Nuclear Structure Effects in Internal Conversion*†

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The problem of the influence of nuclear structure on the calculation of conversion coefficients is discussed and a distinction is made between those effects (static) which are essentially calculable without a detailed nuclear model and the effects (dynamic) for which such a model is mandatory. The theory of the conversion process is developed in such a way as to provide formulas wherein both types of effects are exhibited. No attempt is made to provide specific numerical results for the dynamic effects but instead a framework is provided within which the internal conversion predictions of any nuclear model can be readily tested. In order to do this it is necessary to know not merely the conversion coefficient but the matrix elements for the separate final states as complex numbers. This information, which is available at present for only the K shell, is utilized to provide numerical results for the K shell and for several values of transition energy and atomic number. Electric and magnetic 2^L-pole transitions with $1 \leqslant L \leqslant 5$ are considered.

I. INTRODUCTION

UNTIL the effect of the nonzero radius of the nucleus was taken into account it was considered that internal conversion coefficients were essentially independent of nuclear structure. While this is true for light nuclei and for many transitions in heavy nuclei as well, it is now realized that appreciable structure effects can appear in the conversion process for some heavy nuclei.

Nuclear structure can be said to affect the 2^L-pole conversion coefficients in a "static" way and in a "dynamic" way. It gives rise to a static effect through the average nuclear charge distribution which acts on the atomic electrons in the stationary states of the electron-nucleus system. It gives rise to a dynamic effect in the sense that the nuclear matrix elements for atomic electron ejection are different from those for gamma-ray emission. Thus the static effect, on the one hand, can be expressed entirely in terms of the electron wave functions. The dynamic effect, on the other hand, involves the explicit appearance of the nuclear wave functions and the operators governing electromagnetic transitions in nuclei.

It is generally accepted that the static effect can be dealt with adequately without recourse to a detailed model of nuclear structure by introducing a reasonable charge distribution from which to calculate the electrostatic potential produced by the nucleus in its stationary states. There is reason to believe, however, that in a number of cases the dynamic effect can only be treated adequately with a rather complete nuclear model. Indeed the general failure of very simple nuclear models to predict many gamma-ray lifetimes lends strong support to this view.

The static nuclear structure effect has been treated for the K shell by Sliv[1] and for the K and L shells by Rose.[2] In each calculation the nucleus was considered to be a sphere of constant charge density. Except in a few cases the conversion coefficients are smaller than those calculated for a point nucleus.[3] The discrepancy amounts to a few percent for atomic number Z less than 60. For larger values of Z considerably larger reductions can occur.

In the above-mentioned calculations of conversion coefficients the authors differ somewhat in their management of the dynamic structure effect. Sliv treats this effect by supposing that the nuclear currents are confined to the surface of the nucleus. Rose adopts a different approximation which will be described in Sec. IIIb. For a finite nucleus the two calculations lead to conversion coefficients which differ by less than five percent in almost all cases of physical importance. The result of either treatment is that the part of the conversion coefficient which is dependent on dynamic structure effects is but a few percent of the part which is independent of these effects.

Now, as was pointed out by Church and Weneser,[4] nuclear models which allow for shell structure effects can lead in some cases to (dynamic) structure-dependent contributions to the internal conversion coefficients which are sizable fractions or perhaps even large multiples of the structure-independent contributions. Fur-

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[1] L. A. Sliv, J. Exptl. Theoret. Phys. U.S.S.R. **21**, 770 (1951); L. A. Sliv and M. Listengarten, J. Exptl. Theoret. Phys. U.S.S.R. **22**, 29 (1952); L. A. Sliv and I. M. Band, "Coefficients of internal conversion of gamma radiation," Academy of Science, U.S.S.R., 1956, reproduced in the United States as Report 571CCK1 of the Department of Physics of the University of Illinois.

[2] M. E. Rose (unpublished). These results will appear in a forthcoming book.

[3] M. E. Rose. Some of these results have been published in *Beta- and Gamma-Ray Spectroscopy*, edited by K. Siegbahn (North-Holland Publishing Company, Amsterdam, 1955), Appendix IV. Other results have been circulated privately.

[4] E. Church and J. Weneser, Phys. Rev. **104**, 1382 (1956).

thermore the experimental values[5–8] of the conversion coefficients for a few transisitions, notably $M1$ and $E2$ transitions, seem to be in disagreement with the results of Sliv's calculation.

Recently, it has been shown by Reiner[9] that the unified model can give $M1$ matrix elements which can increase the calculated conversion coefficient by a factor of about 20 in the case of the 482-kev transition in Ta^{181}. This compares with an experimental ratio of 5 ± 2. For nuclei near closed shells smaller but significant changes can be expected in some cases. Thus, calculations of Kisslinger[10] indicate that configuration interaction in Tl^{203} should reduce the computed coefficients by about 10%. In the majority of cases it is to be expected that the dynamic effects of nuclear structure are very small. Nevertheless, the occasional occurrence of large dynamic effects (particularly in strongly deformed nuclei) can now be considered as fairly well established.

It should be recognized that the dynamic contribution to the conversion coefficient adds coherently to the static contribution. It is therefore necessary to exhibit the latter as a set of amplitudes for each final state of the electron and these amplitudes must have the correct absolute phase. Instead of one real number, the conversion coefficient, it is necessary to know two complex numbers in those transitions for which the angular momentum of the electron is $\frac{1}{2}$ and in general it is necessary to know $2j_i+1$ complex numbers in order to incorporate the dynamic effects in the calculation of the conversion coefficients. This program can be carried out for the K shell, thanks to an unpublished tabulation of the matrix element for that case,[11] and the present paper is addressed to this task.

Our aim is, therefore, to develop the theory of internal conversion in such a way that the dynamic structure effects are explicitly exhibited in a formal way without specifying the nuclear model to be used eventually. When a specific model is adopted, the relevant contribution to the final state amplitudes can be added to the calculated amplitudes which represent the static effect only. In any event, it will be appreciated that the calculation of the static effect, which is essentially independent of the uncertainty introduced by nuclear models and can be calculated with a large degree of reliability, should be separated from that part pertaining to the dynamic structure effects. To determine the existence of the latter type of effect the experimental results should always first be compared to the calculations with only static effects included.[12]

In the following, the conventional theory of internal conversion is formulated in such a way as to exhibit the dynamic structure effect as a series of ratios of nuclear matrix elements. These appear with certain coefficients which can be calculated without specification of a model and these coefficients are tabulated so that they may be used in conjunction with any model. This makes it possible to study the dynamic effect with various nuclear models without redoing each time the very tedious part of the calculation which involves the static effect alone.

In this paper we shall consider pure multipoles only. This is no loss of generality since, as is well known, the contributions of two multipoles which mix are added incoherently.[13] Strictly speaking, the mixing ratio is also calculable in terms of the nuclear model and the phenomenon of mixing could be referred to as a nuclear structure effect. However, it seems more appropriate to separate this problem from the coherent effects of nuclear structure to which we have already referred. If desired, the incoherent effect can be taken care of by the introduction of empirically adjusted mixing ratios; these mixing ratios can also be measured in the angular correlation process. There is no corresponding way to represent the coherent effects nor would it be desirable to introduce too many empirically adjusted constants in any event.

II. DEVELOPMENT OF THE THEORY

Consider a neutral atom of atomic number Z whose nucleus is in a low-lying excited state. Selection rules permitting, such a nucleus can undergo a transition to a state of lower energy either by emitting a gamma ray or by ejecting an orbital electron from the atom. Let the transition rates for these two processes be called N_γ and N_e, respectively. Then, by definition, the internal conversion coefficient is the ratio of N_e to N_γ. The starting point of our development is the following pair of equations for N_γ and N_e.[14]

$$N_\gamma=\frac{8\pi k\alpha}{2J_i+1}\sum_{M_f}\sum_{M_i}\sum_{L=1}^{\infty}\sum_{M=-L}^{L}\sum_{\sigma=1}^{2}\times\left|\int_{V_N}d^3x\,\mathbf{J}_N(\mathbf{x})\cdot\mathbf{A}_L{}^M(\mathbf{x};\sigma)\right|^2, \tag{1}$$

$$N_e=\frac{2\pi\alpha^2}{2J_i+1}\sum_{M_f}\sum_{M_i}\sum_{\mu'}\sum_{\kappa}\sum_{\mu}|M_{fi}|^2, \tag{2}$$

[5] F. K. McGowan and P. H. Stelson, Phys. Rev. **107**, 1674 (1957).
[6] K. Siegbahn (private communication).
[7] G. S. Goldhaber (private communication).
[8] A. H. Wapstra and G. J. Nijgh (private communication).
[9] A. S. Reiner, *Proceedings of the 1957 International Conference on Nuclear Structure, Rehovoth, Israel* (North-Holland Publishing Company, Amsterdam, to be published). See also forthcoming publication in Nuclear Physics.
[10] L. S. Kisslinger (unpublished).
[11] This tabulation will be included in the publication referred to in reference 2.
[12] See M. E. Rose, *Proceedings of the 1957 International Conference on Nuclear Structure, Rehovoth, Israel* (North-Holland Publishing Company, Amsterdam) (to be published).
[13] M. E. Rose, *Multipole Fields* (John Wiley and Sons, Inc., New York, 1955). We shall henceforth designate this reference by the symbol I.
[14] The derivation of Eqs. (1), (2), and (3) will be discussed as soon as the notation has been explained. We use units such that $\hbar=m=c=1$.

where

$$M_{fi}=\int_{VN} d^3x \int_{Ve} d^3y\{\mathbf{J}_N(\mathbf{x})\cdot\mathbf{J}_e(\mathbf{y})-\rho_N(\mathbf{x})\rho_e(\mathbf{y})\} \times r^{-1}\exp(ikr). \quad (3)$$

In the above equations the symbols have the following meaning. The constant k, a positive number, is the difference in energy between the two nuclear states; α is the fine structure constant. J_f and J_i are the total angular momentum quantum numbers of the final and initial nuclear states and M_f and M_i are the corresponding projection quantum numbers. The vectors $\mathbf{x}$ and $\mathbf{y}$ are position vectors in the three-dimensional spaces whose volume elements are d^3x and d^3y. The variable, r, is equal to $|\mathbf{x}-\mathbf{y}|$. $\mathbf{J}_N(\mathbf{x})$ and $\rho_N(\mathbf{x})$ are the transiton current and charge densities which describe the interaction of the nucleus with the electromagnetic field. Similarly, $\mathbf{J}_e(\mathbf{y})$ and $\rho_e(\mathbf{y})$ describe interaction of the ejected electron with the electromagnetic field. These densities are assumed to obey the usual continuity equations.

$$\mathrm{div}\mathbf{J}_N=ik\rho_N, \quad \mathrm{div}\mathbf{J}_e=-ik\rho_e. \quad (4)$$

$\mathbf{J}_e$ and ρ_e are defined in the conventional manner in terms of the Dirac matrices and the Dirac one electron wave functions of the initial and final states. These functions are calculated for a spherically symmetrical potential which is determined by supposing that the nucleus is a sphere of constant charge density and radius, $R=1.2A^{\frac{1}{3}}\times10^{-13}$ cm, where A is the nuclear mass number. Atomic screening is accounted for according to the Thomas-Fermi-Dirac model. The wave functions are represented in the manner described on pages 65 and 66 of I and the continuum functions are normalized on the energy scale. The parameter κ is a nonzero integer which specifies the total angular momentum quantum number $j=|\kappa|-\frac{1}{2}$ and the parity quantum number $l=j+\kappa/2|\kappa|$ for the final electron state. The angular momentum projection quantum number for this state is μ.[15] The initial electron state is analogously described by quantum numbers κ', j', l', and μ'.

We shall not write down specific formulas for the nuclear charge and current densities since none will be needed for our purposes. The reader should perhaps be warned that our notation for these densities is extremely compressed. In the case of most nuclear models, the integrals of the densities will be off-diagonal nuclear matrix elements of sums of single-nucleon operators.[16] In addition to supposing that the densities satisfy Eq. (4), we shall in the course of the development make a few further reasonable assumptions about them which are as nonrestrictive as possible.

[15] The notation is the same as that discussed on p. 65 of I.

[16] R. K. Osborne and L. L. Foldy, Phys. Rev. **79**, 795 (1950). These authors show how to construct appropriate charge and current operators phenomenologically.

The vectors $\mathbf{A}_L{}^M(\mathbf{x};\sigma)$ are defined by the relations

$$\begin{aligned}
\mathbf{A}_L{}^M(\mathbf{x};1)&=-j_L(kx)\mathbf{T}_{L,L}{}^M(\theta,\phi),\\
\mathbf{A}_L{}^M(\mathbf{x};2)&=\left(\frac{L+1}{2L+1}\right)^{\frac{1}{2}}j_{L-1}\mathbf{A}_{L,L-1}{}^M\\
&\quad-\left(\frac{L}{2L+1}\right)^{\frac{1}{2}}j_{L+1}\mathbf{T}_{L,L+1}{}^M, \quad (5)\\
\mathbf{A}_L{}^M(\mathbf{x};3)&=\left(\frac{L}{2L+1}\right)^{\frac{1}{2}}j_{L-1}\mathbf{T}_{L,L-1}{}^M\\
&\quad+\left(\frac{L+1}{2L+1}\right)^{\frac{1}{2}}j_{L+1}\mathbf{T}_{L,L+1}{}^M,
\end{aligned}$$

in which spherical coordinates, $x=|\mathbf{x}|$, θ, ϕ have been introduced. $j_L(kx)$ is the spherical Bessel function of order L and the $\mathbf{T}_{L,\lambda}{}^M$ are the vector spherical harmonics defined by Rose.[17]

Before proceeding to use Eqs. (1), (2), and (3) we shall remark briefly on the manner in which they have been established. Derivations of these equations have been given by several authors, each of whom adopted a different representation of quantum electrodynamics. Tralli and Goertzel[18] and Kramer[19] use the Schrödinger representation. They differ, however, in their treatment of the photon field. Coester[20] employs the interaction representation. We have verified by direct comparison that all three methods lead to expressions for the conversion coefficients which agree with those obtained from Eqs. (1), (2), and (3).

The rest of this section will now be devoted to the simplification of Eqs. (1), (2), and (3). We will introduce the usual multipole expansion of M_{fi}, carry out a necessary integration by parts, do the angle dependent parts of the integrals in $\mathbf{x}$ and $\mathbf{y}$ space and carry out the projection quantum number sums which appear in Eqs. (1) and (2). This will lead to expressions in which only the radial integrals remain to be carried out. It will then be seen that it is possible to decompose each of the conversion coefficients for pure 2^L-pole radiation into a part which is independent of dynamic structure effects and a part which is not. Explicit formulas for each of these parts will be given.

In order to make effective use of the conservation of angular momentum it is essential to express the function $r^{-1}\exp(ikr)$ in terms of angular momentum eigenfunc-

[17] See pp. 22, 23, and 30 of I. For purposes of comparison it is important to note that the vector fields of Eq. (5) above are equal to $(\pi/2)^{\frac{1}{2}}$ times the corresponding fields of I.

[18] N. Tralli and G. Goertzel, Phys. Rev. **83**, 399 (1951).

[19] G. Kramer, Z. Physik **146**, 187 (1956). Kramer points out a number of valid objections to the method used by Tralli and Goertzel. However his Eq. (38) leads to a result for the conversion coefficient which is the same as that obtained by Tralli and Goertzel. Also see G. Kramer, Z. Physik **147**, 628 (1957) for additions and an important correction to the above article.

[20] F. Coester (unpublished). Coester uses the S-matrix formalism in the interaction representation.

tions by using the well-known expansion,[21]

$$\frac{e^{ikr}}{4\pi ikr}=\sum_{L=0}^{\infty}\sum_{M=-L}^{L}Y_L{}^{M*}(\theta,\phi)Y_L{}^{M}(\theta',\phi')\times\begin{cases}j_L(kx)h_L{}^{(1)}(ky); & y>x\\ h_L{}^{(1)}(kx)j_L(ky); & y<x.\end{cases}\tag{6}$$

In Eq. (6), $(y=|\mathbf{y}|,\theta',\phi')$ are the spherical coordinates of the point $\mathbf{y}$. An asterisk means complex conjugate. The above expansion does not converge uniformly with respect to both variables $\mathbf{x}$ and $\mathbf{y}$ for unrestricted values of these variables. Consequently the term-by-term integration of the series has to be introduced with appropriate care. The result of the above transformation can be expressed as follows.

$$M_{fi}=\sum_{L=0}^{\infty}\sum_{M}\{M_L{}^{M}(m)+M_L{}^{M}(e)\},\tag{7}$$

where

$$\begin{aligned}(4\pi ik)^{-1}M_L{}^{M}(m)=&\int_{VN}d^3x\,\mathbf{J}_N(\mathbf{x})\cdot\mathbf{B}_L{}^{M*}(\mathbf{x};1)\\&\times\int_0^x d^3y\,\mathbf{J}_e(\mathbf{y})\cdot\mathbf{A}_L{}^{M}(\mathbf{y};1)\\&+\int_{VN}d^3x\,\mathbf{J}_N(\mathbf{x})\cdot\mathbf{A}_L{}^{M*}(\mathbf{x};1)\\&\times\int_x^\infty d^3y\,\mathbf{J}_e(\mathbf{y})\cdot\mathbf{B}_L{}^{M}(\mathbf{y};1),\end{aligned}\tag{8}$$

and

$$\begin{aligned}(4\pi ik)^{-1}M_L{}^{M}(e)=&\sum_{\sigma=2}^{3}\Big\{\int_{VN}d^3x\,\mathbf{J}_N(\mathbf{x})\cdot\mathbf{B}_L{}^{M*}(\mathbf{x};\sigma)\\&\times\int_0^x d^3y\,\mathbf{J}_e(\mathbf{y})\cdot\mathbf{A}_L{}^{M}(\mathbf{y};\sigma)\\&+\int_{VN}d^3x\,\mathbf{J}_N(\mathbf{x})\cdot\mathbf{A}_L{}^{M*}(\mathbf{x};\sigma)\\&\times\int_x^\infty d^3y\,\mathbf{J}_e(\mathbf{y})\cdot\mathbf{B}_L{}^{M}(\mathbf{y};\sigma)\Big\}\\&-\int_{VN}d^3x\,\rho_N(\mathbf{x})Y_L{}^{M*}(\theta,\phi)h_L{}^{(1)}(kx)\\&\times\int_0^x d^3y\,\rho_e(\mathbf{y})Y_L{}^{M}(\theta',\phi')j_L(ky)\\&-\int_{VN}d^3x\,\rho_N(\mathbf{x})Y_L{}^{M*}(\theta,\phi)j_L(kx)\\&\times\int_x^\infty d^3y\,\rho_e(\mathbf{y})Y_L{}^{M}(\theta',\phi')h_L{}^{(1)}(ky).\end{aligned}\tag{9}$$

[21] See Eq. (1.22) of I. In this equation the sum is multiplied by $2\pi^2ik$ rather than $4\pi ik$ because Rose uses radial functions which differ ours by a factor $(2/\pi)^{\frac{1}{2}}$.

In Eqs. (8) and (9) the vectors $\mathbf{B}_L{}^{M}(\mathbf{x};\sigma)$ are obtained from the vectors $\mathbf{A}_L{}^{M}(\mathbf{x};\sigma)$ of Eq. (5) by replacing $j_L(kx)$ by $h_L{}^{(1)}(kx)$. An asterisk applied to either kind of vector means take the complex conjugate of its spin-angle factors $\mathbf{T}_{L,\lambda}{}^{M}$. The notation $\int_0^x d^3y$ means integration over the region for which $0\leqq y\leqq x$; the notation $\int_x^\infty d^3y$ means integration over the region for for which $x\leqq y<\infty$. In obtaining Eqs. (8) and (9) we employed our equivalent of Eq. (4.5) of I. This equation provides an expansion of the unit dyadic times $r^{-1}\exp(ikr)$ in terms of the vectors $\mathbf{A}_L{}^{M}(\mathbf{x};\sigma)$ and $\mathbf{B}_L{}^{M}(\mathbf{x};\sigma)$. An examination of the integrands which appear in Eqs. (8) and (9) shows that because of the Hankel functions the integrals are improper. In each case the integrals are convergent.

In Eq. (7) we have expressed M_{fi} as a sum of terms $M_L{}^{M}(e)$ and $M_L{}^{M}(m)$ which can be referred to as electric 2^L-pole and magnetic 2^L-pole terms, respectively. This decomposition is based on the nuclear angular momentum and parity selection rules which arise from the solid angle integrations implied by Eqs. (8) and (9). For given L, greater than zero, these selection rules are just the well-known rules for electric 2^L-pole gamma radiation and magnetic 2^L-pole gamma radiation which are also contained in the integrals of Eq. (1) for N_γ.

For $L=0$ there is no gamma radiation. Furthermore $M_0{}^{0}(m)$ is zero. However, $M_0{}^{0}(e)$ is not zero and gives rise to what may be termed electric monopole transitions.[22] We shall not concern ourselves with such transitions here since our primary interest lies in the conversion coefficient which implies the possibility of radiative transitions.

Reference to Eq. (1) shows that our expression for N_γ does not involve the nuclear charge density ρ_N. This quantity does appear, however, in our expression for $M_L{}^{M}(e)$ as can be seen from Eq. (9). In order to discuss the nuclear structure dependence of the conversion coefficient, we shall want to compare the structure dependence of N_γ with that of N_e. This comparison is facilitated if both quantities are expressed entirely in terms of $\mathbf{J}_N$. The required transformation can be carried out as follows. In each of the last two terms of Eq. (9) ρ_N is replaced by $(ik)^{-1}\,\mathrm{div}\mathbf{J}_N$ according to Eq. (4). The derivatives are then removed from $\mathbf{J}_N$ by permissible partial integrations, and finally the readily verified relations,[23]

$$\mathbf{A}_L{}^{M*}(\mathbf{x};3)=k^{-1}\,\mathrm{grad}(j_LY_L{}^{M*})\tag{10a}$$

and

$$\mathbf{B}_L{}^{M*}(\mathbf{x};3)=k^{-1}\,\mathrm{grad}(h_L{}^{(1)}Y_L{}^{M*}),\tag{10b}$$

are employed. It is then found that the sum, S, of the

[22] E. Church and J. Weneser, Phys. Rev. **103**, 1035 (1956).

[23] See p. 30 of I.

last two terms of Eq. (9) can be written in the form

$$S=-i\int_{VN} d^3x\, \mathbf{J}_N(\mathbf{x})\cdot\mathbf{B}_L{}^{M*}(\mathbf{x};3)\times\int_0^x d^3y\, \rho_e(\mathbf{y})Y_L{}^M(\theta',\phi')j_L(ky)$$
$$-i\int_{VN} d^3x\, \mathbf{J}_N(\mathbf{x})\cdot\mathbf{A}_L{}^{M*}(\mathbf{x};3)\times\int_x^\infty d^3y\, \rho_e(\mathbf{y})Y_L{}^M(\theta',\phi')h_L{}^{(1)}(ky), \quad (11)$$

which no longer involves ρ_N. This makes it possible to express the nuclear dependence of N_e in terms of $\mathbf{J}_N$ alone.

The next step in the calculation is the evaluation of the solid angle parts of the volume integrals which appear in Eqs. (1), (8), (9), and (11). This is accomplished by first expressing the vectors $\mathbf{A}_L{}^M(\mathbf{x};\sigma)$ and $\mathbf{B}_L{}^M(\mathbf{x},\sigma)$ in terms of radial functions and the spin-angle functions, $\mathbf{T}_{L,\lambda}{}^M$ of Eq. (5). The spin-angle functions are then expressed in terms of spherical harmonics through the easily verified relations,[24]

$$\mathbf{T}_{L,L}{}^M(\theta,\phi)=i\frac{\mathbf{x}\times\mathrm{grad}Y_L{}^M(\theta,\phi)}{[L(L+1)]^{\frac{1}{2}}},$$
$$\mathbf{T}_{L,L+1}{}^M(\theta,\phi)=\frac{|\mathbf{x}|\ \mathrm{grad}Y_L{}^M-(L+1)\hat{x}Y_L{}^M}{[(L+1)(2L+1)]^{\frac{1}{2}}}, \quad (12)$$
$$\mathbf{T}_{L,L-1}{}^M(\theta,\phi)=\frac{|\mathbf{x}|\ \mathrm{grad}Y_L{}^M+L\hat{x}Y_L{}^M}{[L(2L+1)]^{\frac{1}{2}}}.$$

In Eqs. (12), $\hat{x}=\mathbf{x}/|\mathbf{x}|$. By making use of Eqs. (12) and by introducing the Dirac wave functions of the electron, it is possible to carry out the solid angle integrations in the electron space explicitly.[25] It results therefrom that the M dependence, the μ dependence, and the μ' dependence of each of the integrals over the electron space is contained entirely in a multiplicative Clebsch-Gordan coefficient, which we shall designate by the notation $C(j'Lj;\mu'M)$ used in I. As is well known, this Clebsch-Gordan coefficient contains the angular momentum conservation laws for the electron.

The solid angle integrations cannot be carried out explicitly in the space of the nucleons until the function $\mathbf{J}_N(\mathbf{x})$ is specified. We shall proceed by making the very reasonable assumption that for all nuclear models an angular momentum conserving Clebsch-Gordan coefficient also arises from the integrals over the nucleon space. Specifically, we assume that there exist radial functions $\Phi_\lambda{}^L(x)$ such that

$$x^2\int\int d\theta d\phi(\sin\theta)\mathbf{J}_N(x,\theta,\phi)\cdot\mathbf{T}_{L,\lambda}{}^{M*}(\theta,\phi)=(-)^{1+L-\lambda+M}C(J_iLJ_f;M_i,-M)\Phi_\lambda{}^L(x). \quad (13)$$

The essential part of the assumption is that the functions $\Phi_\lambda{}^L(x)$ are independent of M, M_i, and M_f. These functions will of course depend on the many other quantum numbers necessary to specify the initial and final nuclear configurations. It is readily demonstrated that the existence of the functions $\Phi_\lambda{}^L(x)$ follows from the very reasonable assumption that the nuclear current operator is an irreducible tensor of rank one.[26]

Because of the properties of the Clebsch-Gordan coefficients, it is possible to carry out the sums over the projection quantum numbers which appear in Eqs. (1), (2), and (7). The 2^L-pole matrix elements $M_L{}^M(m)$ and $M_L{}^M(e)$ are first expressed in terms of reduced matrix elements $M_\kappa{}^L(m)$ and $M_\kappa{}^L(e)$ as follows:

$$M_L{}^M(m)=(-)^{1+M}C(J_iLJ_f;M_i,-M)\times C(j'Lj;\mu'M)M_\kappa{}^L(m),$$
$$M_L{}^M(e)=(-)^{M}C(J_iLJ_f;M_i,-M)\times C(j'LJ;\mu'M)M_\kappa{}^L(e). \quad (14)$$

For the clarity of the subsequent formulas, κ has been introduced as a subscript in order to emphasize the dependence of the reduced matrix elements on that parameter. Equations (14), (7), and (3) are then used to evaluate the sums over the projection quantum numbers in Eq. (2). In addition to the angular momentum selection rules, we also postulate that the usual parity selection rules for electromagnetic interactions are valid.

A completely analogous reduction of Eq. (1) can also be carried out without difficulty. One obtains the following formulas for N_γ and N_e.

$$N_\gamma=8\pi k\alpha\frac{2J_f+1}{2J_i+1}\sum_{L=1}^{\infty}\left\{\left|\int_0^R dx\,\Phi_L{}^L(x)j_L(kx)\right|^2+\frac{L+1}{2L+1}\left|\int_0^R dx\left[\Phi_{L-1}{}^L(x)j_{L-1}(kx)-\left(\frac{L}{L+1}\right)^{\frac{1}{2}}\Phi_{L+1}{}^L(x)j_{L+1}(kx)\right]\right|^2\right\}, \quad (15a)$$

$$N_e=2\pi\alpha^2\frac{2J_f+1}{2J_i+1}\sum_\kappa\sum_{L=0}^{\infty}\frac{2j+1}{2L+1}\times\{|M_\kappa{}^L(m)|^2+|M_\kappa{}^L(e)|^2\}. \quad (15b)$$

In Eqs. (15), R stands for the nuclear radius. The

[24] See, for example, M. E. Rose and L. C. Biedenharn, Oak Ridge National Laboratory Report ORNL-1779 (unpublished).
[25] See, for example, pp. 65–68 of I.
[26] See, for example, the discussion relating to Eq. (5.59) of M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley and Sons, Inc., New York, 1957).

summations in Eqs. (15a) and (15b) include only those values of L and κ for which the Clebsch-Gordan coefficients of Eqs. (15a) and (15b) do not vanish. This is ensured by the properties of $\Phi_\lambda{}^L$, $(\lambda=L, L\pm1)$. Thus $|J_f-J_i|\leqq L\leqq J_f+J_i$ and for each such L, $|L-j'|\leqq j\leqq L+j'$. In the expression for N_γ the first integral in the curly bracket refers to magnetic 2^L-pole gamma radiation and the second integral refers to electric 2^L-pole gamma radiation. For a given L one of these terms must be zero because they correspond to different nuclear parity changes. For the same reason, for any given L either $M_\kappa{}^L(m)$ or $M_\kappa{}^L(e)$ is zero.

In accordance with the discussion in the introduction, we now consider pure 2^L-pole transitions, i.e., transitions in which each of the sums over L in Eqs. (15) reduces to a single term. Such a transition can be either electric or magnetic. Following Rose, we use the symbol β_L to designate the conversion coefficient for magnetic radiation and the symbol α_L to designate that for electric radiation. The formulas for β_L and α_L are obtained from Eqs. (7) through (15). We shall consider the magnetic case first.

It is convenient at the outset to transform Eq. (8) in such a way that the comparison with the theory for a point nucleus can be readily made. This is done by extending the domain of integration over $\mathbf{y}$ in the second term of Eq. (8) so that the $\mathbf{y}$ integral is carried out over all space. A compensatory term is then added to the expression for $M_L{}^M(m)$. A study of the Dirac wave functions for small values of y shows that this manipulation is permissible in spite of the singularity of the Hankel functions in $\mathbf{B}_L{}^M(\mathbf{x};1)$. The new second term in $M_L{}^M(m)$ is then the product of the same nuclear integral which appears in N_γ and an electron integral which, in the limit of vanishing nuclear radius, tends toward that obtained by Rose for a point nucleus. The calculation of $M_\kappa{}^L(m)$ from Eqs. (14), (8), (5), and (12) is straightforward but tedious. We find

$$|M_\kappa{}^L(m)|^2=4\pi k^2(2\bar{l}+1)(2l'+1)(2j'+1)\times L^{-1}(L+1)^{-1}(\kappa+\kappa')^2C^2(\bar{l}l'L;00)\times W^2(l'j'\bar{l}j;\tfrac{1}{2}L)\,|N_\kappa{}^L(m,1)+N_\kappa{}^L(m,2)|^2, \quad (16)$$

where $W(l'j'\bar{l}j;\frac{1}{2}L)$ is a Racah coefficient in the notation of I and $\bar{l}(\kappa)\equiv l(-\kappa)$. In Eq. (16)

$$N_\kappa{}^L(m,1)=R_\kappa(m)I^L(m), \quad (17a)$$

and

$$N_\kappa{}^L(m,2)=\int_0^R dx\,\Phi_L{}^L(x)\psi_L(x). \quad (17b)$$

In Eqs. (17a) and (17b)

$$I^L(m)=\int_0^R dx\,\Phi_L{}^L(x)\,j_L(kx), \quad (18)$$

$$R_\kappa(m)=\int_0^\infty dy\,h_L{}^{(1)}(ky)(F_\kappa G_{\kappa'}+G_\kappa F_{\kappa'}), \quad (19)$$

and

$$\psi_L(x)=h_L{}^{(1)}(kx)\int_0^x dy\,j_L(ky)(F_\kappa G_{\kappa'}+G_\kappa F_{\kappa'})-j_L(kx)\int_0^x dy\,h_L{}^{(1)}(ky)(F_\kappa G_{\kappa'}+G_\kappa F_{\kappa'}). \quad (20)$$

The function $\Phi_L{}^L$ is defined by Eq. (13) and the functions $F_\kappa(y)$ and $G_\kappa(y)$ are equal, respectively, to y times the Dirac radial functions $f_\kappa(y)$ and $g_\kappa(y)$ which are defined in Eq. (5.8) of I.

The introduction in Eq. (16) of two terms $N_\kappa{}^L(m,1)$ and $N_\kappa{}^L(m,2)$ separates the conversion coefficient into a part in which only the static effect of nuclear structure plays a role and a part arising from the dynamic effect. This latter term also contains factors whose numerical values are influenced by the static effect. Only the static effect is involved in the case of $N_\kappa{}^L(m,1)$ because, according to Eq. (17a), $N_\kappa{}^L(m,1)$ is the product of the nuclear integral $I^L(m)$ for gamma-ray emission [see Eqs. (15)] and an integral $R_\kappa(m)$ which involves the nucleus only through its effect on the electron wave functions. The nuclear integral cancels out of this part of the conversion coefficient. The second term $N_\kappa{}^L(m,2)$, it will be apparent, represents the dynamic effect of nuclear structure.

An explicit formula for the conversion coefficient is readily obtained from Eqs. (15) through (18). The result is

$$\beta_L=\frac{\pi\alpha k(2j'+1)(2l'+1)}{L(L+1)(2L+1)}\sum_\kappa(2j+1)(2\bar{l}+1)\times C^2(\bar{l}l'L;00)W^2(j\bar{l}j'l';\tfrac{1}{2}L)(\kappa+\kappa')^2\times\left|R_\kappa(m)+\frac{N_\kappa{}^L(m,2)}{I^L(m)}\right|^2. \quad (21)$$

The corresponding formula for a point nucleus is given on page 69 of I. A comparison of this equation with Eq. (21) above shows that in the theory of a point nucleus the ratio $N_\kappa{}^L(m,2)/I^L(m)$ is set equal to zero. The only other difference between the two formulas is that in Eq. (21) the electron integral, $R_\kappa(m)$ (R_1+R_2 in Rose's notation), is to be evaluated for a nucleus of finite extent whereas for a point nucleus Coulomb wave functions are used. Further analysis of the structure-dependent term $N_\kappa{}^L(m,2)/I^L(m)$ will be carried out in Sec. IIIb.

We shall conclude this section by presenting the formulas which apply to electric 2^L-pole conversion. As can be seen from Eq. (15) the principal task is the evaluation of the reduced matrix element $M_\kappa{}^L(e)$ which can be obtained from Eqs. (9) and (11) in the same way that $M_\kappa{}^L(m)$ is obtained from Eq. (8). As in the magnetic case it is convenient to extend the y integrations for which $x\leqq y<\infty$ in such a way that $0\leqq y<\infty$.

Compensating terms are then added to the expression for $M_\kappa{}^L(e)$. We find

$$|M_\kappa{}^L(e)|^2=4\pi k^2(2l+1)(2l'+1)(2j'+1)\times L^{-1}(2L+1)^{-1}C^2(ll'L;00)W^2(jlj'l';\tfrac{1}{2}L)\times|\bar{N}_\kappa{}^L(e,1)+\bar{N}_\kappa{}^L(e,2)|^2, \quad (22)$$

where

$$\bar{N}_\kappa{}^L(e,1)=\int_0^R dx\,\Phi_{L-1}{}^L(x)j_{L-1}(kx)\int_0^\infty dy\,\eta_L{}^-(y)+(L/L+1)^{\frac{1}{2}}\int_0^R dx\,\Phi_{L+1}{}^L(x)j_{L+1}(kx)\times\int_0^\infty dy\,\eta_L{}^+(y), \quad (23)$$

and

$$\bar{N}_\kappa{}^L(e,2)=\int_0^R dx\,\Phi_{L-1}{}^L(x)\psi_L{}^-(x)+(L/L+1)^{\frac{1}{2}}\int_0^R dx\,\Phi_{L+1}{}^L(x)\psi_L{}^+(x). \quad (24)$$

The functions $\Phi_{L\pm1}{}^L(x)$ are defined by Eq. (13), and the functions $\psi_L{}^\pm(x)$ are defined by the relations

$$\psi_L{}^\pm(x)=h_{L\pm1}{}^{(1)}(kx)\int_0^x dy\,\xi_L{}^\pm(y)-j_{L\pm1}(kx)\int_0^x dy\,\eta_L{}^\pm(y), \quad (25)$$

where

$$\xi_L{}^-(y)=(\kappa'-\kappa)(F_\kappa G_{\kappa'}+G_\kappa F_{\kappa'})j_{L-1}-L(F_\kappa G_{\kappa'}-G_\kappa F_{\kappa'})j_{L-1}+L(F_\kappa F_{\kappa'}+G_\kappa G_{\kappa'})j_L, \quad (26)$$

$$\xi_L{}^+(y)=(\kappa'-\kappa)(F_\kappa G_{\kappa'}+G_\kappa F_{\kappa'})j_{L+1}+(L+1)(F_\kappa G_{\kappa'}-G_\kappa F_{\kappa'})j_{L+1}+(L+1)(F_\kappa F_{\kappa'}+G_\kappa G_{\kappa'})j_L. \quad (27)$$

The functions $\eta_L{}^\pm$ are obtained from the functions $\xi_L{}^\pm$ by replacing the Bessel functions by the corresponding Hankel functions.

As in the magnetic case, it is possible to isolate a term in which the only effect of nuclear structure is of the static type. In order to see this, however, it is necessary to employ a theorem which is proved in Appendix A and stated as Eq. (A.5). It follows as a particular case of this theorem that

$$\int_0^\infty dy\,\eta_L{}^+(y)=-\int_0^\infty dy\,\eta_L{}^-(y)+\frac{2L+1}{k}\lim_{a\to0}[(F_\kappa(a)G_{\kappa'}(a)-F_{\kappa'}(a)G_\kappa(a))h_L{}^{(1)}(ka)], \quad (28)$$

and consequently that

$$\bar{N}_\kappa{}^L(e,1)=R_\kappa(e)I^L(e)+\left(\frac{L}{L+1}\right)^{\frac{1}{2}}\int_0^R dx\,\Phi_{L+1}{}^L(x)\times j_{L+1}(kx)\frac{2L+1}{k}\lim_{a\to0}\{[F_\kappa(a)G_{\kappa'}(a)-F_{\kappa'}(a)G_\kappa(a)]h_L{}^{(1)}(ka)\}, \quad (29)$$

where

$$R_\kappa(e)=\int_0^\infty dy\,\eta_L{}^-(y)=(\kappa'-\kappa)(R_5+R_6)+L(R_6-R_5+R_3+R_4), \quad (30)$$

$$I^L(e)=\int_0^R dx\left\{\Phi_{L+1}{}^L(x)j_{L-1}(kx)-\left(\frac{L}{L+1}\right)^{\frac{1}{2}}\Phi_{L+1}{}^L(x)j_{L+1}(kx)\right\}. \quad (31)$$

The radial integrals R_3 to R_6 are defined on page 70 of **I.**

The important result exhibited by Eq. (29) is that the first term of $\bar{N}_\kappa{}^L(e,1)$ is the product of the same nuclear integral $I^L(e)$ which enters into the gamma-ray lifetime [Eqs. (15)], and an integral $R_\kappa(e)$ which involves only the static effect of nuclear structure. The sum, $\bar{N}_\kappa{}^L(e,1)+\bar{N}_\kappa{}^L(e,2)$, can thus be regrouped to form an equal sum $N_\kappa{}^L(e,1)+N_\kappa{}^L(e,2)$, in which the contribution of the first term to the conversion coefficient is independent of the dynamic effect of nuclear structure. It is sufficient to define $N_\kappa{}^L(e,1)$ and $N_\kappa{}^L(e,2)$ by the equations

$$\begin{aligned}N_\kappa{}^L(e,1)&=R_\kappa(e)I^L(e),\\ N_\kappa{}^L(e,2)&=\bar{N}_\kappa{}^L(e,2)+\bar{N}_\kappa{}^L(e,1)-N_\kappa{}^L(e,1).\end{aligned} \quad (32)$$

It will be seen in Sec. III that the contribution of $N_\kappa{}^L(e,2)$ depends on dynamic effects.

By making use of Eqs. (15), (22), and (32) it is easy to obtain a formula for the conversion coefficient. One finds

$$\alpha_L=\frac{\pi\alpha k(2j'+1)(2l'+1)}{L(L+1)(2L+1)}\sum_\kappa(2j+1)(2l+1)C^2(ll'L;00)\times W^2(jlj'l';\tfrac{1}{2}L)\left|R_\kappa(e)+\frac{N_\kappa{}^L(e,2)}{I^L(e)}\right|^2. \quad (33)$$

Although the formulas are more complicated, the result contained in Eqs. (24)–(33) is quite similar to that obtained in the magnetic case. The electric conversion coefficient, calculated for a finite nucleus, differs from that given on page 70 of I for a point nucleus in two respects. First of all, for a point nucleus, Coulomb wave functions are used in the radial integrals. Secondly, for a point nucleus, the structure dependent ratio $N_\kappa{}^L(e,2)/I^L(e)$ is set equal to zero. Further analysis of the structure dependence is carried out in Sec. IIIc.

III. FURTHER ANALYSIS OF THE STRUCTURE-DEPENDENT CONTRIBUTIONS TO THE CONVERSION COEFFICENTS

a. Introductory Remarks

A study of Eqs. (17)–(21) and (23)–(33) shows that the nucleus affects the internal conversion coefficients, (a) through the nuclear functions $\Phi_L{}^L$ and $\Phi_{L\pm1}{}^L$, and (b) through the electron wave functions which are affected by the nuclear charge distribution. The wave functions enter into the radial integrals $R_\kappa(e)$ and $R_\kappa(m)$ and into the electronic functions $\psi_L(x)$ and $\psi_L{}^\pm(x)$. As stated in the introduction, a basic assumption of the present work is that while a detailed nuclear model may be necessary for the evaluation of the nuclear functions $\Phi_L{}^L$ and $\Phi_{L\pm1}{}^L$, such a model is not necessary for an accurate evaluation of the radial integrals and the electronic functions. It is our aim therefore to choose a reasonable nuclear charge distribution and do once and for all as much of the calculation as can be done without specifying the nuclear functions. The problem, then, is basically one of storing information about the electronic aspects of the conversion process. We have chosen to do this by expanding the electronic functions in rather rapidly convergent power series whose coefficients can be tabulated as functions of atomic number Z, photon energy k, and multipole order L. The necessary information is stored in these coefficients.

The nuclear charge distribution adopted in the present work is that of a sphere of constant charge density and radius $R=1.2A^{\frac{1}{3}}\times10^{-13}$ cm. Sliv has shown[27] that various physically reasonably variations from this distribution lead to changes of less than a few percent in the radial integrals. The effect of reasonable changes in the nuclear charge distribution on the functions $\psi_L(x)$, etc. is larger but still less than about 10% in most cases. This point is discussed in greater detail in Sec. IVb.

In concluding the general remarks, let us mention that, as can be seen from Eqs. (21) and (33), it is necessary to know the radial integrals $R_\kappa(e)$ and $R_\kappa(m)$. At present, the only tables of these integrals which are available are those calculated by Rose[11] for the K shell, using a point nucleus and no screening.[28]

The use of an extended nuclear charge distribution leads to rather different K-shell radial integrals for large values of Z, and screening also changes the K-shell radial integrals by a few percent. For this reason it is necessary to apply correction factors to the tabulated radial integrals in order to obtain the ones which are required for the present work. The determination of these correction factors was carried out in part by methods which were of necessity mathematically crude and which may introduce errors of several percent in the final expressions. These correction factors will be discussed briefly in Sec. IVc where a more precise statement about the errors in the calculation will also be given.

No radial integrals have been tabulated for the L and M shells. For this reason we are obliged to restrict our analysis to the K shell.

b. Further Analysis of the Magnetic Conversion Coefficient

We now turn to the details of the calculation and start with the magnetic conversion coefficient. Our basic concern is the transformation of the structure-dependent ratios $N_\kappa{}^L(m,2)/I^L(m)$ which appear in Eq. (21). As the first step, all the functions which appear in the definition of $\psi_L(x)$ [Eq. (20)] are expanded in absolutely and uniformly convergent power series. In this way a uniformly convergent power series for $\psi_L(x)$ is obtained. It then follows from Eq. (17b) that if $\Phi_L{}^L(x)$ is bounded,

$$N_\kappa{}^L(m,2)=\sum_{n=0}^{\infty}a_n(\kappa)\int_0^R dx\,\Phi_L{}^L(x)(x/R)^{L+2+2n}, \quad (34)$$

where $a_n(\kappa)$ is the coefficient of $(x/R)^{L+2+2n}$ in the expansion of $\psi_L(x)$.

It is also possible to expand the Bessel function which appears in $I^L(m)$ [see Eq. (18)] in powers of x/R. In this way one immediately obtains the result that

$$I^L(m)=\sum_{n=0}^{\infty}\frac{(-)^n(kR)^{L+2n}}{2^n(2L+2n+1)!!n!}\times\int_0^R dx\,\Phi_L{}^L(x)\left(\frac{x}{R}\right)^{L+2n}, \quad (35)$$

where $(2l+1)!!\equiv1\times3\times5\times\cdots\times(2l+1)$.

Now, $(kR)^2$ is less than 2×10^{-3} for almost all transitions of interest in internal conversion. Hence under almost all physically conceivable conditions the first term in the expansion of $I^L(m)$ will be an excellent approximation. This is true, for example, even if

$$\int_0^R dx\,\Phi_L{}^L(x/R)^{L+2}\sim10^2\int_0^R dx\,\Phi_L{}^L(x/R)^L. \quad (36)$$

For this reason we shall henceforth set

$$I^L(m)=\frac{(kR)^L}{(2L+1)!!}\int_0^R dx\,\Phi_L{}^L(x)\left(\frac{x}{R}\right)^L. \quad (37)$$

Equations (34) and (37) show that $N_\kappa{}^L(m,2)/I^L(m)$ can be expressed as a sum of ratios of nuclear matrix elements. If this result is inserted into Eq. (21) it is

[27] See the introduction to the tables of internal conversion coefficients by Sliv and Band (reference 1).

[28] For purposes of obtaining information about nuclear structure, the restriction to the K shell is not serious except for transitions for which K conversion is energetically impossible. In order to treat these cases it is necessary to have an analysis of the L-shell conversion coefficients. Plans to carry out such a program are now being formulated by one of us (M. E. R.).

easy to show that β_L can be written in the form

$$\beta_L=\bar{C}_L\{|1-\rho_{-L}\Sigma_{-L}\exp(i\phi_{-L})|^2 +y_L|1-\rho_{L+1}\Sigma_{L+1}\exp(i\phi_{L+1})|^2\}, \quad (38)$$

where

$$\rho_\kappa\exp(i\phi_\kappa)=-\frac{a_0(\kappa)(2L+1)!!}{(kR)^L R_\kappa(m)}, \quad \rho_\kappa>0, \quad (39)$$

$$\Sigma_\kappa=\sum_{n=0}^{\infty}\frac{a_n(\kappa)}{a_0(\kappa)}R(L+2+2n;L), \quad (40)$$

and

$$y_L=\frac{L}{L+1}\frac{|R_{L+1}(m)|^2}{|R_{-L}(m)|^2}. \quad (41)$$

The nuclear ratios $R(a;b)$ are defined by the relation

$$R(a;b)=\frac{\int_0^R dx\,\Phi_L{}^L(x/R)^a}{\int_0^R dx\,\Phi_L{}^L(x/R)^b}=\frac{\int_{V_N}d^3x\,\mathbf{J}_N\cdot\mathbf{T}_{L,L}{}^{M*}(x/R)^a}{\int_{V_N}d^3x\,\mathbf{J}_N\cdot\mathbf{T}_{L,L}{}^{M*}(x/R)^b}. \quad (42)$$

The last equality in Eq. (42) follows from Eq. (13).

TABLE I. ρ_{-L}. Multiply each entry by 10 to the number in parentheses.

Z	k	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
96	5.0	7.76(−2)	4.57(−2)	3.58(−2)	3.12(−2)	2.87(−2)
	1.8	5.50(−2)	3.55(−2)	2.97(−2)	2.72(−2)	2.58(−2)
	0.5	4.46(−2)	3.21(−2)	2.95(−2)	2.91(−2)	2.97(−2)
78	5.0	3.96(−2)	2.42(−2)	1.94(−2)	1.69(−2)	1.57(−2)
	1.8	2.77(−2)	1.86(−2)	1.58(−2)	1.46(−2)	1.39(−2)
	0.5	2.21(−2)	1.64(−2)	1.50(−2)	1.45(−2)	1.44(−2)
64	5.0	2.13(−2)	1.41(−2)	1.15(−2)	1.03(−2)	9.48(−3)
	1.8	1.54(−2)	1.09(−2)	9.44(−3)	8.76(−3)	8.35(−3)
	0.5	1.23(−2)	9.58(−3)	8.80(−3)	8.50(−3)	8.35(−3)
30	5.0	3.34(−3)	2.68(−3)	2.39(−3)	2.20(−3)	2.09(−3)
	1.8	2.72(−3)	2.23(−3)	2.04(−3)	1.95(−3)	1.89(−3)
	0.5	2.31(−3)	2.00(−3)	1.91(−3)	1.84(−3)	1.81(−3)
	0.3	2.23(−3)	1.97(−3)	1.89(−3)	1.84(−3)	1.82(−3)

A formula for the parameter $\bar{C}_L$ is readily obtained from a comparison of Eqs. (38) and (21). $\bar{C}_L$ does not involve any nuclear matrix elements.

Equation (38) is suitable for application in connection with specific nuclear models since all of the quantities in it except the ratios $R(a;b)$ can be tabulated once and for all. Hence, given a nuclear model it is possible to evaluate the ratios $R(L+2+2n;L)$ of Eq. (40) and obtain a predicted value for β_L

For practical purposes Eq. (38) can be further simplified, for it turns out that y_L and the ratio ρ_{L+1}/ρ_{-L} are generally so small that for photon energies, k less than 2, and atomic numbers Z greater than 60 the terms containing ρ_{L+1} in Eq. (38) are less than 1.5% of those containing ρ_{-L} for almost any conceivable nuclear model.[29] For this reason we shall henceforth drop the terms in ρ_{L+1} entirely.

TABLE II. y_L. Multiply each entry by 10 to the number in parentheses.

Z	k	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
96	5.0	6.54(−2)	5.11(−2)	4.92(−2)	4.69(−2)	4.46(−2)
	1.8	1.72(−2)	1.69(−2)	1.63(−2)	1.51(−2)	1.38(−2)
	0.5	1.92(−3)	1.27(−3)	1.01(−3)	8.06(−4)	6.55(−4)
78	5.0	1.27(−1)	1.04(−1)	9.89(−2)	9.44(−2)	8.93(−2)
	1.8	4.38(−2)	4.19(−2)	4.02(−2)	3.76(−2)	3.48(−2)
	0.5	8.23(−3)	6.45(−3)	5.52(−3)	4.67(−3)	3.96(−3)
64	5.0	1.87(−1)	1.65(−1)	1.57(−1)	1.51(−1)	1.43(−1)
	1.8	7.92(−2)	7.70(−2)	7.42(−2)	7.03(−2)	6.53(−2)
	0.5	2.00(−2)	1.74(−2)	1.53(−2)	1.34(−2)	1.16(−2)
30	5.0	3.60(−1)	3.95(−1)	4.05(−1)	3.99(−1)	3.96(−1)
	1.8	2.46(−1)	2.69(−1)	2.74(−1)	2.72(−1)	2.68(−1)
	0.5	1.21(−1)	1.29(−1)	1.28(−1)	1.19(−1)	1.11(−1)
	0.3	8.27(−2)	8.41(−2)	8.03(−2)	7.32(−2)	6.69(−2)

Let $\beta_L(\Sigma)$ and $\beta_L(\Sigma^0)$ designate the conversion coefficients for two nuclear models for which the structure-dependent sums are Σ_κ and $\Sigma_\kappa{}^0$, respectively. It follows from Eq. (38) that

$$\frac{\beta_L(\Sigma)}{\beta_L(\Sigma^0)}=\frac{|1-\rho_{-L}\Sigma_{-L}\exp(i\phi_{-L})|^2+y_L}{|1-\rho_{-L}\Sigma_{-L}{}^0\exp(i\phi_{-L})|^2+y_L}. \quad (43)$$

Equation (43) is the final result and it is used as follows. The parameters ρ_{-L}, y_L, and ϕ_{-L} are tabulated as functions of Z, k, and L in Tables I, II, and III and hence their values for any given nuclear transition can be determined by interpolation and extrapolation. (For most practical purposes ϕ_{-L} can be set equal to zero.) Σ_{-L} and $\Sigma_{-L}{}^0$ are determined from Eqs. (40) and (42).

The purpose of introducing a reference model is to provide a calculated value of the conversion coefficient $\beta(\Sigma^0)$ without resorting to the complications which a realistic model would entail. Then $\beta_L(\Sigma)$ is the magnetic conversion coefficient for the realistic model. Accordingly, for Σ^0 one adopts a simple model which permits the evaluation of $\Sigma_{-L}{}^0$ in a trivial way. In this case the dynamic effects in the reference model will be relatively small and the comparison of the observations with the reference model will indicate the presence or absence of strong dynamic effects.

TABLE III. ϕ_{-1}. For given values of Z and k $\cos\phi_{-L}$ is a nondecreasing function of L.

Z	k	$\cos\phi_{-1}$	Z	k	$\cos\phi_{-1}$
96	5.0	0.986	64	5.0	0.968
	1.8	0.999		1.8	0.998
	0.5	1.000		0.5	1.000
78	5.0	0.973	30	5.0	0.982
	1.8	0.999		1.8	0.998
	0.5	1.000		0.5	1.000

[29] This point is discussed in greater detail by T. A. Green and M. E. Rose, Oak Ridge National Laboratory Report ORNL-2395 (unpublished), Sec. III. We shall henceforth designate this reference by the symbol II.

TABLE IV. $-a_1(-L)/a_0(-L)$. Multiply each entry by 10 to the number in parentheses. Linear interpolation in k is accurate to 1% or better.

Z	k	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
96	5.0	1.87(−1)	1.83(−1)	1.80(−1)	1.77(−1)	1.75(−1)
	0.5	1.80(−1)	1.79(−1)	1.76(−1)	1.74(−1)	1.73(−1)
78	5.0	1.48(−1)	1.49(−1)	1.49(−1)	1.48(−1)	1.48(−1)
	0.5	1.44(−1)	1.47(−1)	1.47(−1)	1.47(−1)	1.46(−1)
64	5.0	1.22(−1)	1.27(−1)	1.29(−1)	1.30(−1)	1.30(−1)
	0.5	1.20(−1)	1.26(−1)	1.28(−1)	1.29(−1)	1.30(−1)
30	5.0	7.84(−2)	8.91(−2)	9.45(−2)	9.76(−2)	9.95(−2)
	1.8	8.09(−2)	9.15(−2)	9.67(−2)	9.95(−2)	1.01(−1)

Two procedures have been adopted for the reference model calculations. These are: (1) *"No penetration"* model. Here one assumes all dynamic effects are absent; i.e., $\Sigma_\kappa{}^0=0$. This model has been used by Rose to obtain the coefficients referred to in reference 2. (2) *Surface current model.* This is the basis of Sliv's results.[1] One takes

$$\mathbf{J}_N=\mathbf{J}^0(\phi,\theta)\delta(x-R), \quad \mathbf{x}\cdot\mathbf{J}^0=0, \tag{44}$$

for both electric and magnetic transitions. For Sliv's model the pertinent result here is that $R(L+2+2n;L)=1$. As indicated, the difference in results provided by these two models is small. It is at most 13% for large Z and k and in the large majority of cases is of order 1 or 2%.

Of the coefficients $a_n(\kappa)/a_0(\kappa)$ which appear in Eq. (40), the first is unity and the second is tabulated in Table IV. It is expected that the use of just two terms in the sum of Eq. (40) will usually be a good approximation since it can be shown that the third coefficient which is positive, is less than 0.05 for Z less than 96, less than 0.03 for Z less than 80 and less than 0.02 for Z less than 60.

c. Further Analysis of the Electric Conversion Coefficient

The analogous formulas for electric conversion will now be developed. As in the magnetic case, the results are expressed in terms of sums, Σ_κ of nuclear ratios. However, as can be seen from Eqs. (24) and (31) there are two nuclear functions $\Phi_{L-1}{}^L$ and $\Phi_{L+1}{}^L$ which enter into the basic formulas. Consequently the results involve two types of nuclear ratios. These will be seen to be

$$S(a;b) \equiv \frac{\int_0^R dx\,\Phi_{L-1}{}^L(x/R)^a}{\int_0^R dx\,\Phi_{L-1}{}^L(x/R)^b} = \frac{\int_{V_N} d^3x\,\mathbf{J}_N\cdot\mathbf{T}_{L,L-1}{}^{M*}(x/R)^a}{\int_{V_N} d^3x\,\mathbf{J}_N\cdot\mathbf{T}_{L,L-1}{}^{M*}(x/R)^b}, \tag{45}$$

and

$$T(a;b) \equiv \frac{\int_0^R dx\,\Phi_{L+1}{}^L(x/R)^a}{\int_0^R dx\,\Phi_{L-1}{}^L(x/R)^b} = \frac{\int_{V_N} d^3x\,\mathbf{J}_N\cdot\mathbf{T}_{L,L+1}{}^{M*}(x/R)^a}{\int_{V_N} d^3x\,\mathbf{J}_N\cdot\mathbf{T}_{L,L-1}{}^{M*}(x/R)^b}. \tag{46}$$

It is simplest to begin the analysis by considering the nuclear integral $I^L(e)$. The Bessel functions which appear in Eq. (31) can be expanded in the usual uniformly convergent power series. Moreover, just as in the magnetic case, the smallness of $(kR)^2$ shows that it will almost certainly be a very good approximation to use only the term of lowest order in $(kR)^2$. Thus we set

$$I^L(e)=\frac{(kR)^{L-1}}{(2L-1)!!}\int_0^R dx\,\Phi_{L-1}{}^L\left(\frac{x}{R}\right)^{L-1}. \tag{47}$$

This is, of course, the "long-wavelength" approximation usually made in the theory of gamma-ray emission.

TABLE V. ω_{-L-1}. Multiply each entry by 10 to the number in parentheses.

Z	k	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
96	5.0	1.03(−2)	5.63(−3)	3.92(−3)	3.01(−3)	2.48(−3)
	1.8	4.37(−3)	2.59(−3)	1.88(−3)	1.53(−3)	1.32(−3)
	0.5	2.08(−3)	1.42(−3)	1.17(−3)	1.07(−3)	1.04(−3)
78	5.0	4.69(−3)	2.60(−3)	1.79(−3)	1.39(−3)	1.14(−3)
	1.8	1.87(−3)	1.11(−3)	8.06(−4)	6.48(−4)	5.52(−4)
	0.5	8.16(−4)	5.22(−4)	4.14(−4)	3.63(−4)	3.35(−4)
	0.3	6.39(−4)	4.34(−4)	3.70(−4)	3.45(−4)	3.33(−4)
64	5.0	2.51(−3)	1.42(−3)	9.91(−4)	7.63(−4)	6.24(−4)
	1.8	9.82(−4)	5.81(−4)	4.21(−4)	3.37(−4)	2.85(−4)
	0.5	3.99(−4)	2.52(−4)	1.96(−4)	1.68(−4)	1.52(−4)
	0.3	3.02(−4)	1.99(−4)	1.62(−4)	1.45(−4)	1.35(−4)
30	5.0	5.26(−4)	3.11(−4)	2.18(−4)	1.69(−4)	1.38(−4)
	1.8	1.95(−4)	1.17(−4)	8.47(−5)	6.69(−5)	5.58(−5)
	0.5	6.81(−5)	4.09(−5)	3.07(−5)	2.52(−5)	2.17(−5)
	0.3	4.53(−5)	2.86(−5)	2.20(−5)	1.84(−5)	1.62(−5)

TABLE VI. ω_L. Multiply each entry by 10 to the number in parentheses.

Z	k	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
96	5.0	6.79(−2)	6.81(−2)	6.72(−2)	6.70(−2)	6.81(−2)
	1.8	6.79(−2)	6.24(−2)	6.28(−2)	6.56(−2)	6.94(−2)
	0.5	6.27(−2)	5.73(−2)	6.30(−2)	7.37(−2)	8.73(−2)
78	5.0	2.61(−2)	2.61(−2)	2.62(−2)	2.64(−2)	2.68(−2)
	1.8	2.18(−2)	2.16(−2)	2.23(−2)	2.33(−2)	2.45(−2)
	0.5	1.66(−2)	1.63(−2)	1.81(−2)	2.05(−2)	2.33(−2)
	0.3	1.48(−2)	1.51(−2)	1.78(−2)	2.14(−2)	2.53(−2)
64	5.0	1.18(−2)	1.23(−2)	1.25(−2)	1.26(−2)	1.28(−2)
	1.8	9.22(−3)	9.55(−3)	9.97(−3)	1.05(−2)	1.09(−2)
	0.5	6.18(−3)	6.51(−3)	7.21(−3)	8.07(−3)	9.01(−3)
	0.3	5.24(−3)	5.65(−3)	6.54(−3)	7.61(−3)	8.78(−3)
30	5.0	1.42(−3)	1.59(−3)	1.65(−3)	1.69(−3)	1.73(−3)
	1.8	9.83(−4)	1.11(−3)	1.19(−3)	1.24(−3)	1.29(−3)
	0.5	5.15(−4)	5.94(−4)	6.55(−4)	7.13(−4)	7.65(−4)
	0.3	3.86(−4)	4.54(−4)	5.12(−4)	5.68(−4)	6.20(−4)

Now consider the model-dependent ratio

$$N_\kappa{}^L(e,2)/I^L(e)$$

which appears in Eq. (33). From Eqs. (23)–(33) and Eq. (47), it is not hard to see that the above ratio can be expressed in the form

$$\frac{N_\kappa{}^L(e,2)}{I^L(e)}=\sum_{n=0}^{\infty}[b_n(\kappa)S(L+1+2n;\,L-1)+c_n(\kappa)T(L+1+2n;\,L-1)], \quad (48)$$

where the coefficients $b_n(\kappa)$ and $c_n(\kappa)$ can be determined from the series expansions of the Bessel functions, the Hankel functions, and the electron wave functions.

The next step in the analysis is the use of Eq. (48) for the further reduction of Eq. (33). In treating the magnetic case at this point, we factored out the radial integrals $R_\kappa(m)$. In the electric case for $\kappa=L$,[30] this factorization is not appropriate because $R_L(e)$ vanishes at certain photon energies. For this reason we factor out $R_\kappa(e)$ only for $\kappa=-L-1$ and write Eq. (33) in the form

$$\alpha_L=D_L[\,|\,1-\omega_{-L-1}\Sigma_{-L-1}\exp(i\tau_{-L-1})\,|^2+|\,U_L-\omega_L\Sigma_L\exp(i\tau_L)\,|^2]. \quad (49)$$

The nuclear structure dependence of α_L is entirely contained in the sums Σ_κ, which are defined by the equation

$$\Sigma_\kappa=\sum_{n=0}^{\infty}[\bar{b}_n(\kappa)S(L+1+2n;\,L-1)+\bar{c}_n(\kappa)T(L+1+2n;\,L-1)], \quad (50)$$

where S and T are the matrix element ratios of Eqs. (45) and (46). Equations (49) and (50) involve the parameters D_L, U_L, ω_κ, τ_κ, $\bar{b}_n(\kappa)$, and $\bar{c}_n(\kappa)$. Formulas for these parameters are given in II and they will therefore not be reproduced here. The parameters ω_{-L-1}, ω_L, U_L, $\cos\tau_L$, and $\cos\tau_{-L-1}$ are tabulated in Tables V–VIII. The parameter $\bar{b}_0(-L-1)$ is unity by definition and the parameters $\bar{c}_0(-L-1)$, $\bar{b}_1(-L-1)$, and $\bar{c}_1(-L-1)$ are given in Tables IX–XI. Finally, $\bar{b}_0(L)$, $\bar{c}_0(L)$, $\bar{b}_1(L)$, and $\bar{c}_1(L)$ are given in Tables XII–XV.

From the information contained in Tables IX–XV, Σ_κ can be approximated by the first two terms in each of the sums in Eq. (50). This approximation is expected to be a good one in most cases for as is shown in II the third coefficient in each series is generally less than a few percent of the first.

TABLE VII. U_L. The number in parentheses is the estimated absolute value of the percent error in the tabulated value of U_L. Additional values which are useful for plotting U_L as a function of k or as a function of Z are given below the main group of values.

Z	k	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
96	5.0	0.51(13)	1.09(4)	1.52(4)	1.81(3)	2.05(3)
	1.8	0.28(35)	0.96(5)	1.37(4)	1.69(3)	1.97(3)
	1.0	0.14(50)	0.77(6)	1.15(4)	1.48(4)	1.81(3)
	0.5	0.12(50)	0.49(7)	0.86(5)	1.25(4)	1.65(4)
78	5.0	0.58(7)	0.87(5)	1.13(4)	1.33(3)	1.48(3)
	1.8	0.25(10)	0.62(6)	0.86(4)	1.05(4)	1.21(4)
	1.0	0.05(40)	0.39(6)	0.59(4)	0.77(4)	0.94(4)
	0.5	0.22(7)	0.09(10)	0.28(5)	0.47(3)	0.67(3)
	0.3	0.36(8)	0.10(7)	0.09(5)	0.28(3)	0.48(3)
64	5.0	0.60(5)	0.77(4)	0.94(3)	1.07(3)	1.17(3)
	1.8	0.24(5)	0.46(4)	0.60(3)	0.72(3)	0.82(4)
	1.0	0.02(70)	0.20(3)	0.33(3)	0.43(3)	0.53(4)
	0.5	0.23(7)	0.09(7)	0.02(20)	0.12(3)	0.24(3)
	0.3	0.37(4)	0.27(4)	0.17(4)	0.05(5)	0.06(5)
30	5.0	0.72(3)	0.67(3)	0.69(3)	0.71(3)	0.73(3)
	1.8	0.26(2)	0.24(2)	0.25(2)	0.27(2)	0.28(2)
	1.0	0.00	0.04(2)	0.04(2)	0.02(2)	0.01(2)
	0.5	0.26(2)	0.32(2)	0.32(2)	0.32(2)	0.30(2)
	0.3	0.41(2)	0.47(2)	0.48(2)	0.48(2)	0.46(2)

Z	k	L	U_L	Z	k	L	U_L
96	0.3	1	0.32(40)	64	0.65	2	<0.02
85	0.3	2	<0.03	64	0.45	3	<0.02
78	0.8	1	0.05(40)	64	0.35	4	<0.01
78	0.4	2	<0.02	64	0.23	5	<0.02
74	0.3	3	<0.02	64	0.15	5	0.12(30)
72	0.5	2	<0.03	60	0.5	4	<0.02
68	0.3	4	<0.02	60	0.3	5	<0.02
64	0.9	1	0.02(70)	50	0.5	5	<0.02

[30] The allowed values of κ are L and $-L-1$.

TABLE VIII. $\cos\tau_L$ and $\cos\tau_{-L-1}$ for $L=1$ and $L=2$. Wherever the estimated error in the tabulated values is particularly large it is stated. For $L\geqslant 3$, $\cos\tau_{-L-1}$ is practically unity. $\cos\tau_L$ changes value rapidly from -1 to $+1$ in the neighborhood of $k=\bar{k}$. Values, $\bar{k}$, of k for which $\cos\tau_L=0$ are also listed. (The estimated error in $\bar{k}$ is ± 0.05 in all cases.)

Z	k	$\cos\tau_1$	$\cos\tau_2$	$\cos\tau_{-2}$	$\cos\tau_{-3}$
96	5.0	$0.56_{-0.14}{}^{+0.14}$	0.997	0.92	0.97
	1.8	$0.92_{-0.11}{}^{+0.05}$	0.998	0.96	0.99
	0.5	$-0.88_{-0.07}{}^{+0.60}$	0.997	0.99	1.00
78	5.0	$0.66_{-0.05}{}^{+0.04}$	0.994	0.94	0.98
	1.8	$0.92_{-0.02}{}^{+0.02}$	0.998	0.98	0.99
	0.5	−0.98	0.960	0.99	1.00
	0.3		−0.991		
64	5.0	$0.74_{-0.02}{}^{+0.02}$	0.992	0.95	0.98
	1.8	$0.93_{-0.01}{}^{+0.01}$	0.998	0.98	0.99
	0.5	−0.99	−0.975	0.99	1.00
	0.3		−0.999		
30	5.0	0.94	0.998	0.99	0.99
	1.8	0.98	0.999	1.00	1.00
	0.5	−0.99	−1.000		
	0.3	−1.0	−1.000		

Z	L	$\bar{k}$	Z	L	$\bar{k}$
96	1	0.65	64	5	0.22
78	1	0.90	30	1	1.00
78	2	0.40	30	2	1.10
64	1	0.95	30	3	1.10
64	2	0.65	30	4	1.05
64	3	0.45	30	5	1.00
64	4	0.35			

TABLE IX. $-\bar{c}_0(-L-1)$. This quantity is independent of Z and k.

$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
0.566	0.350	0.257	0.203	0.169

As in the magnetic case it is convenient to make use of a reference model Σ^0 with which to compare any other given nuclear model Σ. For such a pair of models it follows from Eq. (49) that

$$\frac{\alpha_L(\Sigma)}{\alpha_L(\Sigma^0)}=\frac{|1-\omega_{-L-1}\Sigma_{-L-1}\exp(i\tau_{-L-1})|^2+|U_L-\omega_L\Sigma_L\exp(i\tau_L)|^2}{|1-\omega_{-L-1}\Sigma_{-L-1}{}^0\exp(i\tau_{-L-1})|^2+|U_L-\omega_L\Sigma_L{}^0\exp(i\tau_L)|^2}, \tag{51}$$

where Σ_κ and $\Sigma_\kappa{}^0$ are the appropriately calculated sums of matrix element ratios.

If Sliv's surface current model is used for reference the values of $\Sigma_\kappa{}^0$ can be obtained from Eqs. (44), (12), (45), and (46) from which it follows that

$$S(a;b)\equiv 1,$$

$$T(a;b)\equiv(L/L+1)^{\frac{1}{2}}. \quad \text{(Surface current model.)} \tag{52a}$$

Of course, for the "no penetration" model,

$$S(a;b)=T(a;b)=0. \tag{52b}$$

Equations (50), (51), and (52), together with Tables V–XV and the tables[1,2] of conversion coefficients, thus afford a straightforward means of calculating the predicted electric conversion coefficient for any nuclear model. The conclusions which can be drawn from the numerical results are discussed in Secs. Vb and Vc. A low-energy approximation which simplifies Σ_κ is presented in Sec. IIId.

d. A Low-Energy Approximation in which the Σ_κ for Electric Conversion are Expressed in Terms of the Nuclear Charge Density Rather than the Nuclear Current Density

It is well known that in the usual long-wavelength limit the matrix element for electric 2^L-pole gamma radiation can be expressed in terms of the nuclear charge density. We shall now show that for sufficiently low photon energies an analogous result can be obtained for the electron ejection matrix elements which appear in the Σ_κ of Eq. (50).

In Eq. (50) let S and T be expressed as the appropriate ratios of integrals according to Eqs. (45) and (46). In these integrals the $\mathbf{T}_{L,L\pm1}{}^M$ can be expressed in terms of $Y_L{}^M$ according to Eq. (12) and for each n the terms from S and T can be combined. It is then easy to deduce that

$$\Sigma_\kappa=\left\{\int d^3x\,\mathbf{J}_N\cdot\mathrm{grad}\left[Y_L{}^{M*}\left(\frac{x}{R}\right)^L\right]\right\}^{-1}\times\sum_{n=0}^{\infty}\int d^3x\,\mathbf{J}_N\cdot\left\{\left[\bar{b}_n+\left(\frac{L}{L+1}\right)^{\frac{1}{2}}\bar{c}_n\right]\times\mathrm{grad}\left[Y_L{}^{M*}\left(\frac{x}{R}\right)^{L+2+2n}\right]-Y_L{}^{M*}\frac{\hat{x}}{R}\left[(2n+2)\bar{b}_n+(2L+2n+3)\left(\frac{L}{L+1}\right)^{\frac{1}{2}}\bar{c}_n\right]\right\}. \tag{53}$$

Now let $\delta_n(\kappa)$ be defined in such a way that

$$\bar{c}_n=-\left(\frac{L+1}{L}\right)^{\frac{1}{2}}\frac{2n+2}{2L+2n+3}\bar{b}_n[1+\delta_n(\kappa)]. \tag{54}$$

It then follows immediately as the result of integration by parts and the use of Eq. (4) that Σ_κ can be expressed in the form

$$\Sigma_\kappa=\sum_{n=0}^{\infty}\frac{2L+1}{2L+2n+3}\bar{b}_n\frac{\int d^3x\,\rho_N Y_L{}^{M*}(x/R)^{L+2+2n}}{\int d^3x\,\rho_N Y_L{}^{M*}(x/R)^L}+\Delta\Sigma_\kappa, \tag{55}$$

where $\Delta\Sigma_\kappa$ is the part of Σ_κ in which the terms are proportional to $\delta_n(\kappa)$.

A study of the formulas for $\bar{c}_n(\kappa)$ and $\bar{b}_n(\kappa)$ [Eqs. (81), (82), (84), and (85) of II] shows that as kR approaches zero, $\delta_n(\kappa)$ approaches zero and hence $\Delta\Sigma_\kappa$ approaches zero. This clearly suggests that for small kR it will be a good approximation to set $\Delta\Sigma_\kappa$ equal to zero in Eq. (55).

For $\kappa=-L-1$ it can be shown that $\delta_0(\kappa)\equiv 0$. For $\kappa=L$, $\delta_0(\kappa)$ decreases with decreasing k, increasing Z

TABLE X. $-\bar{b}_1(-L-1)$. Linear interpolation in k is accurate to 1% or better. Multiply each entry by 10 to the number in parentheses.

Z	k	$L=1$	$L=2$	$L=3$	$L=5$
96	5.0	1.29(−1)	1.19(−1)	1.14(−1)	1.08(−1)
	0.5	1.16(−1)	1.10(−1)	1.06(−1)	1.03(−1)
78	5.0	8.72(−2)	8.03(−2)	7.64(−2)	7.23(−2)
	0.5	7.75(−2)	7.33(−2)	7.10(−2)	6.85(−2)
64	5.0	6.02(−2)	5.52(−2)	5.24(−2)	4.94(−2)
	0.5	5.27(−2)	4.99(−2)	4.83(−2)	4.65(−2)
30	5.0	1.49(−2)	1.34(−2)	1.26(−2)	1.17(−2)
	1.8	1.29(−2)	1.20(−2)	1.15(−2)	1.09(−2)

TABLE XI. $\bar{c}_1(-L-1)$. Linear interpolation in k is accurate to 1% or better. Multiply each entry by 10 to the number in parentheses.

Z	k	$L=1$	$L=2$	$L=3$	$L=5$
96	5.0	9.54(−2)	6.01(−2)	4.47(−2)	3.00(−2)
	0.5	9.30(−2)	5.94(−2)	4.44(−2)	2.99(−2)
78	5.0	6.38(−2)	4.01(−2)	2.98(−2)	2.00(−2)
	0.5	6.20(−2)	3.96(−2)	2.96(−2)	1.99(−2)
64	5.0	4.35(−2)	2.73(−2)	2.03(−2)	1.36(−2)
	0.5	4.15(−2)	2.69(−2)	2.01(−2)	1.35(−2)
30	5.0	1.01(−2)	6.27(−3)	4.64(−3)	3.09(−3)
	1.8	9.80(−3)	6.19(−3)	4.60(−3)	3.08(−3)

TABLE XII. $\bar{b}_0(L)$. Linear interpolation in k is exact. Multiply each entry by 10 to the number in parentheses.

Z	k	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
96	5.0	1.49(−1)	4.62(−1)	6.09(−1)	6.93(−1)	7.48(−1)
	1.8	2.97(−2)	4.08(−1)	5.75(−1)	6.69(−1)	7.29(−1)
	0.5	−1.86(−2)	3.86(−1)	5.61(−1)	6.59(−1)	7.21(−1)
78	5.0	1.70(−1)	4.70(−1)	6.14(−1)	6.97(−1)	7.51(−1)
	1.8	3.39(−2)	4.09(−1)	5.75(−1)	6.69(−1)	7.29(−1)
	0.5	−2.12(−2)	3.84(−1)	5.60(−1)	6.57(−1)	7.20(−1)
64	5.0	1.92(−1)	4.79(−1)	6.20(−1)	7.01(−1)	7.54(−1)
	1.8	3.83(−2)	4.10(−1)	5.76(−1)	6.69(−1)	7.29(−1)
	0.5	−2.40(−2)	3.82(−1)	5.58(−1)	6.56(−1)	7.18(−1)
30	5.0	3.08(−1)	5.27(−1)	6.49(−1)	7.21(−1)	7.69(−1)
	1.8	6.16(−2)	4.16(−1)	5.78(−1)	6.70(−1)	7.29(−1)

and increasing L. For $L=2$ and $Z>60$ we find that $\delta_0(\kappa)$ is of the order of -0.55 for $k=5$, -0.20 for $k=1.8$, -0.07 for $k=0.5$ and -0.04 for $k=0.3$. For $L=4$ and $Z>60$ we find that $\delta_0(\kappa)$ is of the order of -0.07 for $k=1.8$, -0.02 for $k=0.5$, and -0.01 for $k=0.3$. For $L=1$, $\delta_0(\kappa)$ is not very small compared to unity because the normally dominant terms in αZ happen to cancel out of the coefficients $\bar{b}_n(1)$ and $\bar{c}_n(1)$ with the result that the dominant term is of the order of R. Compared to R, kR is not usually negligible even at threshold.

We have also evaluated $\delta_1(\kappa)$ for both values of κ and shown that $|\delta_1(\kappa)|<0.040$ for $L\geq 2$, $k<0.5$, and $Z>60$. For $L=1$, $\delta_1(\kappa)$ is usually not very small compared to unity.

From what has been said above it is to be expected that for $L\geq 2$ if two terms of the original series (50) are a good approximation to Σ_κ, two terms of the series in Eq. (53) will be a good approximation for low energies, $\Delta\Sigma_\kappa$ being set equal to zero. Evidently the argument given above is not mathematically conclusive since it is conceivable that the sum in Eq. (45) vanishes, making $\Delta\Sigma_\kappa$ dominant. However, in any doubtful case one can always check by calculating $\Delta\Sigma_\kappa$ explicitly. For many, perhaps most, transitions of experimental interest the use of Eq. (55) will be adequate for $L\geq 2$. Its use in the case of $E1$ transitions is harder to justify since, for $L=1$, $\delta_0(L)$ is of the order of 0.5 or larger, even at threshold, for most values of Z which are large enough to be of practical interest.

TABLE XIII. $-\bar{c}_0(L)$. Linear interpolation in k is exact. Multiply each entry by 10 to the number in parentheses.

Z	k	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
96	5.0	−1.52(−1)	8.02(−2)	1.15(−1)	1.16(−1)	1.09(−1)
	1.8	−6.83(−2)	1.14(−1)	1.33(−1)	1.27(−1)	1.17(−1)
	0.5	−3.42(−2)	1.27(−1)	1.40(−1)	1.31(−1)	1.20(−1)
78	5.0	−1.74(−1)	7.18(−2)	1.10(−1)	1.13(−1)	1.07(−1)
	1.8	−7.79(−2)	1.10(−1)	1.31(−1)	1.26(−1)	1.16(−1)
	0.5	−3.09(−2)	1.25(−1)	1.39(−1)	1.31(−1)	1.19(−1)
64	5.0	−1.97(−1)	6.29(−2)	1.05(−1)	1.10(−1)	1.05(−1)
	1.8	−8.81(−2)	1.06(−1)	1.28(−1)	1.24(−1)	1.15(−1)
	0.5	−4.41(−2)	1.23(−1)	1.38(−1)	1.30(−1)	1.19(−1)
30	5.0	−3.16(−1)	1.61(−2)	8.00(−2)	9.38(−2)	9.40(−2)
	1.8	−1.42(−1)	8.50(−2)	1.17(−1)	1.17(−1)	1.10(−1)

TABLE XIV. $-\bar{b}_1(L)$. Where only two entries are tabulated for a given Z and L, linear interpolation in k is accurate to better than 1%. Multiply each entry by 10 to the number in parentheses.

Z	k	$L=1$	$L=2$	$L=3$	$L=5$
96	5.0	2.46(−2)	7.31(−2)	9.63(−2)	1.18(−1)
	1.8	4.65(−3)			
	0.5	−2.84(−3)	5.95(−2)	8.69(−2)	1.12(−1)
78	5.0	1.89(−2)	5.94(−2)	7.98(−2)	9.97(−2)
	1.8	3.54(−3)			
	0.5	−2.15(−3)	4.90(−2)	7.26(−2)	9.51(−2)
64	5.0	1.47(−2)	5.04(−2)	6.90(−2)	8.78(−2)
	1.8	2.72(−3)			
	0.5	−1.65(−3)	4.23(−2)	6.35(−2)	8.43(−2)
30	5.0	5.69(−3)	3.46(−2)	5.08(−2)	6.80(−2)
	1.8	1.02(−3)	3.24(−2)	4.92(−2)	6.68(−2)
	0.3	−8.44(−4)			

IV. DISCUSSION OF THE POWER SERIES EXPANSIONS AND THE CORRECTION FACTORS FOR THE COULOMB RADIAL INTEGRALS

a. Discussion of the Power Series Expansions

As explained in Sec. IIIa, the information about the electronic aspects of the conversion process is stored in the coefficients of the power series expansions of the electron wave functions for the initial and final states.

In the present application, values of the radial variable, x, which are less than R are of primary interest and for these values the Dirac equation determines the coefficients in the power series expansions of F_κ and G_κ to within a single multiplicative normalization factor. It is convenient to express this normalization factor in terms of the value of the "large" radial function at the nuclear radius R.[31]

For the initial state the determination of the large radial function at the nuclear radius can be carried out using the method of Brysk and Rose[32] which includes the effect of electron screening on the normalization. We departed from this method only in that we determined the energy eigenvalue by matching inside and

TABLE XV. $\bar{c}_1(L)$. Where only two entries are tabulated for a given Z and L, linear interpolation in k is accurate to better than 1%. Multiply each entry by 10 to the number in parentheses.

Z	k	$L=1$	$L=2$	$L=3$	$L=5$
96	5.0	−2.67(−2)	2.43(−2)	3.25(−2)	3.12(−2)
	1.8	−1.21(−2)			
	0.5	−6.59(−3)	3.10(−2)	3.57(−2)	3.24(−2)
78	5.0	−2.05(−2)	2.05(−2)	2.74(−2)	2.66(−2)
	1.8	−9.20(−3)			
	0.5	−5.00(−3)	2.56(−2)	2.99(−2)	2.76(−2)
64	5.0	−1.59(−2)	1.82(−2)	2.43(−2)	2.37(−2)
	1.8	−7.09(−3)			
	0.5	−3.84(−3)	2.22(−2)	2.62(−2)	2.44(−2)
30	5.0	−6.16(−3)	1.53(−2)	1.95(−2)	1.91(−2)
	1.8	−2.64(−3)	1.65(−2)	2.01(−2)	1.93(−2)
	0.3	−1.23(−3)			

[31] For $\kappa<0$, G_κ is the large function; for $\kappa>0$, F_κ is the large function.

[32] H. Brysk and M. E. Rose, Oak Ridge National Laboratory Report ORNL-1830, 1956 (unpublished).

outside solutions at the nuclear radius instead of using perturbation theory.

For the final (continuum) state, the value of the large radial function was determined by the method of Rose and Holmes.[33] This method does not allow for the effect of screening on the normalization factor and for this reason the normalization-dependent factors ω_κ and ρ_κ may be in error by several percent at low energies.

In the application of the two methods described above, the many necessary approximations were introduced in such a way that (without the aforementioned screening error) the product of the initial and final state normalization factors is accurate to better than $\pm 2\%$. A detailed discussion of the power series expansions, the normalization factors, and the errors in the coefficients is given in Sec. IV of II. This section also contains formulas for the coefficients $a_n(\kappa)$, $\bar{b}_n(\kappa)$, $\bar{c}_n(\kappa)$ and other quantities of importance in the theory.

b. Dependence of the Results on the Choice of Nuclear Radius

It may very well turn out that the experimental evidence from electron scattering, etc., will indicate that the electrostatic potential is better approximated within the framework of a constant charge density model by the use of a different radius R than that used here. For this reason it is of some importance to study the sensitivity of the parameters of the theory to 5 or 10% variations in R. The results may be summarized as follows.

With one exception the coefficients, $a_n(\kappa)$, $\bar{b}_n(\kappa)$, and $\bar{c}_n(\kappa)$ are very insensitive to changes in R, a 10% change in R generally leading to changes in the coefficients of less than 1%. The exception occurs in the case of the coefficients $\bar{b}_n(1)$ and $\bar{c}_n(1)$ for $E1$ transitions. In this case the normally dominant term in αZ disappears through cancellation and as a result the coefficients are proportional to R.

The coefficients ω_κ and ρ_κ are more sensitive to variations in R since they involve the electron normalization factors. It is not hard to show that these coefficients depend on R primarily through the factor

$$R^{\gamma_1+\gamma_\kappa-L}, \tag{56}$$

where $\gamma_n=(n^2-\alpha^2Z^2)^{\frac{1}{2}}$. For $|\kappa|=L$ the exponent in Eq. (56) is of the order of 0.5 for large Z and small L and of the order of 1 for small Z and large L. For $|\kappa|=L+1$ the exponent varies with L and Z between about 1.5 and 2.

The above information can be used to correct all the parameters of Eqs. (43) and (51) except the phase angles, y_L and U_L. However, as was previously stated in Sec. IIIa, these quantities are insensitive to changes in the nuclear radius.

c. Correction of the Coulomb Radial Integrals

The numerical evaluation of ρ_κ, ω_κ, y_L, U_L, ϕ_κ, and τ_κ require a knowledge of the electron radial integrals $R_\kappa(m)$ and $R_\kappa(e)$, calculated for a nucleus of nonzero extent and including the effects of electron screening. However, as was mentioned in Sec. IIIa, the only available radial integrals are the integrals $R_\kappa{}^c(e)$ and $R_\kappa{}^c(m)$ which were calculated for a point nucleus and neglecting electron screening. It is therefore necessary to apply correction factors to the Coulomb radial integrals in order to get the ones which are needed for the present calculation. Briefly, these correction factors are obtained as follows.

It is first demonstrated that the electron wave functions for an extended nucleus differ appreciably from those for a point nucleus only for values of the radial variable x less than 5 or 10 nuclear radii. It is then noted [see Eqs. (19) and (26)] that the real and imaginary parts of $R_\kappa(e)$ and $R_\kappa(m)$ arise from the spherical Hankel functions, $h_L{}^{(1)}=j_L+in_L$, which multiply various combinations of electron radial functions. Since j_L vanishes at the origin along with the electron wave functions it is clear that contributions from near the origin do add much to the real parts of the radial integrals unless (as is actually unusual) there is extensive cancellation over the rest of the region of integration. For this reason it is a good approximation to equate the real parts of R_κ and $R_\kappa{}^c$ in both magnetic and electric cases.

This result is used as follows. For each case let modulus correction factors ξ_κ be introduced in such a way that $|R_\kappa|=\xi_\kappa|R_\kappa{}^c|$. Also let χ_κ and $\chi_\kappa{}^c$ be the arguments of R_κ and $R_\kappa{}^c$. It then follows from the result of the previous paragraph that

$$\cos\chi_\kappa=(\xi_\kappa)^{-1}\cos\chi_\kappa{}^c. \tag{57}$$

Since the $\chi_\kappa{}^c$ are known, Eq. (57) determines the χ_κ in terms of the ξ_κ. Thus for the magnetic case there are two real correction factors $\xi_{-L}(m)$ and $\xi_{L+1}(m)$ which need to be determined. For the electric case the factors are $\xi_L(e)$ and $\xi_{-L-1}(e)$.

In either case there is one relation between the coefficients from which one of them can be calculated if the other is known. This relation is obtained from the expression for the ratio of Sliv's K-shell conversion coefficients which involve the R_κ to the unscreened K-shell conversion coefficients of Rose *et al.*[34] which involve the $R_\kappa{}^c$. Thus the problem is reduced to evaluating one of the correction factors.

In order to obtain estimates of one of the correction factors the following technique can be employed. For each radial integral the interval of integration $(0,\infty)$ is divided into the subintervals $(0,\bar{R})$ and $(\bar{R},\infty)$, where $\bar{R}\sim 10R$. It is then assumed that the integrals over the region $(\bar{R},\infty)$ are essentially the same for the Coulomb

[33] M. E. Rose and D. K. Holmes, Oak Ridge National Laboratory Report ORNL-1022, 1951 (unpublished).

[34] Rose, Goertzel, Spinrad, Harr, and Strong, Phys. Rev. **83**, 79 (1951).

wave functions as for the finite-size wave functions. In the interval $(0,\bar{R})$ approximate representations of the functions can be used to do the Coulomb and finite-size integrals analytically. In this way estimates of the ξ_κ can be obtained whose accuracy depends mainly on the accuracy with which the integrations over $(0,\bar{R})$ are carried out.

One important result emerges immediately. It is readily seen that for $|\kappa|=L$ the integrands of $R_\kappa{}^c(e)$ and $R_\kappa{}^c(m)$ are singular at the origin and that those of $R_\kappa(e)$ and $R_\kappa(m)$ are not. For this reason the improper but convergent integrals $R_\kappa{}^c(e)$ and $R_\kappa{}^c(m)$, which strongly emphasize the origin, are quite different from the proper integrals $R_\kappa(e)$ and $R_\kappa(m)$ in which the origin does not play nearly as important a role. Thus for $|\kappa|=L$ it is not unusual to find correction factors which differ from unity by as much as 40 or 50% for very large values of Z. For $|\kappa|=L+1$ the situation is rather different because in this case the integrands of all the integrals tend toward zero near the origin. Thus the role played by the origin is less marked and the correction factors differ from unity by a few percent or less.

The methods described above were used to obtain the necessary correction factors and except for $\xi_L(e)$ the factors are accurate to within a few percent. Because of the vanishing of $R_L{}^c(e)$ at certain energies it turned out that near these energies very large errors in the determination of ξ_L were unavoidable. These are reflected in the large errors in U_L and $\cos\tau_1$ which are given in Tables VII and VIII, respectively.

A much more detailed exposition of the material presented in this section is given in Sec. V of II.

V. DISCUSSION AND CONCLUDING REMARKS

a. Magnetic Conversion Coefficients

Although Eq. (43) should be used for quantitative evaluations of the nuclear structure effect, a simpler approximate form of this equation will usually provide reasonably accurate estimates. Let us consider transitions for which $60<Z<96$ and $k\leq 0.5$. For such transitions it follows from Tables I–IV that $y_L\leq 0.02$, $\cos\phi_L=1.000$, $0.0084\leq\rho_{-L}\leq 0.045$, and for Sliv's surface current model $0.82\leq\Sigma_{-L}{}^0\leq 0.88$. Furthermore, it is readily demonstrated that the matrix element ratios $R(L+2+2n;L)$ are real[35] so that Σ_{-L} is a real number. Hence, provided the nuclear structure effect is not too large (say 30% or less), Eq. (43) can be reduced to

$$\beta_L(\Sigma)/\beta_L(\Sigma^0)=1-2\rho_{-L}(\Sigma_{-L}-\Sigma_{-L}{}^0) \tag{58}$$

with an accuracy of 5% or better.

From Eq. (58) and Table I it is readily seen that for an experimentally observable nuclear structure effect (i.e., one which is of the order of 10% or larger) it is necessary that the value of Σ_{-L} be in the neighborhood of 5 or 10. Moreover, from the discussion given in the introduction, it is clear that sufficiently large values of Σ_{-L} actually can occur.[4,9] It is also seen that for a given value of Σ_{-L} large structure effects are usually favored to a certain extent by large values of Z and k and small values of L.

The basic technique employed in our analysis is essentially identical with that used by Church and Weneser.[4] A comparison of our results with theirs is most readily carried out by expressing $\beta_L(\Sigma)/\beta_L(\Sigma^0)$ in the form

$$\frac{\beta_L(\Sigma)}{\beta_L(\Sigma^0)}=[1-C_L(\Sigma_{-L}-\Sigma_{-L}{}^0)]^2+\frac{y_L C_L{}^2(\Sigma_{-L}-\Sigma_{-L}{}^0)^2}{(1-\Sigma_{-L}{}^0\rho_{-L}\cos\phi_{-L})^2}, \tag{59}$$

where

$$C_L=\frac{\rho_{-L}\cos\phi_{-L}(1-\Sigma_{-L}{}^0\rho_{-L}\cos\phi_{-L})}{(1-\Sigma_{-L}{}^0\rho_{-L}\cos\phi_{-L})^2+y_L}. \tag{60}$$

Equation (59) follows from Eq. (43) if the very small terms of the order of $\rho_{-L}{}^2\Sigma_{-L}{}^2\sin^2\phi_{-L}$ are neglected.

It is readily seen that the error made in neglecting the second term in Eq. (59) is of the order of 0.1% in most cases and for this reason Church and Weneser drop this term in their Eq. (7). They use Sliv's conversion coefficients for reference and make the good approximations $\phi_{-1}=0$ and $\Sigma_{-1}{}^0=1$. They also approximate Σ_{-1} by what is effectively the first term in its power series expansion, for in the long-wavelength limit their parameter λ is the same as the ratio $R(3;1)$. The tabulated parameter $C(Z,k)$ of Church and Weneser should be equal to C_1 of our Eq. (60). Actually, as the result of some computational errors the values of $C(Z,k)$ in reference 4 are between 25 and 30% too large.

b. Conversion Coefficients for $E1$ Transitions

Equations (51), (52), (50), (45), and (46) provide the necessary formulas for the analysis of $E1$ transitions. Since the exact formulas involve many parameters, it is convenient for the purpose of discussion to work with a much simpler approximate formula. In Eq. (51) let Sliv's conversion coefficients be used for reference but let the small terms containing $\Sigma_\kappa{}^0$ be neglected. Further, let it be supposed that the nuclear structure effect is not too large (say less than 30%). Finally, let the Σ_κ be approximated by the first terms in their expansions and let $\cos\tau_{-2}$ be approximated by unity. Then, with an accuracy which should be better than 10% in most cases,

$$\frac{\alpha_1(\Sigma)}{\alpha_1(\Sigma^0)}\sim 1-\frac{2\omega_{-2}}{1+U_1{}^2}\left[S(2;0)\left(1+\frac{\bar{b}_0(1)U_1\omega_1\cos\tau_1}{\omega_{-2}}\right)+\bar{c}_0(-2)T(2;0)\left(1+\frac{\bar{c}_0(1)\omega_1U_1\cos\tau_1}{\bar{c}_0(-2)\omega_{-2}}\right)\right]. \tag{61}$$

[35] See, for example, S. P. Lloyd, Phys. Rev. **81**, 161 (1951).

In Eq. (61) the terms proportional to $\cos\tau_1$ are the contributions from the final electron state for which $\kappa=1$. Numerical evaluation shows that in the coefficient of $S(2;0)$ the (positive) contribution for $\kappa=1$ is less than 0.40 for $k<2$; in the coefficient of $T(2;0)$ the contribution decreases with increasing k from about 0.70 to -0.50 for $k<2$. It is fortunate that the state $\kappa=1$ does not play a completely dominant role in Eq. (61) because it is for precisely this state that the uncertainties in the tabulated quantities are very large.

It is easy to see from Eq. (61) that large matrix element ratios are required for an observable structure effect. For a 20% effect it follows from Tables V and VII that the curly bracket in Eq. (61) needs to be of the order of 50 for large values of Z and k and of the order of 100–300 for the more common small values of k. It appears, according to statements made by Nilsson in his paper on nuclear structure effects in internal conversion,[36] that in certain greatly hindered $E1$ transitions matrix element ratios of the necessary magnitude are theoretically possible.

In the above-mentioned paper Nilsson points out the distinct possibility that conventional perturbation theory which uses the same electrostatic potential for both the initial and final nuclear states may be inadequate for the accurate determination of $\bar{b}_n(1)$ and $\bar{c}_n(1)$. Briefly, the reason is that these coefficients are of the form $V'(0)-V(0)+2$ +terms of order k, where $V'(0)$ and $V(0)$ are the values of the electron's potential energy ($\sim-3\alpha Z/2R$) at the origin in the initial and final states. In the conventional perturbation theory the two dominant terms cancel; if even slightly different values of the potential are used, the values of $\bar{b}_n(1)$ and $\bar{c}_n(1)$ are considerably altered. The reader is referred to Nilsson's paper for a more complete discussion. In view of the possible inaccuracy of $\bar{b}_n(1)$ and $\bar{c}_n(1)$ it is fortunate that they do not play a dominant role in Eqs. (51) and (61).

c. Conversion Coefficients for EL Transitions with $L\geqslant 2$

As in the case of $E1$ transitions let Sliv's conversion coefficients be used for reference but let the small terms containing Σ_κ^0 be neglected in Eq. (51). Furthermore, let the discussion be restricted to transitions for which $Z>60$ and $k<0.5$ so that the low-energy approximation of Sec. IIId is a good one. For such transitions it should be a good approximation to set $\Sigma_L=\bar{b}_0(L)\Sigma_{-L-1}$.

Finally let $[\omega_{-L-1}/\bar{b}_0(L)\omega_L]^2$ be neglected in comparison with unity and let $\cos\tau_{-L-1}$ be set equal to unity. (See Tables V, VI, VIII, and XII.) It then follows that

$$\alpha_L(\Sigma)/\alpha_L(\Sigma_0)\sim 1+\epsilon_L, \tag{62}$$

where

$$\epsilon_L=-a_Lx+b_Lx^2, \tag{63}$$

$$a_L=\frac{2}{1+U_L^2}\left(U_L\cos\tau_L+\frac{\omega_{-L-1}}{\bar{b}_0(L)\omega_L}\right), \tag{64}$$

$$b_L=(1+U_L^2)^{-1}, \tag{65}$$

and

$$x=\bar{b}_0(L)\omega_L\Sigma_{-L-1}. \tag{66}$$

The values of the constants a_L and b_L in Eq. (63) can be determined from Tables V, VI, VII, and VIII from which it follows that $-1.1<a_L<1.1$ and $0.2<b_L<1.0$. It can also be shown that a_L attains its extreme values only when U_L is relatively large and hence when b_L is relatively small. When U_L is relatively small a_L is also small and b_L is close to unity. It follows from this that, for a given value of ϵ_L, the smaller of the two values of $|x|$ which satisfy Eq. (63) lies to a good approximation in the range

$$|\epsilon_L|<|x|<\sqrt{\epsilon_L}. \tag{67}$$

The upper limit corresponds to a physical situation for which U_L and a_L are small, and in this case ϵ_L is necessarily positive; the lower limit corresponds to one in which U_L is relatively large. It is thus seen that, for a 20% nuclear structure effect, it is necessary that $0.20<|x|<0.45$.

The relation between x and the matrix element ratios is given by Eq. (66). The value of $\bar{b}_0(L)\omega_L$ can be obtained from Tables VI and XII, from which it can be seen that almost without exception $\bar{b}_0(L)\omega_L$ is an increasing function of Z, k, and L. Clearly, the larger the value of $\bar{b}_0(L)\omega_L$, the smaller the value of $|\Sigma_{-L-1}|$ required for a given value of $|x|$.

Most of the presently available data are for low energies. McGowan and Stelson[5] find, for example, that in the case of the 89 kev, $E2$, transition in Hf^{176}, $\epsilon_2=0.20\pm0.10$. Extrapolation indicates that for this transition $a_2=-0.46$ and $b_2=0.90$. If one assumes that $\epsilon_2=0.20$, it then follows that $x=0.28$ and $\Sigma_{-3}\sim 80$. It is not at all clear how such a value of Σ_{-3} is to be understood from the theoretical point of view, particularly

[36] S. G. Nilsson, University of California Radiation Laboratory Report UCRL-3803, 1957 (unpublished.) The basic Eq. (39) of this paper is incorrect because it was deduced through the use of a partial integration formula [his Eq. (13)] which is not valid for the integrals to which it is applied. For EL transitions with $L\geqslant 2$ and low energies this mistake is unimportant because it merely changes the coefficients $\bar{b}_n$ and $\bar{c}_n$ by amounts of the order of $kR/\alpha Z$. However for $L=1$ it changes the coefficients $\bar{b}_n(1)$ and $\bar{c}_n(1)$ appreciably, the relative error incurred being of the order of k. We wish to acknowledge that it was the study of Nilsson's Eq. (39) which led to a search for and the establishment of the theorem stated as Eq. (55) of our Sec. IIId. See also S. G. Nilsson and J. O. Rasmussen, University of California Radiation Laboratory Report UCRL-3889 (unpublished).

since in this case it is known that the gamma-ray transition rate is enhanced rather than hindered.

In addition to the group of enhanced $E2$ transitions to which the Hf^{176} transition belongs, there is a considerable group of $E3$ transitions in which shell-structure effects are apparent.[37] In this group there are some strongly hindered transitions in odd-proton nuclei which are heavy enough to have structure-sensitive conversion coefficients.

d. Remarks About the Accuracy of the Computation

In carrying out the computation, an effort was made to keep the resultant errors due to the many necessary approximations down to a few percent. This was done in most instances, as can be seen from the analysis given in Sec. V of II. In the case of U_L and $\cos\tau_1$ it turned out that some really large errors were unavoidable. Therefore, in order to warn the user, the errors were indicated along with the tabulated values.

The data in our possession did not permit us to carry out calculations below $k=0.3$, and as a result it is necessary to perform some extrapolation in order to treat many cases of importance. Examples of such extrapolation are given in Figs. 2–8 of II. It will be seen that near threshold the extrapolated values may be quite inaccurate. The extrapolated values of U_L are best obtained from the graphical extrapolation of the monotonic increasing function of k, $U_L\cos\tau_L$.

e. Remark About the Nuclear Models to which the Results of the Present Paper Apply

Evidently any nuclear model will involve some kind of radial wave functions which will, among other things, determine the size of the nucleus. It is of some importance that these wave functions be chosen in such a way that in the nuclear matrix elements the main contribution to the nuclear radial integrals comes from the interval $0\leq x\leq R=1.2A^{\frac{1}{3}}\times10^{-13}$ cm. The reason is that the power series which were used for the electron wave functions converge to these functions only for $0\leq x\leq R$. Thus, strictly speaking, one should use the power series expansion only for that part of the radial integrals for which $x\leq R$. If it is necessary to integrate far beyond R, a different representation of the electron wave functions is required in principle. However, we expect that at present it should be consistent with most nuclear models to use radial functions which are so small for $r>R$ that integration from zero to R is a good approximation.

[37] See, for example, M. Goldhaber and A. W. Sunyar, in *Beta- and Gamma-Ray Spectroscopy*, edited by Kai Siegbahn (North Holland Publishing Company, Amsterdam, 1955), p. 465.

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APPENDIX A. PROOF OF A THEOREM USED IN SECTION II

Let

$$I=\int_a^b dy\, F_\kappa G_{\kappa'} h_{L+1}^{(1)} \quad \text{and} \quad J=\int_a^b dy\, G_\kappa F_{\kappa'} h_{L+1}^{(1)}.$$

We first transform I and J by replacing $h_{L+1}^{(1)}$ by

$$\frac{L}{L+1}h_{L-1}^{(1)}-\frac{2L+1}{L+1}\frac{1}{k}\frac{dh_L^{(1)}}{dy}.$$

We then integrate by parts, an operation which is permissible since F, G, and their first derivatives are continuous. We thus obtain

$$\begin{aligned} I=&\frac{L}{L+1}\int_a^b dy\, F_\kappa G_{\kappa'} h_{L-1}^{(1)} \\ &+\frac{2L+1}{(L+1)}\int_a^b dy\frac{h_L^{(1)}}{k}\frac{d}{dy}(F_\kappa G_{\kappa'}) \\ &-\frac{2L+1}{(L+1)k}h_L^{(1)}F_\kappa G_{\kappa'}\Big]_a^b, \\ J=&\frac{L}{L+1}\int_a^b dx\, F_{\kappa'} G_\kappa h_{L-1}^{(1)} \\ &+\frac{2L+1}{(L+1)}\int_a^b dy\frac{h_L^{(1)}}{k}\frac{d}{dy}(F_{\kappa'} G_\kappa) \\ &-\frac{2L+1}{(L+1)k}h_L^{(1)}F_{\kappa'} G_\kappa\Big]_a^b. \end{aligned} \tag{A1}$$

Next we carry out the differentiations in the second term of I and J using the radial equations on page 66 of I. Finally we replace $[(2L+1)/ky]h_L^{(1)}$ by $h_{L-1}^{(1)}+h_{L+1}^{(1)}$. There results the equations

$$
\begin{aligned}
&I(L+1-\kappa+\kappa') \\
&\quad=(L+\kappa-\kappa')\int_a^b dy\, F_\kappa G_{\kappa'} h_{L-1}{}^{(1)} \\
&\qquad+(2L+1)\int_a^b dy\,\frac{h_L{}^{(1)}}{k} F_\kappa F_{\kappa'}(W'+1-V) \\
&\qquad-(2L+1)\int_a^b dy\,\frac{h_L{}^{(1)}}{k} G_{\kappa'} G_\kappa (W-1-V) \\
&\qquad\qquad -\frac{2L+1}{k} h_L{}^{(1)} F_\kappa G_{\kappa'}\Big]_a^b, \\
&J(L+1-\kappa'+\kappa) \\
&\quad=(L+\kappa'-\kappa)\int_a^b dy\, F_{\kappa'} G_\kappa h_{L-1}{}^{(1)} \\
&\qquad+(2L+1)\int_a^b dy\,\frac{h_L{}^{(1)}}{k} F_\kappa F_{\kappa'}(W+1-V) \\
&\qquad-(2L+1)\int_a^b dy\,\frac{h_L{}^{(1)}}{k} G_{\kappa'} G_\kappa (W'-1-V) \\
&\qquad\qquad -\frac{2L+1}{k} h_L{}^{(1)} F_{\kappa'} G_\kappa\Big]_a^b.
\end{aligned}
\tag{A2}
$$

Now according to Eq. (27) $\eta_L{}^+$ is given by

$$
\begin{aligned}
\eta_L{}^+=&[(\kappa'-\kappa+L+1)F_\kappa G_{\kappa'}+(\kappa'-\kappa-L-1)F_{\kappa'} G_\kappa] h_{L-1}{}^{(1)} \\
&+(L+1)(F_\kappa F_{\kappa'}+G_\kappa G_{\kappa'}) h_L{}^{(1)}.
\end{aligned}
\tag{A3}
$$

Consequently it follows from Eq. (A2) that

$$
\begin{aligned}
\int_a^b dy\,\eta_L{}^+=-\int_a^b dy[&(\kappa'-\kappa-L)F_\kappa G_{\kappa'} h_{L-1}{}^{(1)} \\
&+(\kappa'-\kappa+L)F_{\kappa'} G_\kappa h_{L-1}{}^{(1)} \\
&+L(F_\kappa F_{\kappa'}+G_\kappa G_{\kappa'}) h_L{}^{(1)}] \\
&-\frac{2L+1}{k}(F_\kappa G_{\kappa'}-F_{\kappa'} G_\kappa) h_L{}^{(1)}(ky)\Big]_a^b.
\end{aligned}
\tag{A4}
$$

However, from Eq. (26) it is seen that the term in the square brackets is just $\eta_L{}^-$. Hence

$$
\begin{aligned}
\int_a^b dy\,\eta_L{}^+=-\int_a^b dy\,&\eta_L{}^- \\
&-\frac{2L+1}{k}(F_\kappa G_{\kappa'}-F_{\kappa'} G_\kappa) h_L{}^{(1)}\Big]_a^b.
\end{aligned}
\tag{A5}
$$

The theorem can be extended to $\xi_L{}^+$ and $\xi_L{}^-$ by replacing $h_L{}^{(1)}$ by j_L on the right-hand side of Eq. (A5).