Conductivity of Plasmas to Microwaves*

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A relation [Eq. (13)] between the current-induced part of the distribution function (f_1) and the isotropic part (f_0), which was previously derived with limiting restrictions, is here shown to be valid at moderate electric fields for any form of f_0, e.g., for the case where an agency other than the passing microwaves determines it. The complex conductivity is then computed for a variety of functions f_0 (δ function, step function, Maxwellian distribution) and for different assumptions as to the dependence of the collision frequency on electron speed. The integrals encountered in this work are shown to be computable with good accuracy by means of a saddle-point procedure which is described.

THE present study concerns a situation in which microwaves pass through a neutral plasma containing electrons in quasi-equilibrium with their surroundings; i.e., their distribution in space and in velocity does not change in time over an interval long compared with the period of the microwave. The velocity distribution, however, need not correspond to the temperature of the ambient molecules and ions, nor must it be Maxwellian. Under these conditions the conductivity depends on four things: (1) the number density of electrons, n, (2) the frequency ω of the microwave, (3) the distribution-in-velocity of the electrons, and (4) the functional dependence of the collision cross section q, or, in another version, the collision frequency ν, upon electron velocities. In the following we give attention to all of these factors, selecting for treatment in connection with item 3 as special cases a δ function, a step function, and a Maxwell distribution for any temperature. Under 4, we assume a linear dependence of ν on electron speed v, or of q on $1/v$. The special cases of constant ν and of constant mean free path λ are singled out for special attention.

In earlier publications[1–6] the full range of variability among all parameters has not been surveyed. Special favor seems to have been conferred upon the case of constant ν which, as will be seen again in this study, is in a sense a trivial one since it always leads to the Lorentz formula. Reference 1 dealt with the conductivity of a gas of free electrons colliding with molecules, on the assumption that the electrons obtain *all* their energy from an alternating electric field. This implies a special distribution function, calculated in that paper. Here we suppose the presence of another agency, perhaps of the nature of a dc field generating a discharge, perhaps an intense photon field or, indeed, a shock wave, which generates electrons and impresses a characteristic prior velocity distribution upon them, a distribution which is but slightly modified by the passing microwave. We shall use the method and the notation of reference 1.

GENERAL CONSIDERATIONS

If $E\cos\omega t$ is the electric field strength (directed along x) of the microwaves, the Boltzmann transfer equation may be written in the form

$$\gamma\cos\omega t\frac{\partial f}{\partial v_x}+\frac{\partial f}{\partial t}=\frac{Df}{Dt},$$

where $\gamma\equiv eE/m$, $f=$distribution function, and Df/Dt is the rate of change of f resulting from collisions and all other present agencies, including the "prior" one. In the notation of reference 1,

$$f(\mathbf{v})=f_0(v)+\gamma v_x[f_1(v)\cos\omega t+g_1(v)\sin\omega t],\qquad(1)$$

and in place of Eq. (7) of reference 1 we now write

$$Df_0/Dt=P(f_0),\qquad(2)$$

P being some (usually a differential) operator. If the electrons make elastic collisions with molecules of mass M having a Maxwell distribution at temperature T,[7]

$$P(f_0)=\frac{1}{v^2}\frac{\partial}{\partial v}\left[\frac{v^3}{M\lambda}\left(mvf_0+kT\frac{\partial}{\partial v}f_0\right)\right],\qquad(3)$$

but this is a very special case.

The calculation of Df/Dt is composed additively of two contributions, one to be denoted by D_cf/Dt and coming from collisions with atoms, ions, and molecules; and another, called D_af/Dt, made by other agencies. As to the first, it is noted that D_cf_0/Dt, $D_c(v_xf_1)/Dt$, and $D_c(v_xg_1)/Dt$ are all expected to be finite. The same is true for D_af_0/Dt because the other agencies have an important effect upon the isotropic part of the distribution function. But $D_a(v_xf_1)/Dt$ and $D_a(v_xg_1)/Dt$ are zero if these agencies are indifferent to the slight asymmetries in v-space which are set up by the micro-

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waves, that is, if they do not tend to destroy them (as collisions very effectively do). This latter condition will be assumed to be true.

Let us then compute first $D_c f/Dt$.[2] Suppose that in a collision of type r an electron of initial speed v' loses an amount of energy ϵ_r. The cross section for this process is $q_r(v')$, the mean free path $\lambda_r=(nq_r)^{-1}$ and the frequency of r-collisions for one electron $\nu_r'=v'nq_r(v')$. If the final velocity of the electron of initial velocity $\mathbf{v}'$ after an r-collision is $\mathbf{v}$, the rate of increase of electrons at $\mathbf{v}$ by this type of encounter is

$$v'nq_r(\mathbf{v}')f(\mathbf{v}')d\mathbf{v}',$$

and the rate of loss at $\mathbf{v}$,

$$vnq_r(\mathbf{v})f(\mathbf{v})d\mathbf{v}.$$

Hence the net increase at $\mathbf{v}$ as a result of r-collisions is

$$\left(\frac{D_c f}{Dt}\right)_r d\mathbf{v}=n[v_r'q_r(\mathbf{v}_r')f(\mathbf{v}_r')d\mathbf{v}_r'-vq_r(\mathbf{v})f(\mathbf{v})d\mathbf{v}]. \quad (4)$$

This formula can in principle be used for the evaluation of $(D_c f_0/Dt)_r$ and then, by summing over all r, $D_c f_0/Dt$ could be computed. When furthermore $D_a f_0/Dt$, available only when the agencies producing the electrons are specified, is added to $D_c f_0/Dt$, the operator $P(f_0)$ defined in Eq. (2) is obtained. But we shall not need $P(f_0)$ and dispense with this difficult calculation.

In place of it, we use (4) to calculate

$$\left(\frac{D_c(v_x f_1)}{Dt}\right)_r d\mathbf{v}$$
$$=n[v_r'q_r(\mathbf{v}_r')v_{xr}'f_1(v_r)d\mathbf{v}_r'-vq_r(\mathbf{v})v_x f_1(v)d\mathbf{v}],$$

and sum this expression over r. At this point it will be assumed that $q_r(\mathbf{v})=q_r(v)$ is isotropic, so that for every r there occur pairs of equally likely collisions with positive and negative v_{xr}'. The sum over the first term on the right is then zero, and the equation reads

$$D(v_x f_1)/Dt=-nvq(v)v_x f_1(v)=-\nu v_x f_1, \quad (5)$$

because $\sum_r q_r=q$. In the same way we find

$$D(v_x g_1)/Dt=-\nu v_x g_1. \quad (6)$$

The procedure employed in reference 1 leads to the following equations:

$$\frac{\gamma^2}{6}f_1=v^{-3}\int_0^v v^2 P(f_0)dv, \quad (7)$$

$$g_1=\frac{\omega}{\nu}f_1, \quad (8)$$

$$\frac{1}{v}\frac{\partial f_0}{\partial v}=-\nu f_1-\omega g_1. \quad (9)$$

In the absence of E, we see from the Boltzmann equation and from (2) that $P(f_0)=0$. Indeed if $P(f_0)$ has the special form (3), f_0 is Maxwellian. Let us write

$$P(f_0^0)=0. \quad (10)$$

The electric field adds to f_0^0 a small perturbation f_0' so that

$$f_0=f_0^0+f_0'. \quad (11)$$

The function f_1 can now be computed from Eqs. (7)–(9) in two ways. According to (7),

$$f_1=\frac{6}{\gamma^2 v^3}\int_0^v v^2 P(f_0')dv. \quad (12)$$

But from (8) and (9), with the legitimate neglect of f_0' against f_0^0,

$$f_1=-\frac{\nu}{v}\frac{\partial f_0^0}{\partial v}\Big/(\nu^2+\omega^2). \quad (13)$$

Equation (12), which is not useful here because P is unknown, relates f_1 to f_0', which is likewise unknown. For present purposes, Eq. (13) is the important one. It has the same form as in reference 1; but we have now shown that its validity is wider than was previously demonstrated, and that it links f_1, the function determining the conductivity of the electron gas, to any prior distribution function f_0^0 which satisfies the conditions here explicitly stated.

The current density is

$$I=ne\bar{v}_x=ne\gamma\int v_x^2(f_1\cos\omega t+g_1\sin\omega t)d\mathbf{v}$$

$$=-\frac{ne\gamma}{3}\int\frac{\nu\cos\omega t+\omega\sin\omega t}{\nu^2+\omega^2}\frac{\partial f_0^0}{\partial v}vd\mathbf{v} \quad (14)$$

$$=\frac{ne\gamma}{3}\int\frac{\partial}{\partial v}\left(\frac{\nu\cos\omega t+\omega\sin\omega t}{\nu^2+\omega^2}v^3\right)4\pi f_0^0 dv, \quad (15)$$

provided

$$\int 4\pi v^2 f_0^0 dv=1.$$

From the last result it is seen at once that the Lorentz formula holds for arbitrary f_0^0, provided only ν is constant with respect to v; for then

$$I=\frac{ne\gamma}{\omega^2+\nu^2}(\nu\cos\omega t+\omega\sin\omega t). \quad (16)$$

For the sake of completeness we mention here that Lorentz derived this formula as a solution of the differential equation

$$md^2x/dt^2+F=eE\cos\omega t,$$

choosing for the frictional force on the electron $F=\nu m\dot{x}$. When this is solved for $ne\bar{v}_x$, Eq. (16) results.

SPECIAL DISTRIBUTIONS

A. All Electrons Have a Single Energy

If all electrons have the same speed v', $f_0^0=(4\pi v'^2)^{-1}\times\delta(v,v')$, in terms of the Dirac δ function. In view of Eq. (15), for any f_0^0,

$$I=ne\gamma(J_1\cos\omega t+J_2\sin\omega t), \tag{17}$$

$$J_1=\frac{4\pi}{3}\int f_0^0\frac{\partial}{\partial v}\left(\frac{\nu v^3}{\omega^2+\nu^2}\right)dv;$$

$$J_2=\frac{4\pi}{3}\omega\int f_0^0\frac{\partial}{\partial v}\left(\frac{v^3}{\omega^2+\nu^2}\right)dv. \tag{18}$$

When the δ function is introduced one finds

$$J_1=(\omega^2+\nu'^2)^{-1}\left[\nu'+\frac{v'}{3}\left(\frac{\partial\nu}{\partial v}\right)'\frac{\omega^2-\nu'^2}{\omega^2+\nu'^2}\right],$$

$$J_2=\omega(\omega^2+\nu'^2)^{-1}\left[1-\tfrac{2}{3}v'\nu'\left(\frac{\partial\nu}{\partial v}\right)'\Big/\omega^2+\nu'^2\right].$$

Primed quantities are evaluated at v'. Henceforth in this section we drop the primes but retain this understanding.

J_1 and J_2 take simple forms for special cases of interest. When $\partial\nu/\partial v=0$ we are led back to (16), but for the case of constant mean free path, where $\nu=\text{constant}\times v$,

$$I(\lambda=\text{const})=\frac{ne\gamma}{(\omega^2+\nu^2)^2}\times[(\tfrac{4}{3}\omega^2+\tfrac{2}{3}\nu^2)\nu\cos\omega t+(\omega^2+\tfrac{1}{3}\nu^2)\omega\sin\omega t].$$

If the microwave frequency is low, or the pressure high, *or the electrons are very fast* ($\nu^2=v^2/\lambda^2\gg\omega^2$), the current remains in phase with the microwave field and is given essentially by a dc formula

$$I=\frac{2}{3}\frac{ne\gamma\lambda}{v}\cos\omega t, \tag{19}$$

differing from (16) by the factor $\frac{2}{3}$. Under the reverse condition (slow electrons), the limiting form is identical with that of (16):

$$I=(ne\gamma/\omega)\sin\omega t. \tag{20}$$

B. Electrons are Uniformly Distributed in Energy

If $f_0^0=3C/4\pi$ when $v_1\leq v\leq v_2$ and is zero outside this range, with $C=(v_2^3-v_1^3)^{-1}$, the integrals (18) become

$$J_1=C\left(\frac{\nu v^3}{\omega^2+\nu^2}\right)\Bigg|_{v_1}^{v_2};\quad J_2=C\omega\left(\frac{v^3}{(\omega^2+\nu^2)}\right)\Bigg|_{v_1}^{v_2}.$$

The natural assumption is to take $v_1=0$, in which case

$$J_1=\frac{\nu_2}{\omega^2+\nu_2^2}\quad\text{and}\quad J_2=\frac{\omega}{\omega^2+\nu_2^2}. \tag{21}$$

We are thus led back once more to the Lorentz formula (16); the collision frequency appearing in it must, however, be taken to be that of the *fastest* electrons. Worth noting also is the fact that this is quite independent of the manner in which ν depends on v.

C. Electron Distribution is Maxwellian

The assumption of a Maxwellian distribution

$$f_0^0=(m/2\pi kT)^{\frac{3}{2}}\exp(-mv^2/2kT),$$

with an arbitrary temperature T fits many physical conditions. Under it the J integrals have the form

$$J_1=\frac{8}{3\sqrt{\pi}}\int_0^\infty\exp(-u^2)\frac{\nu u^4}{\omega^2+\nu^2}du, \tag{22}$$

$$J_2=\frac{8}{3\sqrt{\pi}}\omega\int_0^\infty\exp(-u^2)\frac{u^4}{\omega^2+\nu^2}du. \tag{23}$$

They do not depend on T except through ν, which is a function of v. The variable u is defined as $u=v/v_0$ with $v_0=(2kT/m)^{\frac{1}{2}}$, so that $\nu=\nu(u,v_0)$. A good approximation to J_1 and J_2 is furnished by the saddle-point method.

Let

$$\frac{\nu}{\omega^2+\nu^2}\equiv g_1(u),\quad\frac{1}{\omega^2+\nu^2}\equiv g_2(u).$$

We wish to compute

$$K=\int_0^\infty e^{-f(u)}du,\quad\text{with}\quad f(u)=u^2-4\ln u-\ln g.$$

The maximum of f occurs at u_0 which is determined by

$$\frac{g'(u_0)}{g(u_0)}=2u_0-\frac{4}{u_0}. \tag{24}$$

(1) *If the logarithmic derivative of g is small*, then in first approximation $u_0\cong\sqrt{2}$. In the next approximation, which suffices here, we find

$$u_0^2=2+2^{-\frac{1}{2}}\frac{g'(\sqrt{2})}{g(\sqrt{2})}. \tag{25}$$

We shall also need the second derivative:

$$f''(u)=2+\frac{4}{u^2}-\frac{g''}{g}+\left(\frac{g'}{g}\right)^2, \tag{26}$$

$$f''(u_0)\cong 4-2^{-\frac{1}{2}}\frac{g'}{g}-\frac{g''}{g} \tag{27}$$

in the same approximation, and the g-symbols in the last line are evaluated for $u=\sqrt{2}$.

On writing

$$f(u)=f(u_0)+\tfrac{1}{2}f''(u_0)(u-u_0)^2,$$

the integral becomes

$$K=e^{-f(u_0)}\int_0^\infty \exp[-\tfrac{1}{2}f''(u_0)(u-u_0)^2]du$$

$$=\left[\frac{2\pi}{f''(u_0)}\right]^{\frac{1}{2}}e^{-f(u_0)}$$

$$=\left(\frac{2\pi}{f''}\right)^{\frac{1}{2}}\exp\left(-2-2^{-\frac{1}{2}}\frac{g'}{g}\right)(4g+2^{\frac{3}{2}}g') \quad (28)$$

$$\doteq(2\pi)^{\frac{1}{2}}\exp\left(-2-2^{-\frac{1}{2}}\frac{g'}{g}\right)\left(2g+\frac{9}{8}\sqrt{2}g'+\tfrac{1}{4}g''\right)$$

because of (25) and (27).

Moreover,

$$\frac{g_1'}{g_1}=\frac{\nu'}{\nu}\frac{\omega^2-\nu^2}{\omega^2+\nu^2}\equiv s_1;\quad \frac{g_1''}{g_1}=\frac{\nu''}{\nu}\frac{\omega^2-\nu^2}{\omega^2+\nu^2}\equiv t_1,$$

$$\frac{g_2'}{g_2}=\frac{-2\nu\nu'}{\omega^2+\nu^2}\equiv s_2;\quad \frac{g_2''}{g_2}=\frac{-2\nu\nu''}{\omega^2+\nu^2}\equiv t_2,$$

if higher powers of ν'/ν than the first are neglected. In the following, the s and t functions here defined are understood to be evaluated at $u=u_0$, i.e., ν, ν', and ν'' are given their values at

$$v=\sqrt{2}v_0=2(kT/m)^{\frac{1}{2}}.$$

Returning now to (22) and (23), we find

$$J_1=\frac{8}{3}\left(2^{\frac{3}{2}}+\frac{9}{4}s_1+2^{-\frac{3}{2}}t_1\right)\exp\left(-2-\frac{s_1}{\sqrt{2}}\right)\frac{\nu}{\omega^2+\nu^2},$$

$$J_2=\frac{8}{3}\left(2^{\frac{3}{2}}+\frac{9}{4}s_2+2^{-\frac{3}{2}}t_2\right)\exp\left(-2-\frac{s_2}{\sqrt{2}}\right)\frac{\omega}{\omega^2+\nu^2}. \quad (29)$$

The error in the use of the saddle-point method may be judged from the fact that for constant ν the numerical coefficient of J_1 and J_2, which should be 1.000, is actually 1.020.

(2) When the mean free path, λ, is constant the ratio g'/g is not small at the saddle-point, Eq. (26) must be computed without approximation, and the result (29) is not correct. To be sure, this case has been treated exactly by Altshuler and Molmud,[8] for it leads to known functions.

One finds

$$J_1=\tfrac{4}{3}(\pi)^{\frac{1}{2}}\frac{C}{\omega}[1-C^2-C^4\exp(C^2)\,\mathrm{Ei}(-C^2)],$$

$$J_2=\frac{4}{3}\frac{C^2}{\omega}\{(\tfrac{1}{2}-C^2)+(\pi)^{\frac{1}{2}}C^3\exp(C^2)[1-\Phi(C)]\}, \quad (30)$$

where $\mathrm{Ei}(x)$ and $\Phi(x)$ are the exponential integral and error integral, respectively.[9] Our saddle-point method leads to

$$J_1=\frac{8}{3}\left[\frac{2}{f_1''(u_1)}\right]^{\frac{1}{2}}\frac{C}{\omega}\frac{u_1^5\exp(-u_1^2)}{C^2+u_1^2}. \quad (31)$$

Here u_1, is a function of C:

$$2u_1^2=\left(C^4+7C^2+\frac{9}{4}\right)^{\frac{1}{2}}-C^2+\tfrac{3}{2},$$

$$f_1''(u_1)=4u_1^2-14+\frac{20}{u_1^2}+2\frac{3C^2-u_1^2}{(C^2+u_1^2)^2}.$$

Finally,

$$J_2=\frac{8}{3}\left[\frac{2}{f_2''(u_2)}\right]^{\frac{1}{2}}\frac{C^2}{\omega}\frac{u_2^4\exp(-u_2^2)}{C^2+u_2^2}, \quad (32)$$

where

$$2u_2^2=(C^4+6C^2+1)^{\frac{1}{2}}-C^2+1,$$

$$f_2''(u_2)=4u_2^2-14+\frac{20}{u_2^2}+2\frac{C^2-3u_2^2}{(C^2+u_2^2)^2}.$$

Expressions (31) and (32) agree with (30) within 2% over the entire ranges of the parameters. This is being recorded here, not because of the utility of the saddle-point method in this particular problem, but because it will be employed for other more complicated relationships between ν and v which are now under study.

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[8] S. Altshuler and P. Molmud (unpublished). This work came to the author's attention after the present calculation had been completed.

[9] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1953).