Statistical Broadening of Spectral Lines Emitted by Ions in a Plasma*

M. LEWIS AND H. MARGENAU
Yale University, New Haven, Connecticut
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The Holtsmark theory of line broadening omits the factor $e^{-V_{ij}/kT}$ in the evaluation of probabilities, V_{ij} being the Coulomb interaction between the ith and the jth ions. This leads to serious errors in the wings of a line where frequency shifts arise from close encounters which are inhibited by the Boltzmann factor. In part I of the present article a consistent binary approximation to the line shape is given, the assumption being made that only a single perturber is involved in a collision. Part II treats the case of many particles, retains V_{ij} for all perturbers j interacting with a single radiating ion i, but ignores forces between the perturbers themselves. Comparison is made with other treatments.

THE Holtsmark theory of line broadening involves the probability $W(F)$ that the plasma will produce an electric field of magnitude F at the radiating *ion*. $W(F)$ has been calculated by Holtsmark with neglect of the interaction between the ions. This is a good approximation for high temperatures, low density, and small ionic charge, but holds in general only for relatively small values of F.

Methods[1–3] have recently been proposed to permit inclusion of the effect of the interaction between the ions. Mayer[1] employs a single-encounter theory for large fields and assumes, for small fields, among other conditions, that the ions are attracted to their equilibrium positions by a force proportional to their displacement. Broyles[2] uses the method of Bohm and Pines to divide the Coulomb field into a short- and a long-range component. The former acts like a system of particles and the latter like a system of waves. He evaluates the particle part using two different approximations. The first, (a), considers interactions *only* between the radiating ion and the plasma ions while the second, (b), introduces the field but limits interaction to the nearest neighbor. Ecker,[3] following Holtsmark, keeps the statistical independence of the ions but replaces the Coulomb field by a Debye field.

In part I of this note we sketch a fairly obvious generalization of the simple one-perturber form of the Holtsmark theory, compare it with the latter in the range of distances over which both are valid, and draw some conclusions as to the importance of the corrections required. This analysis follows a well-known procedure outlined, for example, in Unsöld's book.[4]

In part II we treat the many-body problem using the following model: the free electrons form a uniformly smeared out negative charge and interactions are considered only between the radiating ion and the plasma ions; we thus ignore interactions between the plasma ions themselves. This model is similar to Broyles' approximation (a) except that we apply it to the actual Coulomb interactions of the ions rather than to the short-range forces that arise in the Bohm-Pines method. The advantage of our method, we believe, is that it avoids some uncertainties inherent in the approximations attending the use of Bohm-Pines' approach, for our approximations are clear in their physical meaning and the analysis is otherwise exact. The limitation of the present work is, of course, its failure to be applicable to radiating *atoms*; here our results reduce to Holtsmark's.

I.

Let $\bar{P}(r)$ be the probability that there shall be no particle within a sphere of radius r about the radiating ion. The probability that there be no particle in a small spherical shell between r and $r+dr$ is known; it is $1-4\pi Ce^{-a/r}r^2dr$ provided that

$$4\pi\int_0^R Ce^{-a/r}r^2dr=N. \tag{1}$$

Here $a=Q_1Q_2/kT$, Q_1 is the charge on the radiating ion, Q_2 is that on the perturbing ion, N= the total number of perturbing ions present, and R= the radius of the sphere containing them; k is the Boltzmann constant and T the temperature.

Unless they are equal, we assume that the perturbers greatly outnumber the radiators. From (1),

$$C=\frac{N}{4\pi A(a,R)},\quad A(a,r)=\int_0^r e^{-a/r}r^2dr. \tag{2}$$

For very large R, $A(a,R)=R^3/3$, and $C\doteq n$, the number of ions per unit volume.

By the law of combining probabilities

$$\bar{P}(r+dr)=\bar{P}(r)(1-4\pi Ce^{-a/r}r^2dr),$$

or

$$\bar{P}'=-\bar{P}4\pi Ce^{-a/r}r^2,\quad \bar{P}=e^{-4\pi CA(a,r)}.$$

The probability that there be one or more ions in the sphere is

$$P(r)=1-\bar{P},$$

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[1] H. Mayer, Los Alamos Scientific Laboratory Report LA-647, 1947 (unpublished).
[2] A. A. Broyles, Phys. Rev. **100**, 1181 (1955); A. A. Broyles, Phys. Rev. **105**, 347 (1957); A. A. Broyles, Atomic Energy Report RM-1682 (unpublished).
[3] G. Ecker, Z. Physik **148**, 593 (1957).
[4] A. Unsöld, *Physik der Sternatmospharen* (Springer-Verlag, Berlin, 1955), second edition.

and

$$dP(r)=4\pi C\exp[-a/r-4\pi CA(a,r)]r^2dr. \quad (3)$$

This is normalized in the sense that

$$\int_0^R dP(r)=(1-e^{-4\pi CA(a,r)})\Big|_0^R=1-e^{-N}\rightarrow 1.$$

Let us refer to (3) as $dP(a,r)$. Holtsmark's distribution is then $dP(0,r)$. The ratio of the two,

$$S\equiv\frac{dP(a,r)}{dP(0,r)}=\exp\left[-\frac{a}{r}+4\pi n\left(\frac{r^3}{3}-A(a,r)\right)\right]. \quad (4)$$

An integration by parts yields

$$A(a,r)=\tfrac{1}{3}e^{-a/r}(r^3-\tfrac{1}{2}ar^2+\tfrac{1}{2}a^2r)+\tfrac{1}{6}a^3\,\mathrm{Ei}(-a/r),$$

where

$$\mathrm{Ei}(-x)\equiv-\int_x^\infty\frac{e^{-t}}{t}dt.$$

If we introduce a variable $\xi\equiv r/a$, then

$$A(a,\xi a)=\tfrac{1}{3}a^3F(\xi),$$

and

$$F(\xi)=(\xi^3-\tfrac{1}{2}\xi^2+\tfrac{1}{2}\xi)e^{-1/\xi}+\tfrac{1}{2}\,\mathrm{Ei}(-1/\xi). \quad (5)$$

$F(\xi)$ is plotted in Fig. 1.

In order that the nearest-neighbor approximation shall be a valid approach to the Holtsmark distribution, the quantity $\beta=(r_0/r)^2$ must be greater than 7 (see Unsöld[4]), r_0 being given by $0.62n^{-\frac{1}{3}}$. The present results, then, begin to be useful at values of r in the neighborhood of $\frac{1}{4}n^{-\frac{1}{3}}$ and remain so for smaller radii. At this critical r_c, Eq. (4) becomes

$$S=\exp\left[-4an^{\frac{1}{3}}+\frac{4\pi}{192}-\frac{4\pi}{3}na^3F\left(\frac{1}{4an^{\frac{1}{3}}}\right)\right], \quad (6)$$

in view of Eq. (5). The terms in the exponent are appreciable when $4an^{\frac{1}{3}}\geqslant 0.1$. If the equality sign holds, then

$$\ln S=-0.1+0.065-0.055=-0.09,$$

the value of $F(10)$ being 845. For $4an^{\frac{1}{3}}=1$, we have

$$\ln S=-1+0.065-0.017=-0.952,$$

and for larger values of $4an^{\frac{1}{3}}$ only the term $\exp(-4an^{\frac{1}{3}})$ of S remains important.

In the light of these numerical results, we consider two cases. Of astronomical interest is the perturbation of singly ionized helium atoms by protons. Here $Q_1=Q_2=e$. The condition that our nearest-neighbor approximation be significant is that $r<0.62n^{-\frac{1}{3}}$. On the other hand, r must be greater than the size of a helium ion, $r>5\times10^{-9}$ cm. Combining these inequalities we see that $n<10^{24}$ cm^{-3} is the criterion for the existence of a useful range for the single-particle approximation. This, then, implies no physical limitation at all.

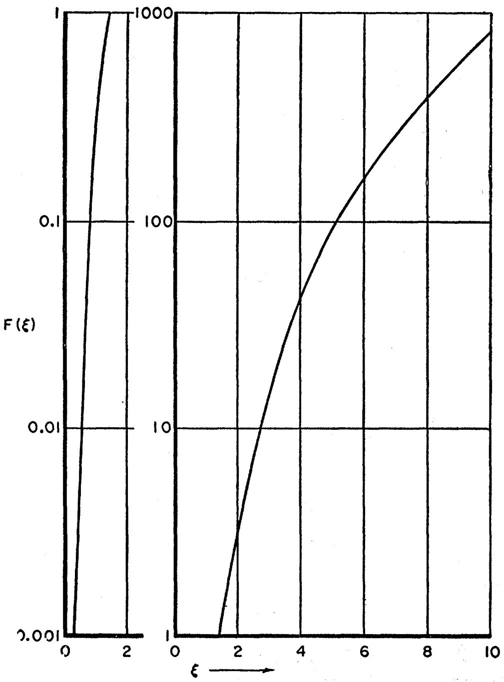

FIG. 1. Graph of $F(\xi)$.

But to have S appreciably smaller than 1, $an^{\frac{1}{3}}$ must be in the neighborhood of 1. This means that $T\approx(e^2/k)n^{\frac{1}{3}}=1.7\times10^{-3}n^{\frac{1}{3}}$ °K and is probably never of interest in astrophysical situations. In a plasma, with a temperature of 1000°K, n has to be of the order 10^{17} cm^{-3} before the correction here computed is important.

The situation is different for conditions of explosions treated in references 1 and 2. An atmosphere of iron ions, each with $Q=23e$, and with $T=10^7$ °K, has $a=8.4\times10^{-8}$ cm. At normal density, $n\approx10^{23}$ cm^{-3}, and the condition upon r is $r<10^{-8}$ cm. Because of the small size of the iron residue, this leaves a considerable range in which the present one-perturber theory is valid. Indeed, at the critical r_c for which formula (6) has been computed, $S\approx10^{-15}$, indicating that the error in the Holtsmark formula is enormous. For smaller r (larger "frequency distances" from the line center), S is accurately given by $e^{-a/r}$. For larger r, a more elaborate analysis of the sort given in reference 2 and in part II of this paper becomes necessary.

II.

If the radiating ion is placed at the origin and there are N ions in a volume V interacting with it, the probability $W_N(\mathbf{F})$ that the field at the origin be $\mathbf{F}$ is given by

$$W_N(\mathbf{F})=\int_V\cdots\int W_N(\mathbf{r}_1\mathbf{r}_2\cdots\mathbf{r}_N)\times\delta\left(\mathbf{F}-\sum_{i=1}^N\mathbf{F}_i\right)\prod_{j=1}^N d\mathbf{r}_j, \quad (7)$$

where $W_N(\mathbf{r}_1\mathbf{r}_2\cdots\mathbf{r}_N)$ is the probability density of finding the N ions in the position $\mathbf{r}_1\cdots\mathbf{r}_N$. In our model, $W_N(\mathbf{r}_1\cdots\mathbf{r}_N)$ is

$$W_N(\mathbf{r}_1\cdots\mathbf{r}_N)=K(V)\exp(-a\textstyle\sum_i r_i^{-1}), \tag{8}$$

where $K(V)$ is a normalization constant. In Eq. (7), δ is the Dirac function and $\mathbf{F}_i$ is the field produced at the origin by the ith ion:

$$\mathbf{F}_i=Q_2\mathbf{r}_i/r_i^3. \tag{9}$$

Equation (7) can be written in a more convenient form. It is best obtained through use of a method given by Chandrasekhar.[5] The results can be summarized as follows.

Denote by $H(s)$ the probability of finding a field of strength s where the dimensionless quantity $s=Fb^{-\frac{2}{3}}$

$$b=(4/15)(2\pi Q_2)^{\frac{3}{2}}n,$$

i.e., $b^{\frac{2}{3}}$ is essentially the field produced on one particle by an ion at the average spacing between ions, and n=number density of ions. We then find[6]

$$H(s)=\frac{2}{\pi s}\int_0^\infty \exp\left[-\left(\frac{x}{s}\right)^{\frac{3}{2}}\eta\right]x\sin x dx, \tag{10}$$

$$\eta=\frac{15}{8}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty dz z^{-7/2}(z-\sin z)\times\exp\left[-\beta z^{\frac{1}{2}}\left(\frac{s}{x}\right)^{\frac{1}{2}}\right], \tag{11}$$

where

$$\beta\equiv(Q_1Q_2^{\frac{1}{2}}/kT)b^{\frac{1}{3}}.$$

It is seen that, as $T\to\infty$, $\eta\to 1$, and Eq. (10) reduces to the Holtsmark distribution.[5] $H(s)$ can be computed directly from Eqs. (10) and (11). However, for certain ranges of β ($\beta\ll 1$ and $\beta\gg 1$) the form of $H(s)$ can be simplified. We shall now obtain expansions for $H(s)$ valid in these two ranges of β and also for the limiting case of finite β and large s.

Limit of $s\to\infty$

It will first be shown that in the limit $s\to\infty$ Eq. (10) reduces to the same limit as the binary theory given in part I. The integral in Eq. (10) can be written as

$$\operatorname{Im}\int_0^\infty \exp\left[-\left(\frac{x}{s}\right)^{\frac{3}{2}}\eta+ix\right]xdx=\operatorname{Im}\int_0^\infty\sum_{n=0}^\infty\frac{1}{n!}\left(\frac{x}{s}\right)^{3n/2}\eta^n(-1)^n xe^{ix}dx.$$

The first term in the expansion is zero; the second term is $\frac{1}{2}\pi cs^{-\frac{5}{2}}e^{-\gamma}$, where $c=(2/\pi)^{\frac{1}{2}}(15/8)$; $\gamma=\beta s^{\frac{1}{2}}$, and the third term for $\gamma\gg 1$ is $c^2\pi\gamma e^{-\gamma}/s^3 4!$ Therefore

$$H(s)=cs^{-\frac{5}{2}}e^{-\gamma}[1+(c/12)(\beta/s)+\cdots], \tag{12}$$

valid for $s\gg\beta$; $\gamma\equiv\beta s^{\frac{1}{2}}\gg 1$. The leading term of Eq. (12) as $s\to\infty$ agrees with the leading term of Eq. (3) in the same limit.

Case 1: $\beta\ll 1$; $\gamma\equiv\beta s^{\frac{1}{2}}\ll 1$

For this case we expand Eq. (5) and, denoting x/s by y, we obtain a series in β:

$$\eta=\frac{15}{8}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty dz z^{-7/2}(z-\sin z)\times\sum_{n=0}^\infty\frac{(-1)^n\beta^n y^{-n/2}z^{n/2}}{n!}, \tag{13}$$

which gives

$$\eta=1-c_1\beta y^{-\frac{1}{2}}+c_2\beta^2 y^{-1}+\cdots, \tag{14}$$

TABLE I. Coefficients of the first two terms of Eq. (15). Values of $(2/\pi s)I_0(s)$ taken from reference 5.

s	$(2/\pi s)I_0(s)$	$(2/\pi s)c_1I_1(s)$
0.1	0.004225	0.00745
0.6	0.129598	0.21264
1.0	0.271322	0.3860
2.0	0.33918	0.1791
3	0.176	−0.08707
6	0.02417	−0.0507
8	0.01038	−0.0273
10	0.00556	−0.0168

where $c_1=\frac{15}{16}(\pi/2)^{\frac{1}{2}}$, $c_2=5/4$. Equation (10) becomes

$$H(s)=\frac{2}{\pi s}\left[I_0(s)+\beta c_1 I_1(s)+\beta^2\left(\frac{c_1^2}{2}I_2(s)-c_2 I_{\frac{1}{2}}(s)\right)\cdots\right], \tag{15}$$

where

$$I_p(s)=\int_0^\infty\exp\left[-\left(\frac{x}{s}\right)^{\frac{3}{2}}\right]x\sin x\left(\frac{x}{s}\right)^p dx. \tag{16}$$

It can be seen that $(2/\pi s)I_0(s)$ is the Holtsmark distribution to which Eq. (15) leads in the limit $T\to\infty$.

The integral of Eq. (16) can be expanded in a series and easily evaluated for $s\leq 3$ and $s\geq 6$. The range from 3 to 6 is difficult because of the slow convergence of the series. The coefficients for the first two terms in Eq. (15) are tabulated in Table I.

Case 2: $\beta\gg 1$; $s\ll\beta$

For this case we treat Eq. (11) in a different manner. If we put $x/s=y$ and $\beta z^{\frac{1}{2}}y^{-\frac{1}{2}}=p$, we have

$$\eta=\frac{15}{4}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{y}{\beta^2}\right)^{-\frac{3}{2}}\int_0^\infty p^{-6}[p^2y\beta^{-2}-\sin(p^2y\beta^{-2})]e^{-p}dp.$$

[5] S. Chandrasekhar, Revs. Modern Phys. 15, 1 (1943).
[6] Our notation is slightly different from Chandrasekhar's.

We expand the sine and obtain the following asymptotic series for η:

$$\eta = K_1 y^{\frac{1}{2}}\beta^{-1} - K_2 y^{\frac{5}{2}}\beta^{-5} + K_3 y^{9/2}\beta^{-9}\cdots,$$

$$K_1 = \frac{15}{4}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{3!},\quad K_2 = \frac{15}{4}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{4!}{5!},$$

$$K_3 = \frac{15}{4}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{8!}{7!}. \quad (17)$$

If only the first term is kept, we have

$$H(s) = \frac{2}{\pi s}\left[s^2\int_0^\infty \exp\left[-y^{\frac{3}{2}}\left(\frac{K_1 y^{\frac{1}{2}}}{\beta}\right)\right] y \sin(sy)dy\right]. \quad (18)$$

Equation (18) can be integrated to give

$$H(s) = \frac{\beta^{\frac{3}{2}} s^2 \exp(-s^2\beta/4K_1)}{2\pi^{\frac{1}{2}} K_1^{\frac{3}{2}}}. \quad (19)$$

TABLE II. Correction to the Holtsmark distribution for $\beta = 2\times 10^{-2}$.

s	Holtsmark	Correction
0.1	0.004225	+0.00015
0.6	0.129598	+0.0042
1.0	0.271322	+0.0078
2	0.33918	+0.0036
3	0.176	−0.0017
6	0.02417	−0.0010
8	0.01038	−0.00054
10	0.00556	−0.00034

Equation (19) is identical with a formula proposed by Mayer[1,2] on the basis of a simpler physical model than ours.

Applications

(a) A typical case of astronomical interest involves the following values of temperature, density and charge: $T = 6000°\text{C}$, $n = 10^{14}$, single charged ions, $Q_1 = Q_2 = e$. For this case $\beta = 2\times 10^{-2}$. Here we can use the results under case 1 for ranges of s from 0.1 to 100. Table II corrects the Holtsmark distribution; it is based only on the first correction term of Eq. (15), for the range $0.1 \leq s \leq 10$.

TABLE III. Comparison of Eq. (19) with Broyles' SRNN.

s	Eq. (19)	SRNN
0.3	0.69	0.63
0.5	1.25	1.13
1.0	0.64	0.66
1.5	0.04	0.147

The percentage error in the Holtsmark distribution is small. It becomes large in the limit of large s, i.e., when the binary theory becomes valid. In the limit of large s, the ratio of H_c, the corrected Holtsmark, to H, the Holtsmark distribution is given by $H_c/H = \exp(-\beta s^{\frac{1}{2}})$; hence the percentage error becomes very great.

(b) A case of recent interest,[1,2] already alluded to in part I, involves: $T = 1.16\times 10^7$ degrees, $n = 8.4\times 10^{22}$ atoms/cc, and $Q_1 = Q_2 = 23e$. For this case, $\beta = 5.5$. For $s \ll 5.5$ we can use the results under case 2. For $s \gg 5.5$ the binary theory can be used.

We have compared (see Table III) the SRNN (short-range nearest-neighbor) result of Broyles[2] with those of Eq. (19) for $0.3 \leq s \leq 1.5$. The percentage error in using Eq. (19) increases with increasing s and is estimated to be less then 10% for $s = 1$. Since the assumption underlying our derivation are quite different from his, one may well have confidence in the practical correctness of this simple result.

For a more detailed comparison of Eq. (19) with Broyles' results and the Holtsmark theory see Fig. 2 of reference 2.[7] The curve marked simple harmonic oscillator is the result of Mayer[1,2] and, as mentioned before, it is identical with our Eq. (19).

ACKNOWLEDGMENT

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[7] In Broyles' notation $\theta = 1$ corresponds to the condition discussed in this section. Broyles' field ϵ, though defined with a different proportionality factor, is numerically almost identical with our s.