all* the partial waves have phase shifts η_l giving $\sin(2\eta_l)$ large and of the same sign. The existence of inelastic processes reduces $\mathrm{Re}f_B(\omega_B)$.

[19] M. L. Goldberger, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics* (Interscience Publishers, Inc., New York, 1956), p. I-14.

[20] J. M. Cassels, *Proceedings of the Seventh Annual Rochester Conference on High-Energy Physics* (Interscience Publishers, Inc., New York, 1957), p. II-5.

[21] These recent Rochester results have been used by Salzman and Schnitzer. They differ slightly from Orear's values $a_1=0.165$, $a_3=-0.105$.

[22] J. Orear, Nuovo cimento 4, 856 (1956).

[23] Nagel, Hildebrand, and Plano, Phys. Rev. **105**, 718 (1956).

[24] L. Marshall, *Proceedings of the Seventh Annual Rochester Conference on High-Energy Physics* (Interscience Publishers, Inc., New York, 1957), p. II-32.

It appears therefore that a high value of the Panofsky ratio would do something, but not enough, to reduce the discrepancy in the π^- dispersion relation around 150 Mev.

The results of this and the previous section's discussion are summarized by suggesting that, in order of importance, experiments to resolve the discrepancy would be (a) redetermination of the value of $\sigma_-(\omega')$ at resonance, and (b) much more accurate differential cross sections at 40 Mev or 60 Mev. This would give accurate information about the scattering lengths.

4. THRESHOLD EFFECTS

At $\omega'=\mu$ both $v_-\sigma(\pi^-+p\rightarrow\pi^0+n)$ and $v_-\sigma(\pi^-+p\rightarrow\gamma+n)$ are finite,[25] therefore the integral in (1) does not converge (for any ω). This is because we forgot that for the charge-exchange process the lower limit of integration is $\omega'\simeq\mu-\Delta m$ where $\Delta m=(m_\pi-m_{\pi'})+(m_p-m_n)\simeq 3$ Mev; for photoproduction the lower limit of integration is $\omega'=\mu^2/2M$ (i.e., at the neutron position in the rest-mass spectrum). With these limits of integration and taking the principal value at $\omega'=\mu$, the integral converges for $\omega>\mu$. Also, it is easy to show that the limit of $\mathrm{Re}f_-(\omega)$ as $\omega\rightarrow\mu$ is finite.

We now consider the corrections to the integral in (1) from these threshold effects. For the charge-exchange process we notice that the usual evaluation[2,4] of the integral for $\mathrm{Re}f_-(\omega)$ assumes that $\sigma_-(\omega')$ is constant for small $(\omega'-\mu)$. Hence we want the correction arising from an increment $\Delta\sigma_{\mathrm{ex}}(\omega')$ in the charge-exchange cross section $\sigma_{\mathrm{ex}}(\omega')$, where

$$\Delta\sigma_{\mathrm{ex}}(\omega')=\left(\frac{v_0-v_-}{v_-}\right)\sigma_{\mathrm{ex}}(\omega'),$$

v_0 and v_- are the c.m. pion velocities, and $\sigma_{\mathrm{ex}}(\omega')$ equals $(8\pi/9)(a_1-a_3)^2$ for $\omega'>\mu$. The integral can be written

$$\frac{k^2}{4\pi^2}\int_{\mu-\Delta m}^{\infty}\frac{d\omega'}{k'^2}\left[\frac{\Gamma(\omega')}{\omega'-\omega}\right], \tag{9}$$

where $\Gamma(\omega')=k'\Delta_{\mathrm{ex}}(\omega')$.

For $\omega\gg\Delta m$, the integrand in (9) decreases rapidly for $\omega'>\mu+\Delta m$ [the factor (v_0-v_-) goes to zero as $(1/k')$ in that region]. At threshold $\Gamma(\mu)=0.03$ unit, but we do not know its form for $\mu-\Delta m<\omega'<\mu$. It is sufficient to assume that in this region $\Gamma(\omega')$ behaves smoothly, going to zero at $\omega'=\mu-\Delta m$.

If $\Gamma(\omega')$ were equal to $\Gamma(\mu)$ throughout $\mu-\Delta m<\omega'<\mu+\Delta m$, the contribution to (9) from this energy interval would be of order $(k^2/\omega)(0.003)\Delta m$. Allowing for the increase in $\Gamma(\omega')$ from 0 at $(\mu-\Delta m)$ to $\Gamma(\mu)$, and the subsequent decrease for $\omega'>\mu+\Delta m$, we get an upper limit[26] $(k^2/\omega)(0.005)\Delta m$ for the contribution to $\mathrm{Re}f_-(\omega)$ from $|\omega'-\mu|\leq\Delta m$. Multiplying by 5 will adequately cover to contribution to (9) from $(\omega'-\mu)>\Delta m$. It follows that for the values of ω in which we are interested ($\omega\gtrsim 120$ Mev), the charge-exchange threshold effect is quite negligible.[27]

The effect on $\mathrm{Re}f_-(\mu)$ need not be negligible. However, this contribution is included in Noyes' correction[28] which was discussed in the preceding section.

The radiative correction arising from $\sigma(\pi^-+p\rightarrow\gamma+n)$ has been evaluated by Agodi *et al.*[29]; the contribution to $\mathrm{Re}f_-(\omega)$, at the values of ω we are interested in, is unimportant.

Coulomb effects are not included in the dispersion relations (1). At $\omega\gtrsim 100$ Mev the Born approximation can be used to subtract the Coulomb contribution from the observed scattering. Because the Coulomb term is important only close to the forward direction, the subtraction can be made more or less independent of the phase-shift analysis.[15] This procedure is simpler than trying to work with dispersion relations which include the Coulomb interaction (if such relations exist).

Agodi *et al.*[29] have also shown that K-meson and hyperon effects do not change the dispersion relations (1). Since we insert the total observed cross sections in (1), the only place where an extra term due to strange particle effects can occur is in the unphysical region $0<\omega'<\mu$. However, the conservation of strangeness ensures that the only bound states if the neutron; this is already included in (1).

5. CHARGE INDEPENDENCE

The derivation of the dispersion relations (1) does not require charge independence; only charge symmetry need be used.[30] It follows that a breakdown of charge independence for pion scattering near the resonance ($\sim$175 Mev) would not account for the discrepancy in $\mathrm{Re}f_-(\omega)$.

On the other hand, there is a dispersion relation which can be used as a test of charge independence; this is the charge-exchange relation: the scattering amplitude $f_{\mathrm{ex}}(\omega)$ for forward charge exchange scattering is given by

$$f_{\mathrm{ex}}(\omega)=-\sqrt{2}f^{(2)}(\omega), \tag{10}$$

where

$$f^{(2)}(\omega)=\tfrac{1}{2}[f_-(\omega)-f_+(\omega)].$$

Also $f^{(2)}(\omega)$ must obey

$$\frac{1}{\omega}\mathrm{Re}f^{(2)}(\omega)=\frac{1}{\mu}\mathrm{Re}f^{(2)}(\mu)-\frac{2f^2}{\mu^2}\frac{k^2}{\omega^2-(\mu^2/2M)^2}+\frac{k^2}{4\pi^2}\int_{\mu}^{\infty}\frac{d\omega'}{k'}\left[\frac{\sigma_-(\omega')-\sigma_+(\omega')}{\omega'^2-\omega^2}\right]. \tag{11}$$

[25] v_- is the π^- velocity.

[26] This is based on assuming that $\Gamma(\omega')$ has the slope $\Gamma(\mu)/\Delta m$ throughout $|\omega'-\mu|\leq\Delta m$. In fact the slope is negative for $\omega'>\mu$.

[27] Using a fixed-source Hamiltonian, A. Agodi and M. Cini [Nuovo cimento **5**, 1256 (1957)] reach the same conclusion.

[28] H. P. Noyes, Phys. Rev. **101**, 320 (1956).

[29] Agodi, Cini, and Vitale, Phys. Rev. **107**, 630 (1957).

[30] A. Agodi and M. Cini, Nuovo cimento **6**, 686 (1957).

From the observed differential cross section for charge exchange scattering, we deduce $|f_{ex}(\omega)|^2$. Using

$$\mathrm{Im} f_{ex}(\omega) = \frac{k}{4\pi\sqrt{2}}[\sigma_+(\omega) - \sigma_-(\omega)],$$

we deduce $\mathrm{Re} f_{ex}(\omega)$. This observed value should fit (11).

Among the observations of charge-exchange scattering, only the results of Ashkin *et al.*[15] would seem to be sufficiently accurate to compare with (11). The 150-Mev data give $\mathrm{Re} f_{ex}(\omega) = 0 \pm 0.2$; this is to be compared with the theoretical value -0.2. The 170-Mev charge-exchange data are in disagreement with charge independence as they give $\mathrm{Re} f_{ex}(\omega) = (-0.08 \pm 0.04)^{\frac{1}{2}}$. The theoretical value is close to zero.

Although there are larger quoted errors in the charge-exchange data at 120, 144, 165, and 220 Mev,[31] they give values of $\mathrm{Re} f_{ex}(\omega)$ which are in reasonable agreement with (11). However, the accuracy of these data is not sufficiently great for us to use (10) as a useful check on charge independence.

Obviously it would be valuable as a check on $\mathrm{Re} f_-(\omega)$ to have other accurate charge-exchange differential cross sections in the 150-Mev and 300-Mev ranges.

Finally we comment on the form [Eq. (1)] of the dispersion relations. It is well known that in the general case it may be necessary to add[32] to $\mathrm{Re} f_\pm(\omega)$ in dispersion relations a real finite polynomial in ω. This polynomial could arise from an indeterminancy in the scattering amplitude contribution from $\epsilon(x_0)[j_\alpha(x), j_{\alpha'}(0)]$ on the light cone, or it could arise from the term $\delta(x_0)[j_\alpha(x), \phi_{\alpha'}(0)]$ which should be included if there were a direct pion-pion interaction.[33] The explicit form of this polynomial cannot be found by general arguments, but fortunately it disappears from the dispersion relations if they are made sufficiently convergent[32] (for example, by the subtraction procedure). It is probable that Eqs. (1) are sufficiently convergent to eliminate the polynomial.[34]

The most straightforward way of investigating the polynomial would be to examine the observed values of $\mathrm{Re} f_\pm(\omega)$ for large ω. However, it seems possible even with the present data to exclude such an addition to the right-hand side of (1) as an explanation of the discrepancy in $\mathrm{Re} f_-(\omega)$. The relations (1) contain one subtraction (at $\omega = \mu$), so the polynomial must be of the form $c_1(\omega-\mu) + c_2(\omega-\mu)^2 + \cdots$. At least two terms would be required to give a discrepancy which changes sign between 150 Mev and 300 Mev. This implies, at the least, that $\mathrm{Re} f_-(\omega) \sim c_2\omega^2$ for large ω. Such a large value of $\mathrm{Re} f_-(\omega)$ for large ω seems quite unreasonable.[35]

6. CONCLUSIONS

Salzman and Schnitzer's use of the σ_- cross-section values of Anderson has removed a good deal of the π^- discrepancy, but a careful examination shows that the remaining disagreement cannot be removed without violating some apparently accurate experimental results.

It would be useful to know (i) the exact value of $\sigma_-(\omega')$ at resonance; (ii) the forward amplitude D_-^B at a point in the 40–60 Mev range (to an accuracy of ± 0.01 unit); (iii) more accurate charge-exchange reresults both in the 150-Mev and 300-Mev ranges.

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[31] H. L. Anderson and M. Glicksman, Phys. Rev. **100**, 268 (1955); H. L. Anderson *et al.*, Phys. Rev. **91**, 155 (1953).

[32] K. Symanzik, Phys. Rev. **105**, 743 (1957).

[33] The notation is that of M. L. Goldberger, Phys. Rev. **99**, 979 (1955).

[34] K. Symanzik, Nuovo cimento **5**, 659 (1957); E. Kazes, Phys. Rev. **108**, 123 (1957). Convergence of the integrals in (1) is necessary but not sufficient to exclude the polynomial.

[35] See the argument in footnote 18. Also, notice that the existence of the sum rule (7) implies that (1) is correct as it stands.