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# Velocity-Dependent Forces and Nuclear Structure. II. Spin-Dependent Forces

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Recent investigations have shown the presence of velocity-dependent forces in the two-nucleon interaction, increasing the number of parameters in the two-nucleon potential, and making more difficult the determination of these parameters in a unique way. In view of the successes of the shell model and of the assumptions that the same interactions hold between nucleons inside nuclear matter as between free nucleons, it is of interest to explore the possible restrictions on the two-nucleon potential that follow from level arrangements and separations in nuclear shell theory. In the present paper we carry out this exploration for velocity-dependent forces, starting with the two-body spin-orbit force, two forms of which have recently been proposed, and considering also the simplest velocity-dependent forces that depend on the second power of the momentum, which include the velocity-dependent tensor force recently introduced by Breit. The two-body spin-orbit force of Gammel and Thaler has a very short range, and, taking advantage of this fact, we show in Sec. 2 that the interaction energy for two nucleons in the same shell is proportional to the interaction energy for the zero-range velocity-dependent central force discussed previously by the author. The simple expression for the interaction energy allows us to compare in Sec. 3 the level separation due to the spin-orbit force of Gammel and Thaler and that due to the spin-orbit force of Signell and Marshak. We also compare the effects of both types of spin-orbit forces with the interaction energy due to the central even singlet force of Gammel, Christian, and Thaler. In Sec. 4 we analyze a velocity-dependent central force that acts only in the triplet state. In Sec. 5 we discuss the velocity-dependent tensor potential in the long-range approximation, and show the restrictions that follow on the strength of this potential from the assumption that the separation between levels should be small compared with the separation between shells. In Sec. 6 we discuss the velocity-dependent tensor potential in the short-range approximation, and obtain restrictions on the product of strength and range of this potential. The interaction energies for all short-range velocity-dependent potentials show similarities, suggesting the possibility of finding simple closed expressions for the interaction energies for all short-range forces.

## 1. INTRODUCTION

In recent publications it has become clear that velocity-dependent forces play an important role in the interactions between nucleons. In particular, the work of Signell and Marshak[1] and of Gammel and Thaler[2] has shown that the experimental data on the two-nucleon interaction require for its explanation potentials that contain, besides ordinary forces (both central and tensor), a strong spin-orbit coupling force.

The work of Brueckner,[3] Bethe,[4] and their collaborators has shown that it is possible to assume for the interaction between nucleons inside nuclear matter the same potentials as between free nucleons. It is therefore of interest to see the effect of velocity-dependent forces between nucleons on nuclear structure, and particularly on nuclear shell theory. For the two-particle spin-orbit coupling force a general discussion has been given by Hope and Longdon[5] and by Hope.[6] The results of Hope and Longdon are rather complex, as they wanted them to apply to a spin-orbit force whose range was arbitrary. The analysis of Gammel and Thaler[2] shows, however, that the spin-orbit force has a very short range. This property permits a considerable simplification in the analysis, and we shall show that the effects of a short-range spin-orbit force in nuclear shell theory, are similar to those of the short-range velocity-dependent central force discussed by the author in a previous paper[7] (to be referred to as I). The explicit expressions for the interaction energy could be useful to discriminate between the different forms[1,2] of the spin-orbit coupling forces being proposed, particularly as it is possible to take now into account[8] the repulsive hard core present in the force introduced by Gammel and Thaler.[2]

It is well known that the only interaction between like nucleons that satisfies the invariance requirements[9] and depends on the first power of the momentum, is the spin-orbit coupling force. If a further dependence on the velocity is found for the interactions between

[1] P. S. Signell and R. E. Marshak, Phys. Rev. **106**, 832 (1957).
[2] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 (1957).
[3] K. A. Brueckner, Phys. Rev. **97**, 1353 (1955).
[4] H. A. Bethe, Phys. Rev. **103**, 1353 (1956).
[5] J. Hope and L. W. Longdon, Phys. Rev. **102**, 1124 (1956).
[6] J. Hope, Phys. Rev. **106**, 771 (1957).
[7] M. Moshinsky, Phys. Rev. **106**, 117 (1957), to be referred to as I.
[8] M. Bauer and M. Moshinsky, Nuclear Phys. 4, 615 (1957).
[9] L. Rosenfeld, *Nuclear Forces* (North-Holland Publishing Company, Amsterdam, 1948), p. 313.

like nucleons, it is expected to be a force depending on the second or higher power of the momentum. The simplest of these forces have been discussed by the author[10] and they have the forms

$$\tfrac{1}{8}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)^2[\mathbf{p}'\cdot V(r')\mathbf{p}'], \tag{1.1}$$

$$[1-\tfrac{1}{8}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)^2][\mathbf{p}'\cdot V(r')\mathbf{p}'], \tag{1.2}$$

$$-\tfrac{1}{2}V_T(r')[3(\boldsymbol{\sigma}_1\cdot\mathbf{p}')(\boldsymbol{\sigma}_2\cdot\mathbf{p}')-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)p'^2], \tag{1.3}$$

where $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ are the spin matrices of particles 1 and 2, and

$$\mathbf{p}'=\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2), \quad r'=|\mathbf{r}'|=|\mathbf{r}_1-\mathbf{r}_2|. \tag{1.4}$$

An interaction of the form (1.1) will act only on the triplet state where it reduces to $\mathbf{p}'\cdot V(r')\mathbf{p}'$, which is the central velocity-dependent force discussed in I. An interaction of the form (1.2) will act only on the singlet state where it again reduces to $\mathbf{p}'\cdot V(r')\mathbf{p}'$. Finally, an interaction of the form (1.3), when properly made Hermitian, represents a velocity-dependent tensor force of a form similar to the one recently derived (together with the spin-orbit coupling force) by Breit,[11] from a natural modification of the pseudoscalar theory of nuclear forces.

Besides the spin-orbit coupling force, we shall discuss in this paper the effect on the level arrangement in nuclear shell theory of velocity-dependent forces of the above form. Since perturbation methods are used in the determination of these level arrangements, the effects of the different types of forces are additive, and, comparing with the experimental level arrangement, one could obtain restrictions on the strength and range of the velocity-dependent forces.

As the addition of (1.1) and (1.2) gives the interaction potential discussed in I, we can restrict ourselves to the potentials (1.1) and (1.3). As in I, we shall discuss the interaction in the limits of very long and very short ranges, compared with the radius $R_0$ of the nucleus. We shall see that in the short-range approximation, the spin-orbit coupling force, and the central and tensor velocity-dependent forces, give rise to very similar expressions for the interaction energy.

## 2. SPIN-ORBIT COUPLING FORCE IN THE SHORT-RANGE APPROXIMATION

In this section we shall discuss the effect in nuclear shell theory of a two-particle spin-orbit force of the form

$$\hbar^{-1}\xi(r')\mathbf{l}'\cdot\mathbf{S}, \tag{2.1}$$

where

$$\mathbf{l}'=\mathbf{r}'\times\mathbf{p}', \quad \mathbf{S}=\tfrac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2), \tag{2.2}$$

and $\mathbf{r}'$, $\mathbf{p}'$ are defined in (1.4). The radial part $\xi(r')$ of the force will be assumed of short range, and we shall give below a specific definition of what we mean by short range. We do not intend $\xi(r')$ to be a $\delta$ function, as in this case $\mathbf{l}'$ would give zero.

We shall take for the common potential of the nucleons in the nucleus a harmonic oscillator potential of frequency $\omega$. It is well known[12,13] that this potential has definite advantages from the standpoint of calculation, while it does not represent a too strong idealization of the physical situation.

The wave function associated with a given nucleon, will be designated by the bracket notation of Dirac as

$$|nlm\rangle=\mathfrak{R}_{nl}(\nu,r)Y_{lm}(\theta,\varphi), \tag{2.3}$$

where the $Y_{lm}$ are the spherical harmonics, and $\mathfrak{R}_{nl}(\nu,r)$ are the radial functions[12,13] of the harmonic oscillator

$$\mathfrak{R}_{nl}(\nu,r)=N_{nl}\exp(-\tfrac{1}{2}\nu r^2)r^l v_{nl}(\nu r^2). \tag{2.4}$$

In (2.4), $N_{nl}$ is a normalization constant, $\nu\equiv(m\omega/\hbar)$ where $m$ is the mass of the nucleon, and $v_{nl}(\nu r^2)$ is a polynomial of order $n$ in $\nu r^2$ that starts with 1, i.e., $v_{nl}(0)=1$.

For nucleons 1 and 2, the corresponding wave functions will be designated by $|n_1l_1m_1\rangle$ and $|n_2l_2m_2\rangle$. Introducing the total orbital angular momentum $\mathbf{L}=\mathbf{l}_1+\mathbf{l}_2$, we can construct an eigenfunction of $L^2$, $L_z$ in the form

$$|n_1l_1,n_2l_2,LM\rangle = \sum_{m_1m_2}\{|n_1l_1m_1\rangle|n_2l_2m_2\rangle\langle l_1l_2m_1m_2|LM\rangle\}, \tag{2.5}$$

where $\langle l_1l_2m_1m_2|LM\rangle$ is a Clebsch-Gordan coefficient. Combining these wave functions with those of the total spin $\mathbf{S}$, we see that the general matrix element for the spin-orbit force, in the two particle configuration with $LS$ coupling, is

$$\begin{aligned}\langle n_1l_1,n_2l_2,L,S,J|\hbar^{-1}\xi(r')\mathbf{l}'\cdot\mathbf{S}|n_1'l_1',n_2'l_2',L',S',J\rangle \\ =\hbar^{-1}(-1)^{L+1-J}\langle n_1l_1,n_2l_2,L\|\xi(r')\mathbf{l}'\|n_1'l_1',n_2'l_2',L'\rangle \\ \times\langle 1\|\mathbf{S}\|1\rangle\delta_{S1}\delta_{S'1}W(LL'11;1J),\end{aligned} \tag{2.6}$$

where $W$ indicates a Racah coefficient, and the $\delta$'s show that the interaction takes place only in the triplet state.

The matrix element of $\xi(r')\mathbf{l}'$ that appears in (2.6), could also be considered in terms of the wave functions for the relative coordinate $\mathbf{r}'=\mathbf{r}_1-\mathbf{r}_2$, and the center-of-mass coordinate $\mathbf{r}''=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$. We shall designate these wave functions by the ket $|n'l'm'\rangle$, which has the form (2.3) when $\mathbf{r}$ is replaced by $\mathbf{r}'$ and $\nu$ by $\nu'=(\nu/2)$, and by the ket $|n''l''m''\rangle$, which also has the form (2.3) with $\mathbf{r}$ replaced by $\mathbf{r}''$ and $\nu$ by $\nu''=2\nu$. As the corresponding angular momenta $\mathbf{l}'$, $\mathbf{l}''$ add up to the total angular momentum $\mathbf{L}$ given before, i.e.,

$$\mathbf{l}'+\mathbf{l}''=\mathbf{L}=\mathbf{l}_1+\mathbf{l}_2, \tag{2.7}$$

[10] M. Moshinsky, J. phys. radium **15**, 264 (1954).
[11] G. Breit (private communication).
[12] I. Talmi, Helv. Phys. Acta **25**, 185 (1952).
[13] R. Thieberger, Nuclear Phys. **2**, 533 (1956).

we can construct the wave function

$$|n'l',n''l'',LM\rangle = \sum_{m'm''} \{ |n'l'm'\rangle |n''l''m''\rangle \langle l'l''m'm''|LM\rangle \}. \quad (2.8)$$

Because of (2.7), the transformation bracket taking us from (2.5) to (2.8) would contain the same $L$, and we could write

$$|n_1l_1,n_2l_2,LM\rangle = \sum_{n'l',n''l''} \{ |n'l',n''l'',LM\rangle \times \langle n'l',n''l'',L|n_1l_1,n_2l_2,L\rangle \}, \quad (2.9)$$

where the fact that the wave functions must correspond to the same energy eigenvalues restricts $n'l'$, $n''l''$ to non-negative integers that satisfy

$$\hbar\omega(2n'+l'+2n''+l''+3) = \hbar\omega(2n_1+l_1+2n_2+l_2+3). \quad (2.10)$$

With the help of (2.9) we could express the matrix element for $\xi(r')\mathbf{l}'$ in (2.6) in terms of matrix elements involving the wave functions of the coordinates $\mathbf{r}'$ and $\mathbf{r}''$. We shall restrict ourselves to the case of identical nucleons in the same shell. We have then $n_1=n_2=n_1'=n_2'=n$, $l_1=l_2=l_1'=l_2'=l$, and $J$ takes only even values as the wave function is antisymmetric. As $L+S$, $L'+S'$ are also even,[14] and $S=S'=1$, we see that $L$, $L'$ are odd. Furthermore, from the Racah coefficient in (2.6) we see that $L'=L\pm1$, $L$, which together with the previous remark, implies that $L'=L$. Under these restrictions, the general matrix element for $\xi(r')\mathbf{l}'$ takes the form[15,8]

$$\langle (nl)^2L\|\xi(r')\mathbf{l}'\|(nl)^2L\rangle = \sum_{n'l',n''l''} [\langle n'l',n''l'',L|nl,nl,L\rangle]^2 \times(-1)^{l'+1-l''-L}\hbar[l'(l'+1)(2l'+1)]^{\frac{1}{2}} \times(2L+1)W(l'l'LL;1l'')I_{n'l'}, \quad (2.11)$$

where $I_{n'l'}$ stands for the integral

$$I_{n'l'} = \int_0^\infty r'^2[\Re_{n'l'}(\nu',r')]^2\xi(r')dr'. \quad (2.12)$$

So far we have made no use of the fact that $\xi(r')$ is of short range. From (2.12) and (2.4) we see that if the range of $\xi(r')$ tends to zero, $I_{n'l'}$ diminishes rapidly as $l'$ increases. If, for example, we take for $\xi(r')$ the Gaussian form

$$\xi(r') = V_0 \exp[-(r'/b)^2], \quad (2.13)$$

then, as shown by Talmi,[12] $I_{0l'}$ becomes

$$I_{0l'} \equiv I_{l'} = V_0\left(\frac{b^2\nu'}{1+b^2\nu'}\right)^{l'+\frac{3}{2}}. \quad (2.14)$$

If we understand now by a short-range potential, a potential for which

$$b(\nu')^{\frac{1}{2}} \equiv b(m\omega/2\hbar)^{\frac{1}{2}} \ll 1, \quad (2.15)$$

then from (2.14) we see that the predominating terms in (2.11) are those for which $l'$ is as small as possible. We cannot take $l'=0$, because from (2.8) $l''=L$ and $W(00LL;1L)=0$. If we take $l'=1$, then $l''=L\pm1$, $L$, but because of parity[12] considerations $(-1)^{l'+l''}=(-1)^{2l}$, so that $l''$ is odd, and as $L$ is odd, $l''=L$. As[16]

$$W(11LL;1L) = [6L(L+1)(2L+1)]^{-\frac{1}{2}} \neq 0, \quad (2.16)$$

we could restrict the summation in (2.11) to $l'=1$, $l''=L$, if $\xi(r')$ is of short range according to the definition (2.15). Furthermore, we can see from (2.12) and the form (2.4) of the radial function, that $I_{n'l'}$ can be expressed as a linear combination of $I_{l'}$, $I_{l'+1}$, $\cdots I_{l'+2n'}$. Again we should only keep $I_{l'}$ if the potential is of short range, and taking into account that $v_{n'l'}(0)=1$, the coefficient of $I_{l'}$ becomes

$$(I_{n'l'}/I_{l'})_{b\to0} = (N_{n'l'}{}^2/N_{0l'}{}^2). \quad (2.17)$$

Substituting (2.16) and (2.17) in (2.11), we obtain for a short-range potential the matrix element

$$\langle (nl)^2L\|\xi(r')\mathbf{l}'\|(nl)^2L\rangle = \hbar[(2L+1)/L(L+1)]^{\frac{1}{2}}(I_1/N_{01}{}^2) \times\sum_{n'n''}\{[\langle n'1,n''L,L|nl,nl,L\rangle]^2N_{n'1}{}^2\}, \quad (2.18)$$

where $I_1$ is given by (2.12) with $n'=0$, $l'=1$, and from (2.10), $n'$ and $n''$ are restricted to non-negative integers that satisfy the relation

$$n'+n''=2n+l-\tfrac{1}{2}(L+1). \quad (2.19)$$

It would seem at first sight that we need to determine the transformation brackets involved in (2.18). Fortunately, we shall be able to show that the velocity dependent central force discussed in I, gives in the short range approximation just the summation that appears in (2.18). We have seen in I that the interaction $\mathbf{p}'\cdot\delta(\mathbf{r}')\mathbf{p}'$ gives rise to the matrix element

$$\langle (nl)^2L\|\mathbf{p}'\cdot\delta(\mathbf{r}')\mathbf{p}'\|(nl)^2L\rangle = (\hbar^2/4\pi)A_{nl}H(l,L), \quad (2.20)$$

where $A_{nl}$ is the radial integral

$$A_{nl} = \int_0^\infty [\Re_{nl}(\nu,r)]^4 dr, \quad (2.21)$$

[14] G. Racah, Phys. Rev. **63**, 367 (1943).
[15] G. Racah, Phys. Rev. **62**, 438 (1942).
[16] Biedenharn, Blatt, and Rose, Revs. Modern Phys. **24**, 249 (1952).

and[7]

$$H(l,L)=l(l+1)(2l+1)^3 \times\sum_{L'}[\langle ll00|L'0\rangle]^2[W(llLL';1l)]^2$$
$$\equiv(l+1)(2l+3)(2l+1)^3[\langle l+1l00|L0\rangle]^2 \times[W(ll+1LL;1l)]^2. \quad (2.22)$$

The last form for $H(l,L)$ could be obtained from the first, with the help of the explicit expression[16] for the Clebsch-Gordan and Racah coefficients.

We now express the matrix element (2.20) in a representation employing the wave functions of the relative and center-of-mass coordinates $\mathbf{r}'$ and $\mathbf{r}''$. From (2.9) we see immediately that

$$\langle(nl)^2L\|\mathbf{p}'\cdot\delta(\mathbf{r}')\mathbf{p}'\|(nl)^2L\rangle = \sum_{n'l',n''l''}[\langle n'l',n''l'',L|nl,nl,L\rangle]^2 \times\langle n'l'\|\mathbf{p}'\cdot\delta(\mathbf{r}')\mathbf{p}'\|n'l'\rangle, \quad (2.23)$$

where because of the $\delta$ function we have

$$\langle n'l'\|\mathbf{p}'\cdot\delta(\mathbf{r}')\mathbf{p}'\|n'l'\rangle =\{[\mathbf{p}'\mathfrak{R}_{n'l'}Y_{l'm'}]^*\cdot[\mathbf{p}'\mathfrak{R}_{n'l'}Y_{l'm'}]\}_{r'=0}. \quad (2.24)$$

Writing $\mathbf{p}'$ as

$$\mathbf{p}'=(r')^{-2}[\mathbf{r}'(\mathbf{r}'\cdot\mathbf{p}')]-(r')^{-2}(\mathbf{r}'\times\mathbf{l}'), \quad (2.25)$$

we obtain by an analysis similar to the one given in the appendix of I, that

$$\langle n'l'\|\mathbf{p}'\cdot\delta(\mathbf{r}')\mathbf{p}'\|n'l'\rangle=\{\hbar^2(\partial\mathfrak{R}_{n'l'}/\partial r')^2|Y_{l'm'}|^2 +\hbar^2l'(l'+1)(\mathfrak{R}_{n'l'}/r')^2 \times\textstyle\sum_{mq}[\langle l'1qm|l'm'\rangle]^2|Y_{l'q}|^2\}_{r'=0}. \quad (2.26)$$

Using the explicit form of the radial wave function given in (2.4), we have

$$(\mathfrak{R}_{n'l'}/r')_{r'=0}=(\partial\mathfrak{R}_{n'l'}/\partial r')_{r'=0}=N_{n'1}\delta_{l'1}. \quad (2.27)$$

Furthermore, $|Y_{1q}(0,0)|^2$ vanishes unless $q=0$, in which case it becomes $(3/4\pi)$. We see then that the matrix element (2.26) is independent of $m'$ [as it should be, $\mathbf{p}'\cdot\delta(\mathbf{r}')\mathbf{p}'$ being a scalar], and substituting it in (2.23) we obtain

$$\langle(nl)^2L\|\mathbf{p}'\cdot\delta(\mathbf{r}')\mathbf{p}'\|(nl)^2L\rangle =(3\hbar^2/4\pi)\sum_{n'n''}\{[\langle n'1,n''L,L|nl,nl,L\rangle]^2N_{n'1}^2\}, \quad (2.28)$$

where again because of parity, $l''$ is restricted to odd values so that $l''=L$.

Comparing (2.28) and (2.18), we obtain finally

$$\langle(nl)^2L\|\xi(r')\mathbf{l}'\|(nl)^2L\rangle=\hbar[(2L+1)/L(L+1)]^{\frac{1}{2}} \times[(\pi/2)^{\frac{1}{2}}\nu^{-\frac{3}{2}}A_{nl}]I_1H(l,L), \quad (2.29)$$

where we already introduced the explicit expression[12] for $N_{01}^2$, i.e., $N_{01}^2=\frac{1}{3}(2/\pi)^{\frac{1}{2}}\nu^{\frac{5}{2}}$.

For identical nucleons in a given shell, the matrix element for a short-range spin-orbit force takes a very simple form. In the next section we shall discuss the effects of the different types of spin-orbit forces that have been proposed.

### 3. TWO-PARTICLE SPIN-ORBIT COUPLING AND NUCLEAR SHELL THEORY

From (2.6) and (2.29), the interaction energy due to a short-range spin-orbit force, for two identical nucleons in the same shell and in $LS$ coupling, becomes

$$\langle(nl)^2L,1,J|\hbar^{-1}\xi(r')\mathbf{l}'\cdot\mathbf{S}|(nl)^2L,1,J\rangle =I_1[(\pi/2)^{\frac{1}{2}}(m\omega/\hbar)^{-\frac{3}{2}}A_{nl}] \times\begin{cases}(L+1)^{-1}H(l,L) & \text{if } J=L+1\\ -L^{-1}H(l,L) & \text{if } J=L-1.\end{cases} \quad (3.1)$$

In (3.1), $A_{nl}$ stands for the radial integral (2.21) and $H(l,L)$ is given by (2.22). From the explicit expressions[16] for the Clebsch-Gordan and Racah coefficients appearing in (2.22), it can be easily seen that $H(l,L)$ satisfies the recurrence relation

$$\frac{H(l,L+2)}{H(l,L)}=\frac{L(L+2)}{(L+1)(L+3)}\,\frac{4l(l+1)-(L+1)(L+3)}{4l(l+1)-L(L+2)}, \quad (3.2)$$

where $L$ is restricted to odd values, and that for $L=1$, we have

$$H(l,1)=\tfrac{1}{2}l(l+1)(2l+1). \quad (3.3)$$

The values of $H(l,L)$ for the first shells are given in Table I of reference 7.

The $I_1$ appearing in (3.1), stands for the radial integral (2.12) where $n'=0$, $l'=1$. Introducing the explicit form (2.4) of the wave function, and the change of variable $x=(\nu/2)^{\frac{1}{2}}r'$, we obtain

$$I_1=(8/3)\pi^{-\frac{1}{2}}\int_0^\infty x^4\exp(-x^2)\xi[(\nu/2)^{-\frac{1}{2}}x]dx. \quad (3.4)$$

In both[1,2] of the proposed forms for the spin-orbit force the $\xi(r')$ is negative, and therefore $I_1$ is also negative. From (3.1) and (3.2) we see then that the level ordering induced by the spin-orbit force, would go in order of increasing $L$ when $J=L+1$ and in order of decreasing $L$ when $J=L-1$. For two-particle configurations in $jj$ coupling, and also for configurations involving more than two particles, the level ordering due to the spin-orbit force could be obtained from (3.1), with the help[7] of the $9j$ coefficients,[17] or of the fractional parentage coefficients.[14]

[17] H. Matsunobu and H. Takebe, Progr. Theoret. Phys. (Japan) 14, 589 (1955).

Let us discuss first the effect in nuclear shell theory of the spin-orbit force introduced by Gammel and Thaler.[2] To explain the polarization data in proton-proton scattering, they make use of a spin-orbit force whose radial part is a Yukawa potential with a repulsive core. The parameters of this potential are

$$V_0=7112 \text{ Mev}, \quad a=0.4125\times10^{-13} \text{ cm}, \quad b=0.27\times10^{-13} \text{ cm}, \tag{3.5}$$

where $-V_0$ is the depth and $b$ the range of the Yukawa potential, and $a$ is the range of the repulsive core. It has been shown by Bauer and Moshinsky,[8] that the effect in nuclear shell theory of a repulsive core in the interaction potential, can be taken into account by simply translating the potential by the range of the core. This implies that $\xi$ for the potential of Gammel and Thaler[2] should have the form

$$\xi=-V_0[\beta/(x+\alpha)]\exp[-(x+\alpha)/\beta], \tag{3.6}$$

where

$$x=(\nu/2)^{\frac{1}{2}}r', \quad \alpha=(\nu/2)^{\frac{1}{2}}a, \quad \beta=(\nu/2)^{\frac{1}{2}}b, \tag{3.7}$$

and $r'$ extends to the interval $0\leq r'\leq\infty$.

As in (2.15), we define a short-range potential by the condition $\beta\ll1$. When we substitute (3.6) in (3.4), we notice that the integrand contributes mainly for values $x$ of the order of $\beta$, and if $\beta\ll1$ the $\exp(-x^2)$ in (3.4) could be approximated by 1. In the short-range approximation, $I_1$ can then be written as

$$I_1\simeq-(8/3)\pi^{-\frac{1}{2}}V_0\int_0^\infty x^4[\beta/(x+\alpha)]\exp[-(x+\alpha)/\beta]dx$$
$$=-(8/3)\pi^{-\frac{1}{2}}V_0\beta^5\left\{c^4[-\mathrm{Ei}(-c)] +\exp(-c)\sum_{n=0}^{3}[(-1)^{n+1}n!c^{3-n}]\right\}, \tag{3.8}$$

where

$$-\mathrm{Ei}(-c)\equiv\int_c^\infty x^{-1}\exp(-x)dx; \quad c=(a/b). \tag{3.9}$$

To obtain the value of $\beta$ for the potential of Gammel and Thaler,[2] we need to give the separation between the levels of the harmonic oscillator. Taking $\hbar\omega\simeq10$ Mev as suggested in reference 8, we obtain

$$\beta=0.0938. \tag{3.10}$$

The condition of short range is reasonably satisfied in this case, and as from (3.5), $c=1.53$ we obtain for (3.8) the value[18]

$$I_1\simeq-0.068 \text{ Mev}. \tag{3.11}$$

[18] The value (3.11) of $I_1$ was calculated on the assumption that $\exp(-x^2)$ in (3.4) could be approximated by 1. A more correct calculation could be made if we expand $\exp(-x^2)$ in a power series, in which case we would have a series of integrals of the type (3.8) containing higher powers of $x$. Keeping the first four terms in the expansion of $\exp(-x^2)$, the integral (3.4) takes the value $|I_1|=0.058$ Mev with an error of less than 0.001 Mev. Taking into account though the approximations involved in Sec. 2, it is sufficient for the following analysis to consider $|I_1|$ given by the upper bound (3.11).

TABLE I. Interaction energy for the short-range spin-orbit force (3.1) in units of $I_1$, when $n=0$, $J=L+1$. The interaction energy for $J=L-1$ is obtained multiplying each column by $[-(L+1)/L]$.

| $l$ \ $L$ | 1 | 3 | 5 |
|---|---|---|---|
| 1 | 1/2 | | |
| 2 | 7/8 | 1/8 | |
| 3 | 99/80 | 33/160 | 1/16 |

The radial integral $A_{nl}$ that appears in (3.1) is easily evaluated. In particular for $n=0$, we see from (2.4) and (2.21) that[12]

$$(\pi/2)^{\frac{1}{2}}\nu^{-\frac{3}{2}}A_{0l}=\{[(2l+1)!!]^2 2^{2l-2}\}^{-1}[(4l-1)!!], \tag{3.12}$$

where $(2l+1)!!\equiv1\times3\times5\cdots\times(2l+1)$. From (3.12) and Table I of reference 7, we can give in Table I of the present work the interaction energy (3.1), in units of $I_1$ for a short-range spin-orbit force, where $l=1, 2, 3$ and $n=0$.

To compare the interaction energy due to the spin-orbit force, with the interaction energy due to the singlet even potential of Gammel, Christian, and Thaler,[19] we have to pass to the $jj$ coupling scheme. This is accomplished with the help of the $9j$ coefficients as shown in I. We shall only indicate here that for the two neutrons outside the closed shell in $Ca^{42}$, the interaction energy due to the spin-orbit force of Gammel and Thaler,[2] gives a separation between the levels $J=0$ and $J=2$ of only 0.1 Mev, for $\hbar\omega\simeq10$ Mev, i.e., for $I_1$ having the value (3.11). For the same $\hbar\omega$, the even singlet potential of Gammel, Christian, and Thaler,[19] would give[8] the experimentally observed separation of 1.5 Mev. It is clear therefore, that if the form of the spin orbit force of Gammel and Thaler is accepted, its effect would be small in nuclear shell theory as compared with the effect of the central force.

Let us discuss now the effect of the spin-orbit force introduced by Signell and Marshak.[1] The radial part of their force is given by the derivative of a Yukawa potential with a straight cutoff, so that $\xi(r)$ takes the form

$$\begin{aligned}\xi(r)&=-V_0(b/a)^3[1+(a/b)]\exp(-a/b) && \text{if } 0\leq r\leq a,\\ \xi(r)&=-V_0(b/r)^3[1+(r/b)]\exp(-r/b) && \text{if } a\leq r\leq\infty,\end{aligned} \tag{3.13}$$

where

$$V_0=30 \text{ Mev}, \quad a=0.21\times10^{-13} \text{ cm}, \quad b=1.07\times10^{-13} \text{ cm}. \tag{3.14}$$

If we take as before $\hbar\omega\simeq10$ Mev, we have for $\alpha$, $\beta$ defined in (3.7), and for $c=(a/b)$, the values

$$\alpha=0.073, \quad \beta=0.3715, \quad c=0.196. \tag{3.15}$$

[19] Gammel, Christian, and Thaler, Phys. Rev. 105, 311 (1957).

Comparing the $\beta$ in (3.15) with the value (3.10), we see that the short-range approximation would not be as good for the potential of Signell and Marshak[1] as it is for the potential of Gammel and Thaler.[2] Nevertheless, it is interesting to consider the order of magnitude of its effect in nuclear shell theory in the short-range approximation. We need then to calculate $I_1$ of (3.4) with the $\xi$ of (3.13). We divide the integral into two parts, one for $x$ between 0 and $\alpha$, and the other from $\alpha$ to $\infty$. In view of the small value of $\alpha$ in (3.15), we can replace $\exp(-x^2)$ in the first integral by 1, and with this assumption we obtain straightforwardly

$$I_1=-(8/3)\pi^{-\frac{1}{2}}V_0\{c^{-3}(1+c)\exp(-c)(\alpha^5/5)+\beta^2\exp(1/4\beta^2)[(4\beta^2)^{-1}(\pi^{\frac{1}{2}}/2)\operatorname{erfc}(\gamma)+\tfrac{1}{2}(\beta-\beta^{-1}+\gamma)\exp(-\gamma^2)]\}, \quad (3.16)$$

where

$$\operatorname{erfc}(\gamma)=2\pi^{-\frac{1}{2}}\int_\gamma^\infty \exp(-x^2)dx, \quad \gamma=\alpha+(2\beta)^{-1}. \quad (3.17)$$

For the values of $\alpha$, $\beta$, $c$ given in (3.15) we obtain for $I_1$

$$I_1=-0.48 \text{ Mev}. \quad (3.18)$$

Comparing this value with (3.11), we see that the effect of the potential of Signell and Marshak in nuclear shell theory is considerably larger than the effect of the potential of Gammel and Thaler. In fact, assuming the validity of the short-range approximation, the separation between the levels $J=0$ and $J=2$ in $Ca^{42}$ due to the force of Signell and Marshak would be 0.7 Mev, which is an appreciable fraction of the experimental value of 1.5 Mev.

## 4. VELOCITY-DEPENDENT TRIPLET POTENTIAL

We now turn to the potentials that depend on the second power of momentum, the simplest of which were given in the introduction. As the potential (1.1) acts only in the triplet state, we shall designate it as the velocity-dependent triplet potential. Introducing the total spin $\mathbf{S}$ of (2.2), the triplet potential becomes

$$\tfrac{1}{8}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)^2\mathbf{p}'\cdot V(r')\mathbf{p}' =\mathbf{p}'\cdot V(r')\mathbf{p}'+(\tfrac{1}{2}\mathbf{S}^2-1)\mathbf{p}'\cdot V(r')\mathbf{p}'. \quad (4.1)$$

The interaction energy for a two-particle configuration in $jj$ coupling can again be written in terms of the interaction energy in $LS$ coupling with the help of the transformation bracket[20] (3.2 I). As $\frac{1}{2}\mathbf{S}^2-1$ is 0 in the triplet state, and $-1$ in the singlet state, we obtain from the form (3.8 I) of the transformation bracket for $S=0$, that the interaction energy becomes

$$\langle(nlj)^2JM|\tfrac{1}{2}\mathbf{S}^2\mathbf{p}'\cdot V(r')\mathbf{p}'|(nlj)^2JM\rangle =\langle(nlj)^2JM|\mathbf{p}'\cdot V(r')\mathbf{p}'|(nlj)^2JM\rangle -\tfrac{1}{2}(2l+1)^{-2}[(2l+1)(2j+1)-J(J+1)] \times\langle(nl)^2JM|\mathbf{p}'\cdot V(r')\mathbf{p}'|(nl)^2JM\rangle. \quad (4.2)$$

[20] Equations of reference 7 will be indicated by I following the number of the equation.

From (4.2) we see that the interaction energy associated with the potential (4.1), can be expressed as a linear combination of the matrix element in $jj$ coupling and in the singlet state of $LS$ coupling of the velocity-dependent central potential discussed in I.

As in I, we express the $V(r')$ in (4.1) in terms of the Gaussian potential,

$$V(r')=M_0^{-1}\exp[-b^{-2}(\mathbf{r}_1-\mathbf{r}_2)^2]. \quad (4.3)$$

For the long-range approximation we have, from the last term in (4.2), that the interaction energy will depend on the total angular momentum $J$, even in the limit $b\to\infty$. Taking into account that the matrix element of $\mathbf{p}_1\cdot\mathbf{p}_2$ is zero because of parity, and extending the analysis to $\lambda$ identical particles in the shell by a procedure similar to that outlined in I, we obtain in the long-range approximation:

$$\langle(nlj)^\lambda JM|\tfrac{1}{2}\mathbf{S}^2\mathbf{p}'\cdot V(r')\mathbf{p}'|(nlj)^\lambda JM\rangle =(2M_0)^{-1}\langle p^2\rangle\{\tfrac{1}{2}\lambda(\lambda-1) -\tfrac{1}{4}\lambda(\lambda-1)(2l+1)^{-2}[(2l+1)(2j+1)-2j(j+1)] +\tfrac{1}{2}(2l+1)^{-2}[J(J+1)-\lambda j(j+1)]\}, \quad (4.4)$$

where as in (2.21 I), $\langle p^2\rangle$ stands for the single-particle expectation value of the square of the momentum.

From (4.4) we see that the velocity-dependent triplet potential will give, in the long-range approximation, a level arrangement that goes in the order of increasing (decreasing) values of the total angular momentum $J$ if $M_0$ is positive (negative).

In the short-range approximation we must distinguish in the interaction energy (4.2) between states of isotopic spin 1 and 0. In the first case, $J$ is restricted to even values as the wave function is antisymmetric, and as shown in the appendix of I, the last matrix element in (4.2) is zero. For isotopic spin $T=1$, the interaction energy in the short-range approximation for a velocity dependent triplet potential is then identical to the one given by (3.5 I) and Table II of I. For isotopic spin $T=0$, $J$ is restricted to odd values as the wave function is symmetric, and the first matrix element in (4.2) is given by (3.7 I), so from (4.2) we see that the interaction energy is zero.

For the case of identical particles in a given shell in the short-range approximation, the same restrictions on strength and range that were obtained for the velocity-dependent central potential, hold for the velocity-dependent triplet potential (4.1).

## 5. VELOCITY-DEPENDENT TENSOR POTENTIAL IN THE LONG-RANGE APPROXIMATION

The $V_T(r')$ in (1.3) will also be taken in the form (4.3), replacing only $M_0$ by $M_T$. In the long-range limit $b\to\infty$, $V_T(r')$ reduces to $M_T^{-1}$ and replacing $\mathbf{p}'$ by its value (1.4), we see that because of parity considerations, the terms containing double products $p_{1i}p_{2j}$ give no contribution to the interaction energy. Because of the symmetry of the matrix element under the interchange of the two particles, we obtain for the interaction energy in the long-range approximation:

$$\begin{aligned}E(2,nlj,J)&\equiv\langle(nlj)^2JM|-\tfrac{1}{2}V_T(r')[3(\boldsymbol{\sigma}_1\cdot\mathbf{p}')(\boldsymbol{\sigma}_2\cdot\mathbf{p}')-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)p'^2]|(nlj)^2JM\rangle\\&=\langle(nlj)^2JM|-(4M_T)^{-1}[3(\boldsymbol{\sigma}_1\cdot\mathbf{p}_1)\mathbf{p}_1-\boldsymbol{\sigma}_1p_1^2]\cdot\boldsymbol{\sigma}_2|(nlj)^2JM\rangle\\&=-(4M_T)^{-1}(-1)^{2j-J}\langle nlj\|3(\boldsymbol{\sigma}_1\cdot\mathbf{p}_1)\mathbf{p}_1-\boldsymbol{\sigma}_1p_1^2\|nlj\rangle\\&\qquad\times\langle nlj\|\boldsymbol{\sigma}_2\|nlj\rangle W(jjjj;J1).\end{aligned}\quad(5.1)$$

The interaction energy $E$ is given as a function of the number of particles (2 in this case), the shell quantum numbers, and the quantum numbers necessary to specify the state, in this case only the total angular momentum $J$. The matrix element of $\boldsymbol{\sigma}_2$ can be evaluated by the standard methods of Racah.[15] For the first matrix element in (5.1), we express the spherical components of the vector in the form:

$$[3(\boldsymbol{\sigma}_1\cdot\mathbf{p}_1)\mathbf{p}_1-\boldsymbol{\sigma}_1p_1^2]_m=\sum_{\mu,m'}\langle 21\mu m'|1m\rangle Q_\mu^2\sigma_{1m'},\quad(5.2)$$

where the $\mu=0$ component of the second-order Racah tensor[15] $Q_\mu^2$ is given by

$$Q_0^2=-(5/2)^{\frac{1}{2}}[3p_{1z}^2-p_1^2].\quad(5.3)$$

Using the methods of Jahn and Hope[21] for the evaluation of tensor products of Racah tensors, we obtain:

$$\langle nl\tfrac{1}{2}j\|3(\boldsymbol{\sigma}_1\cdot\mathbf{p}_1)\mathbf{p}_1-\boldsymbol{\sigma}_1p_1^2\|nl\tfrac{1}{2}j\rangle=(2j+1)(\sqrt{3})\langle nl\|Q^2\|nl\rangle\langle\tfrac{1}{2}\|\sigma\|\tfrac{1}{2}\rangle U\begin{Bmatrix}l&l&2\\j&j&1\\\tfrac{1}{2}&\tfrac{1}{2}&1\end{Bmatrix},\quad(5.4)$$

where $U$ is the $9j$ symbol of Wigner in the notation of Matsunobu and Takebe.[17]

For the matrix element of $Q^2$ we have:

$$\langle nl\|-(5/2)^{\frac{1}{2}}[3p_{1z}^2-p_1^2]\|nl\rangle=\langle p^2\rangle(10)^{\frac{1}{2}}\left[\frac{l(l+1)(2l+1)}{(2l-1)(2l+3)}\right]^{\frac{1}{2}},\quad(5.5)$$

where $\langle p^2\rangle$ is the single-particle expectation value of $p^2$ as in (2.21 I). This result can be obtained straightforwardly if we consider the expectation value of (5.3) in momentum space.

From (5.4), (5.5), and the well-known relation

$$\langle\tfrac{1}{2}\|\sigma\|\tfrac{1}{2}\rangle=\sqrt{6},\quad(5.6)$$

we can obtain the energy $E$ for the two-particle configuration. As the dependence of this energy on $J$ takes place only through the Racah coefficient in (5.1) given by[16]

$$W(jjjj;1J)=(-1)^{2j-J-1}[(2j)(j+1)(2j+1)]^{-1}\times[2j(j+1)-J(J+1)],\quad(5.7)$$

[21] H. A. Jahn and J. Hope, Phys. Rev. **93**, 318 (1954).

we can immediately generalize (5.1) to the $\lambda$ particle configuration as done in Sec. 2 of I, and in the long-range approximation we obtain, after some simplification,

$$E(\lambda,nlj,J)=(4M_T)^{-1}\langle p^2\rangle[(2j)(j+1)]^{-1}[1\pm(2l+1)^{-1}]\times[J(J+1)-\lambda j(j+1)],\quad\text{for}\quad l=j\pm\tfrac{1}{2}.\quad(5.8)$$

As in the previous sections, we take for the common potential of the nucleons in the nucleus a harmonic oscillator of frequency $\omega$. In this case, the first term in (5.8) becomes

$$(4M_T)^{-1}\langle p^2\rangle=\tfrac{1}{4}(m/M_T)\hbar\omega(2n+l+\tfrac{3}{2}),\quad(5.9)$$

where $m$ is the mass of the nucleon.

From (5.8) and (5.9) we see that the ordering of the levels due to a long-range velocity-dependent tensor potential, goes with increasing (decreasing) $J$ if $M_T$ is positive (negative), and that the separation between successive levels is of the order of $(m/M_T)\hbar\omega$. As the energy differences between levels in a given shell should be smaller than the separation $\hbar\omega$ between shells, we obtain for a long-range velocity-dependent tensor potential the restriction $(m/M_T)\ll 1$.

The long-range approximation for velocity-dependent tensor forces, will be valid if the range $b$ of the potential (4.3) is small compared with the radius of the nucleus. Therefore, the long-range approximation will hold at most for light nuclei. For the medium and heavy nuclei we shall use rather the short-range approximation developed in the following section.

## 6. VELOCITY-DEPENDENT TENSOR POTENTIAL IN THE SHORT-RANGE APPROXIMATION

We first rewrite the tensor potential (1.3) by a procedure similar to the one discussed in Sec. 2 of I, so as to obtain

$$-V_T(r')[3(\mathbf{S}\cdot\mathbf{p}')^2-S^2p'^2]\equiv-V_T(r')\mathbf{T}^2\cdot\mathbf{X}^2,\quad(6.1)$$

where the $\mu=0$ component of the Racah tensors $\mathbf{T}^2$, $\mathbf{X}^2$ is given by

$$T_0^2=(1/\sqrt{2})(3p_z'^2-p'^2),\quad X_0^2=(1/\sqrt{2})(3S_z^2-S^2).\quad(6.2)$$

In the short-range approximation, $V_T(r')$ could be represented by the $\delta$-function potential

$$V_T(r')=M_T^{-1}\pi^{\frac{3}{2}}b^3\delta(\mathbf{r}_1-\mathbf{r}_2),\quad(6.3)$$

so that $-V_T(r')T_0^2$, when properly made Hermitian,[10] becomes

$$-V_T(r')T_0^2=-(\sqrt{2}M_T)^{-1}\pi^{\frac{3}{2}}b^3[3p_z'\delta(\mathbf{r}_1-\mathbf{r}_2)p_z'-\mathbf{p}'\cdot\delta(\mathbf{r}_1-\mathbf{r}_2)\mathbf{p}'].\quad(6.4)$$

Because of the $\delta$ function appearing in (6.4), it is convenient to write the two-particle interaction energy in $jj$ coupling in terms of the matrix elements in $LS$ coupling, and from (6.1) and the discussion in Sec. 3 of

TABLE II. The two-particle interaction energy (6.8) for the short-range velocity-dependent tensor potential. The unit is $[-(m/M_T)\hbar\omega\beta^3]$ and the radial quantum number $n=0$.

| $l_j \backslash J$ | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $p_{1/2}$ | 5/3 | | | |
| $p_{3/2}$ | 5/6 | 1/6 | | |
| $d_{3/2}$ | 21/8 | 21/40 | | |
| $d_{5/2}$ | 7/4 | 1/4 | 1/4 | |
| $f_{5/2}$ | 99/28 | 2871/3920 | 297/784 | |
| $f_{7/2}$ | 297/112 | 297/784 | 255/784 | 27/112 |

I, we obtain

$$E(2,nlj,J)=3(2j+1)^2\sum_{L,L'}[(2L+1)(2L'+1)]^{\frac{1}{2}}$$
$$\times U\begin{bmatrix} l & l & L \\ j & j & J \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} U\begin{bmatrix} l & l & L' \\ j & j & J \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}(-1)^{L+1-J}$$
$$\times\langle (nl)^2L\|-V_T(r')T^2\|(nl)^2L'\rangle\langle 1\|X^2\|1\rangle$$
$$\times W(L1L'1;J2). \quad (6.5)$$

For the matrix element of $V_T(r')T_0{}^2$, the analysis is carried out in the Appendix, where it is shown that the matrix is diagonal in $L$ and different from zero only for odd $L$, becoming

$$\langle (nl)^2L\|-V_T(r')T^2\|(nl)^2L\rangle$$
$$=-(\sqrt{2}M_T)^{-1}(\hbar^2/4\pi)\pi^{\frac{3}{2}}b^3[(2L-1)(2L+1)(2L+3)]^{\frac{1}{2}}$$
$$\times[L(L+1)]^{-\frac{1}{2}}A_{nl}H(l,L);\quad L \text{ odd},\quad (6.6)$$

where $A_{nl}$ is the radial integral (2.21), and $H(l,L)$ is given either by (2.22) or by the recurrence relations (3.2) and (3.3).

Using the methods of Racah,[15] we have from (6.2) that

$$\langle 1\|X^2\|1\rangle=(15)^{\frac{1}{2}}. \quad (6.7)$$

Substituting (6.6) and (6.7) into (6.5), and making use of the explicit expressions for the Racah[16] and $U$ coefficients,[17] we obtain, after simple reductions

$$E(2,nlj,J)=-(m/M_T)(\hbar\omega)\beta^3[(\pi/2)^{\frac{1}{2}}\nu^{-\frac{5}{2}}A_{nl}]$$
$$\times\{[(2J+5)/(2J+2)]H(l,j,J)-\tfrac{3}{4}[(J+1)(2l+1)^2]^{-1}$$
$$\times[(2j+1\mp J)(2l+1\mp J)]H(l,J-1)\}. \quad (6.8)$$

In (6.8) the minus sign is used if $l=j+\frac{1}{2}$ and the plus sign, if $l=j-\frac{1}{2}$, and $J$ is restricted to even values. The coefficient $H(l,j,J)$ is defined by (3.6 I), and, as in Sec. 3, $\beta$ is

$$\beta=(\nu/2)^{\frac{1}{2}}b=(m\omega/2\hbar)^{\frac{1}{2}}b. \quad (6.9)$$

The coefficients $H(l,J-1)$ and $H(l,j,J)$ are given in Tables I and II, respectively, of reference 7. For $n=0$ the first square bracket in (6.8) has the explicit form (3.12). We can then give in Table II the values of the interaction energy for the two particle configuration in units of $[-(m/M_T)(\hbar\omega)\beta^3]$.

TABLE III. The three-particle interaction energy (6.10) for the short-range velocity-dependent tensor potential. The unit is $[-(m/M_T)\hbar\omega\beta^3]$ and the radial quantum number $n=0$.

| $l_j \backslash J$ | 3/2 | 5/2 | 9/2 |
|---|---|---|---|
| $d_{5/2}$ | 3/4 | 7/4 | 3/4 |
| $f_{5/2}$ | 1485/784 | 99/28 | 2673/1960 |

For the three-particle configurations, the interaction energy $E(3,nlj,J)$ could be obtained from (6.8) with the help of the fractional parentage coefficients, in the form[7]

$$E(3,nlj,J)=\sum_{J'}3[\langle (j)^2J',j,J|\}(j)^3J\rangle]^2E(2,nlj,J'). \quad (6.10)$$

From Table IV of reference 7, we obtain in Table III of this paper the interaction energy for a three-particle configuration in the $j=\frac{5}{2}$ shells, in the same units as in Table II.

Looking at the interaction energies for short-range velocity-dependent tensor forces given in Table II, we notice that for $M_T$ positive (negative) the level ordering for a two-particle configuration goes with increasing (decreasing) $J$, just as in the case of the long-range approximation discussed in the previous section. For the three-particle configuration in the $j=\frac{5}{2}$ shells, we see from Table III that, for $M_T>0$ the lowest state in the short-range approximation is $J=\frac{5}{2}$, while for the long-range approximation, it would be $J=\frac{3}{2}$.

From Tables II and III, we also see that the interaction energy due to a short-range velocity-dependent force increases with $l$. This is also observed for the interaction energy of the short-range spin-orbit force given in Table I. This characteristic is reasonable for velocity-dependent forces, as we expect a stronger interaction for states of higher angular momentum and kinetic energy. In contrast, for ordinary short-range forces, the interaction energy decreases[22] with increasing $l$.

From Tables II and III we obtain restrictions on the magnitude of the product $(m/M_T)\beta^3$, as the velocity-dependent tensor force should not be strong enough to give by itself the observed separation between levels, in case $M_T>0$, or to invert the normal level ordering, in case $M_T<0$. For example, for two identical nucleons in the $n=0$, $f_{7/2}$ shell, we see from Table II that the separation between levels $J=0$ and $J=2$ due to the velocity-dependent tensor force is

$$2.27(m/M_T)\beta^3(\hbar\omega). \quad (6.11)$$

Considering the case of $Ca^{42}$, where the separation between the levels[22] is 1.51 Mev, and taking as before

[22] N. Zeldes, Nuclear Phys. 2, 1 (1956).

$\hbar\omega \simeq 10$ Mev, we obtain the restriction

$$(m/M_T)\beta^3 < 0.066. \qquad (6.12)$$

Any values for $M_T$ and $b$ in (6.3) that would be consistent with the two-body data, would also have to be tested in connection with inequalities like the above.

It is of interest to notice from (3.1), (6.6), as well as from (3.3 I), that in the short-range approximation the spin-orbit, velocity-dependent tensor, and velocity-dependent central forces contain the same factor $H(l,L)$, which is given by the recurrence relation (3.2) and (3.3). For ordinary short-range forces, we have a similar factor[7] $H'(j,J)$ given by the recurrence relation (5.4 I). This suggests the possibility that for all types of short-range forces, we could express the interaction energy in terms of simple closed expressions.

## APPENDIX

The matrix element (6.6) could be obtained from the expectation value of the operator (6.4) since

$$\langle (nl)^2LM| -V_T(r')T_0^2|(nl)^2L'M\rangle = \langle (nl)^2L\| -V_T(r')T^2\|(nl)^2L'\rangle(2L+1)^{-\frac{1}{2}}\langle L'2M0|LM\rangle$$

$$= -(\sqrt{2}M_T)^{-1}\pi^{\frac{3}{2}}b^3\int\{3(p_z\psi)^*_{1=2}(p_z\psi')_{1=2} - (\mathbf{p}\psi)^*_{1=2}\cdot(\mathbf{p}\psi')_{1=2}\}d\tau, \qquad (A.1)$$

where $(\mathbf{p}\psi)_{1=2}$ is given, as in the Appendix of I, by

$$(\mathbf{p}\psi)_{1=2} = \sum_{m_1m_2}\langle llm_1m_2|LM\rangle[\mathbf{p}\mathfrak{R}_{nl}(r)Y_{lm_1}(\theta,\varphi)][\mathfrak{R}_{nl}(r)Y_{lm_2}(\theta,\varphi)], \qquad (A.2)$$

and $L$ is restricted to odd values.

The $z$ component of $\mathbf{p}$ has the form

$$p_z = (\hbar/i)[\cos\theta(\partial/\partial r) - r^{-1}\sin\theta(\partial/\partial\theta)], \qquad (A.3)$$

and because $L$ is odd, only the second term in (A.3) contributes to (A.2), so that we can write

$$(p_z\psi)_{1=2} = i\hbar r^{-1}[\mathfrak{R}_{nl}(r)]^2\{\sum_{m_1m_2}\langle llm_1m_2|LM\rangle[\sin\theta(\partial Y_{lm_1}/\partial\theta)]Y_{lm_2}\}. \qquad (A.4)$$

The expression inside the curly brackets can be developed in spherical harmonics by the standard methods of Racah,[15] and we obtain

$$(p_z\psi)_{1=2} = i\hbar r^{-1}[\mathfrak{R}_{nl}(r)]^2(4\pi)^{-\frac{1}{2}}[(2l+1)^3(l+1)(2l+3)]^{\frac{1}{2}}\sum_{L''}\{(2L+1)^{\frac{1}{2}}(2L''+1)^{\frac{1}{2}}\langle l+1l00|L''0\rangle$$
$$\times W(ll+1LL'';1l)\langle 1L0M|L''M\rangle Y_{L''M}(\theta,\varphi)\}. \qquad (A.5)$$

In the sum appearing in (A.5), $L''$ is restricted to odd values because the first Clebsch-Gordan coefficient requires $2l+1+L''$ to be even. Furthermore, from the Racah coefficient $L''$ is restricted to $L''=L$, $L\pm1$. As $L$ is odd, we can have only $L''=L$ and the summation reduces to one term. Clearly then, the matrix element in (A.1) will be zero if $L'\neq L$.

Taking (A.5), and the results in the Appendix of I, we obtain from (A.1) that

$$\langle (nl)^2L\| -V_T(r')T^2\|(nl)^2L\rangle = -(\sqrt{2}M_T)^{-1}(\hbar^2/4\pi)\pi^{\frac{3}{2}}b^3A_{nl}[(2L-1)(2L+1)(2L+3)]^{\frac{1}{2}}[L(L+1)]^{-\frac{1}{2}}$$
$$\times(l+1)(2l+3)(2l+1)^3[\langle l+1l00|L0\rangle]^2[W(ll+1LL;1l)]^2, \qquad (A.6)$$

where $A_{nl}$ is the radial integral given in (2.21).