Significance of the Redundant Solutions of the Low-Wick Equation

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The content of the Low-Wick scattering formalism is studied, using the example of a class of exactly soluble meson theories with fixed-source interaction. The theories in question describe a set of harmonic oscillators with an arbitrary distribution of frequencies, coupled to a scalar meson field by means of their total dipole moment. The Low equation for scattering of a meson is shown to possess infinitely many solutions. These are compared with the exact, explicit solution of the same problem, and it is shown that there is a one-to-one correspondence between a particular choice of theory (number of oscillators and their frequencies) and a given one of the aforementioned solutions of the Low equation. A similar situation is shown to obtain for symmetric pseudoscalar theory, and it is made plausible thereby that Chew and Low have chosen the particular solution appropriate to their Hamiltonian.

I. INTRODUCTION

THE solution of the Low equation[1] obtained by Chew and Low[2] for a fixed-source theory and with the use of the one-meson approximation has been shown[3] not to be unique but rather only one of an infinite number of solutions. Since this solution is distinguished from the others only in that it has the fewest number of zeros in the scattering amplitude, there appears a need for justifying its choice. It has been variously suggested that there may exist a general principle which would require all the other solutions to be eliminated on the grounds that they are "non-physical." On the other hand, it may be that the Low equation can not imply a particular solution because it does not manifest the full physical content of the system to which it is applied, and, in this case, other properties of the system would have to be utilized before a unique solution could be obtained.

There exists one class of theories for which the correctness of this latter point of view has been demonstrated.[4] However, as pointed out by Dyson,[4] the models comprising this class do not have the property of "crossing-invariance,"[5] and it is conceivable therefore that the conclusions drawn from a study of these theories may in some measure be attributable to this lack. In order to clear up this point and in order to acquire some further insight into the correct significance of the solutions of the Low equation, we devote ourselves in this paper to the study of a class of theories which do possess "crossing-invariance" and at the same time ones for which both the Low equation and the equations of motion can be solved exactly.

In Sec. II the Low equation is derived, and it is demonstrated that the same equation describes every model of the class. The solution for the scattering amplitudes are then obtained by solving the Low equation and also, in Sec. III, by solving the equations of motion. In Sec. IV these two results are compared, and the conclusions drawn from this comparison are used as the basis for a conjecture concerning the meaning of the additional solutions of the Low equation for the symmetrical, pseudoscalar theory.

II. SOLUTION OF THE LOW EQUATION

A typical theory of the class has the Hamiltonian

$$H=H_0+H_I, \tag{1}$$

where

$$H_0=\int d\mathbf{k}\, a^*(\mathbf{k})\omega(k)a(\mathbf{k})+\sum_{i=1}^{N}\left(\frac{\mathbf{p}_i^2}{2m}+\tfrac{1}{2}m\omega_i^2r_i^2\right), \tag{2}$$

$$H_I=g\int d\mathbf{x}\,\rho(x)\left[\nabla\varphi(\mathbf{x})\cdot\sum_{i=1}^{N}\mathbf{r}_i\right]-E_s, \tag{3}$$

while the whole class is generated as N is allowed to assume all integral values from one to infinity. In addition to the free meson field, the unperturbed Hamiltonian in Eq. (2) describes a matter system composed of N particles each of mass m and each bound harmonically with its own resonant frequency ω_i. Equation (3) characterizes the interaction of this matter system with the scalar meson field φ. $\rho(x)$ is the spherically symmetric form factor common to all the bound particles, g is the common coupling constant, $a(\mathbf{k})[a^*(\mathbf{k})]$ is the destruction [creation] operator for a single meson of momentum $\mathbf{k}$, and E_s is a c number introduced to fix the ground state of the complete Hamiltonian at zero energy. For $N=1$, the model is similar to that of a harmonically bound electron interacting with the radiation field in the electric dipole approximation.[6] The theory considered here, however, is not burdened with additional computational involvement arising from the boson polarization.

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[1] F. E. Low, Phys. Rev. **97**, 1392 (1955).

[2] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570, 1579 (1956).

[3] Castillejo, Dalitz, and Dyson, Phys. Rev. **101**, 453 (1956); see also A. Klein, Phys. Rev. **104**, 1336 (1956).

[4] F. J. Dyson, Phys. Rev. **106**, 157 (1957). See also R. Haag, Nuovo cimento **5**, 203 (1957).

[5] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954).

[6] N. G. Van Kampen, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 15 (1951); see also K. McVoy and H. Steinwedel, Nuclear Phys. **1**, 164 (1956).

By employing the methods and notation of Wick,[7] one readily obtains for the S matrix

$$(\mathbf{k}'|S|\mathbf{k})=\delta(\mathbf{k}'-\mathbf{k})-2\pi i\delta(\omega'-\omega)(\mathbf{k}'|T|\mathbf{k}), \quad (4)$$

where the scattering amplitude $(\mathbf{k}'|T|\mathbf{k})$ is given by

$$(\mathbf{k}'|T|\mathbf{k})=(\Psi^{(-)}(\mathbf{k}'),j^*(k)\Psi_0) \quad (5)$$

$$=\left(\Psi_0,\, j^*(\mathbf{k})\frac{1}{-\omega-H}j(\mathbf{k}')\Psi_0\right)+\left(\Psi_0,\, j(\mathbf{k}')\frac{1}{\omega+i\eta-H}j^*(\mathbf{k})\Psi_0\right), \quad (6)$$

and where

$$j^*(\mathbf{k})=-j(\mathbf{k})\equiv[H_I,a^*(\mathbf{k})]=\frac{ig\rho(k)}{(2\omega)^{\frac{1}{2}}}\mathbf{k}\cdot\sum_{i=1}^{N}\mathbf{r}_i. \quad (7)$$

Ψ_0 is the ground state of the complete Hamiltonian H, and the symbol $+i\eta$ indicates that the $\lim\eta\rightarrow 0_+$ is to be taken after the performance of all integrations.

Noting from the form of H_I in Eq. (3) that only the partial waves of unit angular momentum can interact, we take the projection of Eq. (6) onto the states of $l=1$ by means of the relation

$$T(\omega)=\frac{\omega k}{3}\sum_{m=-1}^{1}\int d\Omega\, d\Omega'\, Y_{1m}^*(\hat{k}')Y_{1m}(\hat{k})(\mathbf{k}'|T|\mathbf{k}), \quad (8)$$

where $\hat{k}$ is a unit vector in the direction of $\mathbf{k}$, and obtain, after combining the terms,

$$T(\omega)=\frac{4\pi g^2k^3|\rho(k)|^2}{9}\sum_n\frac{\omega_n}{\omega^2-\omega_n{}^2+i\eta}\left|\left(n\left|\sum_{i=1}^{N}\mathbf{r}_i\right|\Psi_0\right)\right|^2. \quad (9)$$

The sum in Eq. (9) is over any complete set of eigenstates of H. However we will assume[8] that there exist no bound states, from which it follows that only the one-meson intermediate states will contribute. The validity of the one-meson approximation for this theory can be verified by noting that the Hamiltonian in Eq. (1) is a quadratic function of the operators for the meson and harmonic oscillator quanta. Due to this fact the Hamiltonian can be diagonalized such that the operators for the physical quanta are *linear* combinations of the operators for the free quanta. A similar situation exists in the case of pair theory.[9]

With the choice of the incoming-wave eigenstates $\Psi^{(-)}(\mathbf{k})$ for $|n)$, Eq. (9) can be rewritten as

$$T(\omega)=\frac{4\pi g^2k^3|\rho(k)|^2}{9}\int\frac{d\mathbf{k}'\omega'\left|\left(\Psi^{(-)}(\mathbf{k}'),\sum_{i=1}^{N}\mathbf{r}_i\Psi_0\right)\right|^2}{\omega^2-\omega'^2+i\eta}, \quad (10)$$

which, by virtue of Eqs. (5), (7), and (8) becomes

$$T(\omega)=2k^3|\rho(k)|^2\int\frac{d\mathbf{k}'|T(\omega')|^2}{(\omega^2-\omega'^2+i\eta)k'^2|\rho(k')|^2}. \quad (11)$$

If we now define the function $h(\omega)$,

$$h(\omega)\equiv\frac{e^{i\delta(\omega)}\sin\delta(\omega)}{k^3|\rho(k)|^2}=\frac{-\pi T(\omega)}{k^3|\rho(k)|^2}, \quad (12)$$

and use it to rewrite Eq. (11), we obtain

$$h(\omega)=-\frac{2}{\pi}\int\frac{d\mathbf{k}'k'^4|\rho(k')|^2|h(\omega')|^2}{\omega^2-\omega'^2+i\eta}, \quad (13)$$

which is the Low scattering equation. We note that since the parameter "N" [see Eqs. (1), (2), and (3)] does not appear, every member of the infinite class of theories is described by this same Low equation.

Our job now is to obtain the solution of Eq. (13). We first extend the argument $\omega^2+i\eta$ of Eq. (13) into the complex z plane and thereby obtain the function

$$\kappa(z)=-\frac{2}{\pi}\int\frac{d\mathbf{k}'k'^4|\rho(k')|^2|h(\omega')|^2}{z-\omega'^2}, \quad (14)$$

which obviously satisfies

$$h(\omega)=\kappa(\omega^2)=\lim_{\eta\rightarrow 0+}\kappa(\omega^2+i\eta). \quad (15)$$

The additional relevant properties of $\kappa(z)$ are the following:

(a) $\kappa(z)$ is analytic everywhere in the z plane except along the real axis between μ^2 and infinity [μ is the rest mass of the mesons and hence the lower limit of ω' in the integral in Eq. (14)].

(b) Between μ^2 and infinity along the positive real axis, $\kappa(z)$ has a branch line with a discontinuity given by

$$\kappa(x+i\eta)-\kappa(x-i\eta)=2i(x-\mu^2)^{\frac{3}{2}}|\rho[(x-\mu^2)^{\frac{1}{2}}]|^2|\kappa(x)|^2. \quad (16)$$

(c) As z goes to infinity $\kappa(z)$ goes to zero as z^{-1}.

(d) The imaginary part of $\kappa(z)$ satisfies

$$\mathrm{Im}\kappa(z)=\left(\frac{1}{\pi}\int_{\mu^2}^{\infty}\frac{d(\omega'^2)k'^3|\rho(k')|^2|h(\omega')|^2}{|z-\omega'^2|^2}\right)\mathrm{Im}z, \quad (17)$$

and since the coefficient within the parenthesis is positive definite, $\kappa(z)$ is nonvanishing off the real axis. More precisely, $\mathrm{Im}\kappa(z)$ is positive (negative) definite in the upper (lower) half-plane.

(e) $\kappa(z)$ is a real function of z; that is $\kappa^*(z)=\kappa(z^*)$. If we now write

$$H(z)=1/\kappa(z), \quad (18)$$

the function $H(z)$ is well defined everywhere that $\kappa(z)$ is nonvanishing. From the properties (a) and (d) of

[7] G. C. Wick, Revs. Modern Phys. 27, 339 (1955).

[8] This assumption is discussed in somewhat more detail in Sec. III.

[9] See, for example, G. Wentzel, Helv. Phys. Acta 15, 111 (1942).

$\kappa(z)$, it follows that $H(z)$ is a well-defined, analytic function off the real axis. Furthermore, from (b) and (e), we obtain that

$$H(x+i\eta)-H(x-i\eta)=\frac{\kappa(x-i\eta)-\kappa(x+i\eta)}{|\kappa(x+i\eta)|^2}$$
$$=-2i(x-\mu^2)^{\frac{3}{2}}|\rho(\sqrt{(x-\mu^2)})|^2, \quad (19)$$

for every value of x which does *not* satisfy $\kappa(x)=0$. We now have established:

(α) $H(z)$ is an analytic function of z off the real axis.

(β) $H(z)$ has a branch line along the real axis from μ^2 to infinity, and except for possible singular points, the discontinuity across this branch line is given by Eq. (19). It should be remarked that we are neglecting the possibility that $\kappa(x)$ vanishes over finite intervals because of the nonphysical nature of the resulting phase shift. For the same reason we also do not consider the consequence of the zeros of $\kappa(x)$ having a cluster point.[3] From the property (c) of $\kappa(z)$ we also have that

(γ) $H(z)$ increases linearly with z as z goes to infinity, while the remaining necessary property of $H(z)$ is that

(δ) the singularities of $H(z)$ all lie on the real axis between μ^2 and infinity, and at each of them $H(z)$ has only a simple pole. The first part of this statement can be seen immediately from Eq. (14) by noting that $\kappa(x)$ is positive definite for $-\infty<x<\mu^2$. The fact that all the poles are simple can be proved as follows[10]:

(i) Let x_i be a singularity of $H(z)$ so that, in view of the discussion under property (β), $H(z)\sim a_i(z-x_i)^{-n}$ in the neighborhood of $z=x_i$.

(ii) Letting $z-x_i=re^{i\theta}$ and $a_i=|a_i|e^{-i\varphi}$, $H(z)\sim|a_i|r^{-n}e^{-i(n\theta+\varphi)}$ in the neighborhood of $z=x_i$.

(iii) It now follows that $n=1$ and $\varphi=0$ since, from property (d) of $\kappa(z)$, $\mathrm{Im}H(z)=-\mathrm{Im}\kappa(z)|\kappa(z)|^{-2}$ is negative (positive) definite in the upper (lower) half plane.

From the property (α) of $H(z)$, we have

$$\oint_C\frac{dz'[H(z')-Az'-B]}{z'-z}=2\pi i[H(z)-Az-B], \quad (20)$$

where C is the contour shown in Fig. 1. Furthermore, by choosing for the constants A and B,

$$A=\lim_{z\to\infty}H(z)/z, \quad (21)$$

$$B=\lim_{z\to\infty}H(z)-Az, \quad (22)$$

it follows from property (γ) that the integral over the infinite arc can be neglected. Hence, making use of Eq. (19) to express the discontinuity in $H(z)$, we can

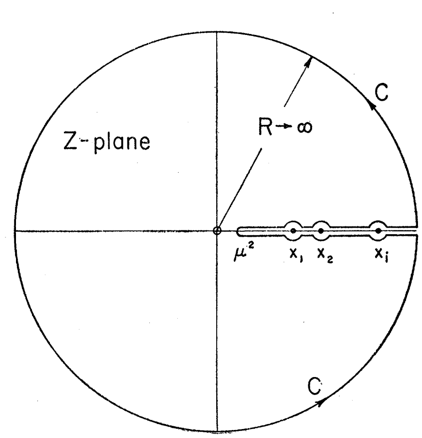

FIG. 1. The contour C for the evaluation of the integral in Eq. (20).

rewrite Eq. (20) as

$$H(z)=Az+B+\sum_i\frac{a_i}{z-x_i}-\frac{1}{\pi}\int_{\mu^2}^{\infty}\frac{dx(x-\mu^2)^{\frac{3}{2}}|\rho[(x-\mu^2)^{\frac{1}{2}}]|^2}{x-z}, \quad (23)$$

where a_i is the residue of $H(z)$ at its pole located at $z=x_i$, and the sum is over all the singularities of $H(z)$. From Eqs. (15) and (18), Eq. (23) can be rewritten as

$$\frac{1}{h(\omega)}=A\omega^2+B+\sum_i\frac{a_i}{\omega^2-x_i}-\frac{1}{\pi}\int_{\mu^2}^{\infty}\frac{2\omega'd\omega'k'^3|\rho(k')|^2}{\omega'^2-\omega^2-i\eta}, \quad (24)$$

and by taking the real part of this expression, we obtain

$$k^3|\rho(k)|^2\cot\delta$$
$$=A\omega^2+B+\sum_i\frac{a_i}{\omega^2-x_i}-\frac{1}{2\pi^2}\mathcal{P}\int\frac{d\mathbf{k}'k'^2|\rho(k')|^2}{k'^2-k^2}, \quad (25)$$

by virtue of Eq. (12) for the definition of the phase shift δ.

Equation (25) is the general solution of the Low equation, and, as expected, it does contain an infinite number of arbitrary constants. It is clear that with an appropriate choice of the constants this general solution must reduce to the solution for any one member of the infinite class of theories described in Eqs. (1), (2), and (3). Indeed, in the next section we will see that the solution obtained by any choice of the constants in Eq. (25) is the solution for one of the theories of this type.

III. EXACT CALCULATION OF THE SCATTERING AMPLITUDES

The equations of motion for the theory described by Eqs. (1), (2), and (3) are

$$\frac{d^2\mathbf{r}}{dt^2}+\omega_i^2\mathbf{r}_i=-\left(\frac{g}{m}\right)\int\varphi(\mathbf{x})\nabla\rho(x)d\mathbf{x}, \quad (26)$$

[10] The proof given here is essentially that of E. P. Wigner, [Ann. Math. 53, 36 (1951)].

and

$$(\Box^2-\mu^2)\varphi(\mathbf{x},t)=-g\nabla\rho\cdot\sum_{i=1}^{N}\mathbf{r}_i(t), \tag{27}$$

where the forms of the expressions on the right are the result of integrating by parts the interaction H_I in Eq. (3).

In order to solve these equations we first introduce the Fourier transforms[11]

$$\mathbf{r}_i(k_0)=\frac{1}{(2\pi)^{\frac{1}{2}}}\int e^{ik_0t}\mathbf{r}_i(t)dt, \tag{28}$$

$$\varphi(\mathbf{k},k_0)=\frac{1}{(2\pi)^2}\int e^{i(k_0t-\mathbf{k}\cdot\mathbf{x})}\varphi(\mathbf{x},t)d\mathbf{x}dt, \tag{29}$$

$$\rho(k)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int e^{i\mathbf{k}\cdot\mathbf{x}}\rho(x)d\mathbf{x}. \tag{30}$$

Next, we write ω^2 in place of $k^2+\mu^2$ and use these transforms to re-express the equations of motion (26) and (27) in the form

$$(\omega_i^2-k_0^2)\mathbf{r}_i(k_0)=(g/m)(\nabla\rho,\varphi(k_0)), \tag{31}$$

and

$$(\omega^2-k_0^2)\varphi(k_0)=g\nabla\rho\cdot\sum_{i=1}^{N}\mathbf{r}_i(k_0), \tag{32}$$

where the three spatial coordinates have been suppressed in favor of matrix notation. Inverting Eqs. (31) and (32), we have

$$\mathbf{r}_i(k_0)=\mathbf{r}_{(\ ),i}(k_0)+\frac{g}{m}\frac{(\nabla\rho,\varphi(k_0))}{\omega_i^2-k_0^2} \tag{33}$$

and therefore

$$\varphi(k_0)=\varphi_{(\ )}(k_0)+\frac{g}{\omega^2-k_0^2}\nabla\rho \cdot\sum_{i=1}^{N}\left(\mathbf{r}_{(\ )i}(k_0)+\frac{g}{m}\frac{(\nabla\rho,\varphi(k_0))}{\omega_i^2-k_0^2}\right), \tag{34}$$

where the subscript ( ) on the homogeneous solutions indicate either the "in" or the "out" operators of Yang and Feldman[12] depending upon whether singularities in the nonhomogeneous parts of the solutions are avoided by giving k_0 an infinitesimal positive or negative imaginary part, respectively. The separable integral equation (34) can now be solved with the result that

$$\varphi(k_0)=\Theta_{(\ )}(k_0)\varphi_{(\ )}(k_0), \tag{35}$$

where

$$\Theta_{(\ )}(k_0)=1+\frac{g^2/m}{\omega^2-k_0^2}\nabla\rho\cdot\sum_{i=1}^{N}\frac{1}{\omega_i^2-k_0^2} \times\left[1-\left(\nabla\rho,\frac{g^2/m}{\omega^2-k_0^2}\nabla\rho\right)\sum_{j=1}^{N}\frac{1}{\omega_j^2-k_0^2}\right]^{-1}\cdot\nabla\rho, \tag{36}$$

and where the correspondence between the subscript and the means of avoiding the singularities is the same as above. Note also that the factor in the middle is a dyadic.

Now it is clear that we can write

$$\varphi_{(\ )}(\mathbf{k},k_0)=a_{(\ )}(\mathbf{k},k_0)\delta(k_0^2-\omega^2), \tag{37}$$

so that the positive (+) and negative (−) frequency parts of $\varphi_{(\ )}(k_0)$ take the form

$$\varphi_{(\ )}{}^{(\pm)}(\mathbf{k},k_0)=\frac{a_{(\ )}(\mathbf{k},k_0)}{(2\omega)^{\frac{1}{2}}}\delta(k_0\mp\omega). \tag{38}$$

Consequently, if we use Eq. (38) to obtain the positive-frequency part of Eqs. (35) and (36), and if we write out explicitly the subscripts and the integration prescriptions, the result can be expressed in the form

$$\begin{aligned}\varphi^{(+)}(\mathbf{k},k_0)&=\int d\mathbf{k}'(\mathbf{k}|\Omega_{(+)}|\mathbf{k}')\frac{\delta(k_0-\omega')}{(2\omega')^{\frac{1}{2}}}a_{(\mathrm{in})}(\mathbf{k}',k_0),\\ \varphi^{(+)}(\mathbf{k},k_0)&=\int d\mathbf{k}'(\mathbf{k}|\Omega_{(-)}|\mathbf{k}')\frac{\delta(k_0-\omega')}{(2\omega')^{\frac{1}{2}}}a_{(\mathrm{out})}(\mathbf{k}',k_0),\end{aligned} \tag{39}$$

where

$$(\mathbf{k}|\Omega_{(\pm)}|\mathbf{k}')=\delta(\mathbf{k}-\mathbf{k}')+\frac{g^2}{m}\frac{\mathbf{k}\cdot\mathbf{k}'}{k'^2\pm i\eta-k^2}\frac{\rho(k)\rho^*(k')}{D_{(\pm)}(k'^2)}, \tag{40}$$

and where

$$\frac{1}{D_{(\pm)}(k^2)}=\sum_{i=1}^{N}\frac{1}{D_{(\pm)i}(k^2)}, \tag{41}$$

and

$$D_{(\pm)i}(k^2)=(\omega^2\pm i\eta-\omega_i^2)\left[1-\frac{g^2}{3m}\left(\sum_{j=1}^{N}\frac{1}{\omega^2\pm i\eta-\omega_j^2}\right) \times\int\frac{d\mathbf{k}'k'^2|\rho(k')|^2}{k^2\pm i\eta-k'^2}\right]. \tag{42}$$

As previously mentioned in Sec. II, we are assuming that there are no bound states. In terms of this formalism, this assumption means that we are considering the class of problems for which g, m, and the ω_i have such magnitudes that $D_{(\ )}(k^2)$ is nonvanishing for $k^2<0$. It can be easily verified that values which satisfy this requirement do indeed exist. On the basis of the fact that there are no bound states, it can be shown[13] that the matrices $\Omega_{(+)}$ and $\Omega_{(-)}$ are both unitary. Therefore, equating the two alternative expressions for Eq. (39), we obtain

$$\delta(k_0-\omega)a_{(\mathrm{out})}(\mathbf{k},k_0) =\int d\mathbf{k}'(\mathbf{k}|\Omega_{(-)}{}^*\Omega_{(+)}|\mathbf{k}')\delta(k_0-\omega')a_{(\mathrm{in})}(\mathbf{k}',k_0). \tag{43}$$

[11] The methods which are used here are those of A. Klein and B. McCormick, [Phys. Rev. 98, 1428 (1955)].

[12] C. N. Yang and D. Feldman, Phys. Rev. 79, 972 (1950).

[13] The proof of the unitarity of the $\Omega_{(\ )}$ matrices can be accomplished by a procedure identical to that used for a slightly different problem in Appendix A of reference 11.

If we now express the $\Omega_{(\ )}$ matrices defined by Eq. (40) in a representation in which the angular momentum is given, and restrict our attention to the only subspace of interest, that of $l=1$, we obtain

$$(k|\Omega_{(\pm)}|k')=\delta(k-k')+\frac{4\pi g^2}{3m}\frac{\rho(k)\rho^*(k')k^2k'^2}{(k'^2\pm i\eta-k^2)D_{(\pm)}(k'^2)}. \quad (44)$$

This expression, in turn, can be rewritten as

$$(k|\Omega_{(\pm)}|k')=(k|\Omega_{(1)}|k')\frac{D_{(1)}(k'^2)}{D_{(\pm)}(k'^2)}, \quad (45)$$

where the subscript "(1)" indicates the real part of the quantity to which it is attached. We can now make use of Eq. (45), and our knowledge of the unitarity of the $\Omega_{(\ )}$ matrices, to obtain

$$(k|\Omega_{(-)}{}^*\Omega_{(+)}|k')=\frac{D_{(-)}(k^2)}{D_{(+)}(k^2)}\delta(k-k'), \quad (46)$$

as an expression for the matrix element appearing in Eq. (43). However, since the S matrix is defined to be the matrix which connects the "in" and the "out" operators in the form

$$a_{(\mathrm{out})}(\mathbf{k},k_0)=\int d\mathbf{k}'(\mathbf{k}|S|\mathbf{k}')a_{(\mathrm{in})}(\mathbf{k}',k_0), \quad (47)$$

it follows from Eqs. (43), (46), and (47) that

$$(k|S|k')\equiv e^{2i\delta}\delta(k-k')=\frac{D_{(-)}(k^2)}{D_{(+)}(k^2)}\delta(k-k'). \quad (48)$$

Finally, with the help of Eqs. (41) and (42) which define the $D_{(\ )}$ functions, we obtain from (48) that

$$k^3|\rho(k)|^2\cot\delta=-\frac{3m}{2\pi^2g^2}\bigg[\bigg(\sum_{i=1}^{N}\frac{1}{\omega^2-\omega_i^2}\bigg)^{-1}+\frac{g^2}{3m}\mathcal{P}\int\frac{d\mathbf{k}'k'^2|\rho(k')|^2}{k'^2-k^2}\bigg]. \quad (49)$$

Equation (49) is the exact solution for $\cot\delta$ and should be compared to Eq. (25) which is the general solution of the Low equation for this same quantity.

IV. CONCLUSIONS

First we note that for each value of N, the solution given by Eq. (49) is a special case of the general solution illustrated by Eq. (25). This fact can be seen by expanding

$$\bigg[\sum_{i=1}^{N}(\omega^2-\omega_i^2)^{-1}\bigg]^{-1}-A\omega^2-B$$

into partial fractions and noting the denominator, as a function of ω^2, has only single roots. One can therefore set up a "one-to-one" correspondence between the particular solutions of Eq. (25) and the models of the infinite class which they describe. We conclude, as was done in reference 4, that every solution of the Low equation is equally physical and, in fact, that the physical model corresponding to any one of the solutions differs from the model corresponding to any other solution only by an alteration or extension of those properties of the system which do not manifest themselves in the Low equation.

We observe that for the class of theories considered here, as well as for those studied in reference 4, the significance of the additional solutions of the Low equation is well understood. The solution having the fewest number of arbitrary constants is the analog of the Chew-Low solution and describes that theory of the infinite class which is distinguished by a target system with a minimum of internal structure. All the other solutions correspond to models for which the target has excited states which lead to additional resonances when the interaction with the meson field is included.

It seems reasonable to suggest that there is a similar origin to the many solutions of the Low equation for the symmetrical, pseudoscalar theory.[14] For example, suppose we consider the infinite class of Hamiltonians having a typical member of the form

$$H=H_0+H_I, \quad (50)$$

where

$$H_0=\int d\mathbf{k}a^*(\mathbf{k})\omega(k)a(\mathbf{k})+\sum_{i=0}^{N}\psi_i{}^*M_i\psi_i, \quad (51)$$

$$H_I=(g/2)\sum_{\mu=0}^{N}\psi_\mu{}^*\boldsymbol{\sigma}\boldsymbol{\tau}\psi_0\cdot\int\nabla\boldsymbol{\varphi}(\mathbf{x})\rho(x)d\mathbf{x}+\text{Herm. conj.} \quad (52)$$

For $N=0$ this Hamiltonian describes the usual fixed-source theory for the pion-nucleon interaction, while for $N>0$ the Hamiltonian represents the interaction of a meson field with a "nucleon" capable of existing in N excited states. If the M_i-M_0, for $i>0$, are sufficiently large compared to μ, the excited states of the unperturbed "nucleon" do not introduce any additional bound states in the spectrum of the complete Hamiltonian. The existence of these excited states manifests itself only as resonances in the transition amplitudes. In this case, the same Low equation describes each member of the infinite class of theories, and by analogy with the examples discussed previously, the Chew-Low solution would be expected to correspond to the theory with $N=0$.

[14] This suggestion is also immediately attendant upon the considerations of R. Norton and A. Klein, [Phys. Rev. 109, 584 (1958)].