perturbation result of Salzman by about 30%. The correction part with recoil further reduces the perturbation part by about 20%, so the final value of the non-Foldy term is 5.7 kev for $f^2=0.08$. This amounts to a total reduction of 45%. Treiman and Sachs found that the correction part without recoil reduces the perturbation part by 20%. When the nucleon current is included, Fried's result is reduced by a factor of 2.75. If this can be taken as an indication of the effects of the nucleon current, it is likely that our value of 5.7 kev will be reduced further.

The error of our result, as estimated in Sec. III, was 20%. Thus the final value of the interaction is not as important as the general feature that the contribution of the non-Foldy term to the neutron-electron interaction is considerably reduced by recoil effects, provided that the dispersion relations are meaningful. This result is somewhat encouraging because some doubts might be cast on the validity of the concept of local fields at short distances if a large (positive) contribution of the nucleon current had been needed to cancel nearly all of a large (negative) contribution of the meson current in the non-Foldy part of the electron-neutron interaction.

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Complete Set of Dispersion Relations for a Class of Fixed-Source Meson Theories

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The structure of the transition matrices for all processes that can occur for a class of fixed-source meson theories is studied. The model consists of a scalar meson field coupled to an extended source in such a way that any finite number of quanta can be emitted or absorbed at a given time (multiple vertices), but that all interactions are restricted to be S wave in nature. The general reaction matrix element for m incident and n emergent particles contains many terms describing sequences of independent processes, which must be removed before one obtains a proper object for studies of a dispersion-theoretic nature. It is shown that the ratio of the residual matrix element to a suitable product of source functions possesses those analytic properties, as a function of the total energy of the system, which permit dispersion relations to be stated. Other than for the elastic scattering amplitude the latter make reference to values of the amplitude in a nonphysical energy region. In conjunction with a suitably generalized unitarity condition, however, the scheme, when viewed as a set of coupled integral equations, can be solved by successive approximations in terms of a number of arbitrary constants, essentially equal to the number of coupling constants in the original Hamiltonian (actually one fewer). Nevertheless, it is pointed out that the scheme does not admit a unique solution, and this is illustrated physically by exhibiting an extended class of Hamiltonians which yield the same dispersion relations, but which, as a class, contain more coupling constants than make their appearance in the dispersion relations. Physically, these are connected with the occurrence of resonance scattering.

I. INTRODUCTION

IT has been conjectured recently[1] that the set of all dispersion relations may be complete in the sense that they can be considered in place of the Lagrangian as the starting point to compute all the observables of a theory. However, until now, very little has been done toward exhibiting such a complete set of relations for any nontrivial theory[2] or indicating how they can be used to calculate quantities other than the amplitudes for the simplest reactions.[3] In view of the complexity of the relativistic problem, we have attempted to construct such a complete set of dispersion relations for a comparatively simple fixed-source model, but one which nevertheless contains nonvanishing amplitudes for multiple meson production. In particular the model chosen has a Hamiltonian

$$H=H_0+H_I, \tag{1}$$

where

$$H_0=\sum_{\mathbf{k}} a^*(\mathbf{k})a(\mathbf{k})\omega(\mathbf{k}), \tag{2}$$

$$H_I=\sum_{\mu=1}^{N}\lambda_\mu\left[\int \rho(x)\phi(\mathbf{x})d\mathbf{x}\right]^\mu-E_s, \tag{3}$$

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[1] For example, by J. S. Toll (private communication) and by M. Gell-Mann in *Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics* (Interscience Publishers, Inc., New York, 1956), p. III–30.

[2] See, however, Toll, Wong, and Knight (to be published). For a general approach to the relativistic problem, see J. C. Polkinghorne, Nuovo cimento 4, 216 (1956).

[3] The rapidly accumulating literature on dispersion relations may be traced from the following recent papers: K. Symanzik, Phys. Rev. **105**, 743 (1957); J. S. Toll, Phys. Rev. **104**, 1760 (1956); R. H. Capps and G. Takeda, Phys. Rev. **103**, 1877 (1956); R. Oehme, Nuovo cimento 4, 1316 (1956).

which describes the interaction of a fixed, spherically symmetric source $\rho(x)$ with a neutral, scalar meson field $\phi(\mathbf{x})$. The quantities $a^*(\mathbf{k})$ and $a(\mathbf{k})$ are, respectively, the creation and annihilation operations for single mesons of momentum $\mathbf{k}$, and $\omega(\mathbf{k})=(\mu^2+k^2)^{\frac{1}{2}}$ is the meson energy. E_s is a c number introduced to fix the lowest eigenvalue of H at zero. This theory contains only S waves and for $N>2$ has matrix elements for multiple meson production.

Our treatment can be considered as an extension of the original Chew-Low program of applying fixed-source theory to meson-nucleon scattering.[4] It is interesting to contrast the assumptions of this theory with those made in the relativistic case. They parallel each other except that the statement of microscopic causality in the relativistic theory is replaced here by an explicit statement of the manner in which the meson field is coupled to its source. We are led to construct relations for the ratio of the transition amplitudes to a suitable product of the source functions, motivated to this procedure by the fact that this ratio manifests an explicit dependence upon the total energy of the system. Its behavior in all energy domains, including the high-energy limit, can be inferred immediately and need not be assumed as an additional postulate. This advantage is gained at the expense of actually stating the Hamiltonian. We should remark also that despite the great simplicity of our model the transition amplitudes for multiple boson processes involve, nevertheless, integrals over nonphysical energies, which restrict their direct application to experiment.

In Sec. II we describe the general structure of the S matrix and define an appropriate transition amplitude. This requires, in general, that we separate off from a given S matrix element all terms describing a sequence of independent processes. We then demonstrate an equation satisfied by this transition amplitude which is equivalent to the requirement of unitarity on the S matrix. In Sec. III a complete set of dispersion relations for this transition amplitude is derived and the number of arbitrary constants is shown to be one less than the number of coupling constants λ_i in the Hamiltonian. In Sec. IV we demonstrate an iterative procedure which, if it converges, will yield a solution to the set of coupled integral equations consisting of the dispersion relations and the unitarity expressions. Nevertheless it is made manifest that without further definition, the scheme cannot possess a unique solution, since we are able to exhibit an extended class of Hamiltonians which all satisfy the same dispersion relations. These may be characterized as containing more parameters than are in evidence in the dispersion relations and indeed as many more as one desires. They are obviously connected physically with the occurrence of resonance scattering.

[4] F. E. Low, Phys. Rev. **97**, 1392 (1955); G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570, 1579 (1956).

II. STRUCTURE OF THE S MATRIX

The matrix element of S between an initial state of m mesons with momenta $\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_m$ and a final state of n mesons with momenta $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ is defined to be

$$(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|S|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m) \equiv (\Psi^{(-)}(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n),\Psi^{(+)}(\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)), \quad (4)$$

where $\Psi^{(-)}$ and $\Psi^{(+)}$ are, respectively, the usual[5] incoming and outgoing wave eigenstates of the complete Hamiltonian H.

Denoting the ground state of H by Ψ_0, we can employ the methods and notation of Wick[6] to write[7]

$$\Psi^{(\pm)}(\mathbf{q}_1\mathbf{q}_2\cdots\mathbf{q}_r)=N(q)[a^*(\mathbf{q}_1)a^*(\mathbf{q}_2)\cdots a^*(\mathbf{q}_r)\Psi_0 +\chi^{(\pm)}(\mathbf{q}_1\mathbf{q}_2\cdots\mathbf{q}_r)], \quad (5)$$

where $N(q)$ is the constant of normalization. In order to solve for $\chi^{(\pm)}$, we use (5) and the fact that $H\Psi_0=0$ to write the left side of the Schrödinger equation,

$$H\Psi^{(\pm)}(\mathbf{q}_1\mathbf{q}_2\cdots\mathbf{q}_r)=\sum_{i=1}^{r}\omega(\mathbf{q}_i)\Psi^{(\pm)}(\mathbf{q}_1\mathbf{q}_2\cdots\mathbf{q}_r), \quad (6)$$

in the form

$$\begin{aligned} &H\Psi^{(\pm)}(\mathbf{q}_1\mathbf{q}_2\cdots\mathbf{q}_r)\\ &=[H, a^*(\mathbf{q}_1)\cdots a^*(\mathbf{q}_r)]\Psi_0+H\chi^{(\pm)}(\mathbf{q}_1\mathbf{q}_2\cdots\mathbf{q}_r)\\ &=\sum_{i=1}^{r}\omega(\mathbf{q}_i)a^*(\mathbf{q}_1)\cdots a^*(\mathbf{q}_r)\Psi_0\\ &\quad+[H_I, a^*(\mathbf{q}_1)\cdots a^*(\mathbf{q}_r)]\Psi_0+H\chi^{(\pm)}(\mathbf{q}_1\mathbf{q}_2\cdots\mathbf{q}_r), \end{aligned} \quad (7)$$

so that by comparing (5), (6), and (7) we immediately obtain the relation

$$\begin{aligned} &(\sum_{i=1}^{r}\omega(\mathbf{q}_i)-H)\chi^{(\pm)}(\mathbf{q}_1\mathbf{q}_2\cdots\mathbf{q}_r)\\ &\qquad=[H_I, a^*(\mathbf{q}_1)\cdots a^*(\mathbf{q}_r)]\Psi_0\\ &=\sum_{k=1}^{r}a^*(\mathbf{q}_1)\cdots a^*(\mathbf{q}_{k-1})j^*(\mathbf{q}_k)a^*(\mathbf{q}_{k+1})\cdots a^*(\mathbf{q}_r)\Psi_0, \end{aligned} \quad (8)$$

where $j^*(\mathbf{q}_k)\equiv[H_I,a^*(\mathbf{q}_k)]$, which is equivalent to

$$\begin{aligned} \chi^{(\pm)}(\mathbf{q}_1\mathbf{q}_2\cdots\mathbf{q}_r)&=(\sum_{i=1}^{r}\omega(\mathbf{q}_i)\pm i\eta-H)^{-1}\\ &\times\sum_{k=1}^{r}a^*(\mathbf{q}_1)\cdots a^*(\mathbf{q}_{k-1})j^*(\mathbf{q}_k)a^*(\mathbf{q}_{k+1})\cdots a^*(\mathbf{q}_r)\Psi_0. \end{aligned} \quad (9)$$

[5] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[6] G. C. Wick, Revs. Modern Phys. **27**, 339 (1955).
[7] Extensions of this work to multiple processes have also been considered recently by S. Barshay, Phys. Rev. **103**, 1102 (1956), and by N. Fukuda and J. S. Kovacs, Phys. Rev. **104**, 1784 (1956).

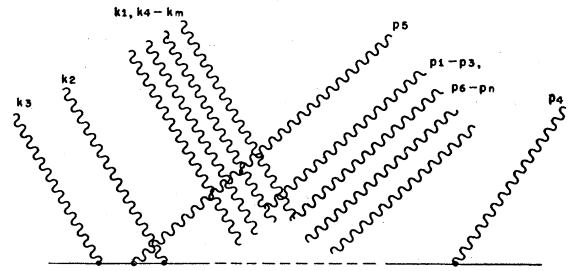

FIG. 1. Diagram corresponding to the matrix element of Eq. (17).

Substitution of (9) into (5) then yields

$$\Psi^{(\pm)}(\mathbf{q}_1\cdots\mathbf{q}_r)$$
$$=N(q)[a^*(\mathbf{q}_1)\cdots a^*(\mathbf{q}_r)\Psi_0+(\sum_{i=1}^{r}\omega(\mathbf{q}_i)\pm i\eta-H)^{-1}$$
$$\times\sum_{k=1}^{r}a^*(\mathbf{q}_1)\cdots j^*(\mathbf{q}_k)\cdots a^*(\mathbf{q}_r)\Psi_0]. \quad (10)$$

It is shown in Appendix A that the $\Psi^{(\pm)}$ satisfy the orthogonality relation

$$(\Psi^{(\pm)}(\mathbf{q}_1\mathbf{q}_2\cdots\mathbf{q}_r),\Psi^{(\pm)}(\mathbf{q}_1'\mathbf{q}_2'\cdots\mathbf{q}_{r'}'))$$
$$=N(q)N(q')\sum_{P(\mathbf{q}')}\delta(r,r')\delta(\mathbf{q}_1-\mathbf{q}_1')$$
$$\times\delta(\mathbf{q}_2-\mathbf{q}_2')\cdots\delta(\mathbf{q}_r-\mathbf{q}_{r'}'), \quad (11)$$

where the symbol $\sum_{P(\mathbf{q}')}$ indicates a sum over all permutations of the $\mathbf{q}'$'s. From (11) it is clear that the normalization constant $N(q)$ in (10) should be $(r!)^{-\frac{1}{2}}$.

From (10) and (11), we can rewrite (4) in the form

$$\begin{aligned}&(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|S|k_1k_2\cdots k_m)\\&=(\Psi^{(-)}(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n),\Psi^{(-)}(\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m))\\&+(\Psi^{(-)}(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n),\Psi^{(+)}-\Psi^{(-)}(\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m))\\&=(n!m!)^{-\frac{1}{2}}[\sum_{P(\mathbf{k})}\delta(m,n)\delta(\mathbf{p}_1-\mathbf{k}_1)\delta(\mathbf{p}_2-\mathbf{k}_2)\cdots\\&\times\delta(\mathbf{p}_n-\mathbf{k}_m)-2\pi i\delta(\sum_{i=1}^{n}\omega(\mathbf{p}_i)-\sum_{j=1}^{m}\omega(\mathbf{k}_j))\\&\times(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|Q|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)],\end{aligned} \quad (12)$$

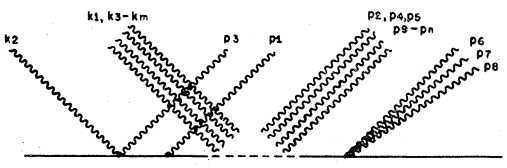

FIG. 2. Diagram corresponding to the matrix element in Eq. (19).

where

$$\begin{aligned}&(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|Q|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)\\&=(\Psi_0,a(\mathbf{p}_1)\cdots a(\mathbf{p}_n)\sum_{s=1}^{m}a^*(\mathbf{k}_1)\cdots a^*(\mathbf{k}_{s-1})\\&\times j^*(\mathbf{k}_s)a^*(\mathbf{k}_{s+1})\cdots a^*(\mathbf{k}_m)\Psi_0)\\&+(\Psi_0,\sum_{l=1}^{n}a(\mathbf{p}_1)\cdots a(\mathbf{p}_{l-1})j(\mathbf{p}_l)a(\mathbf{p}_{l+1})\cdots\\&\times a(\mathbf{p}_n)[\sum_{r=1}^{n}\omega(\mathbf{p}_r)+i\eta-H]^{-1}\sum_{t=1}^{m}a^*(\mathbf{k}_1)\cdots\\&\times a^*(\mathbf{k}_{t-1})j^*(\mathbf{k}_t)a^*(\mathbf{k}_{t+1})\cdots a^*(\mathbf{k}_m)\Psi_0).\end{aligned} \quad (13)$$

In order to eliminate the explicit appearance of the elementary creation and annihilation operators by means of such relations as[6]

$$a(\mathbf{q})\Psi_0=-[\omega(\mathbf{q})+H]^{-1}j(\mathbf{q})\Psi_0, \quad (14)$$

we must manipulate the right side of (13) so that the a's rather than the a^*'s act on the Ψ_0's. This task is readily accomplished by employing repeatedly the

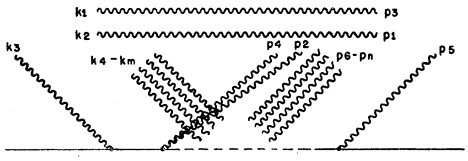

FIG. 3. Diagram corresponding to the matrix element in Eq. (20)

equalities

$$a(\mathbf{p})\frac{1}{E+i\eta-H}=\frac{1}{E-\omega(p)+i\eta-H}j(\mathbf{p})\frac{1}{E+i\eta-H}$$
$$+\frac{1}{E-\omega(p)+i\eta-H}a(\mathbf{p}), \quad (15)$$

and

$$\frac{1}{E+i\eta-H}a^*(\mathbf{k})=\frac{1}{E+i\eta-H}j^*(\mathbf{k})\frac{1}{E-\omega(\mathbf{k})+i\eta-H}$$
$$+a^*(\mathbf{k})\frac{1}{E-\omega(\mathbf{k})+i\eta-H}, \quad (16)$$

where the derivation of (15) [the derivations of (14) and (16) are similar] is the following:

$$\begin{aligned}(E+i\eta-H)a(\mathbf{p})&\frac{1}{E+i\eta-H}\\&=-[H,a(\mathbf{p})]\frac{1}{E+i\eta-H}+a(\mathbf{p})\\&=[\omega(\mathbf{p})a(\mathbf{p})+j(\mathbf{p})]\frac{1}{E+i\eta-H}+a(\mathbf{p}),\end{aligned}$$

so that

$$[E-\omega(\mathbf{p})+i\eta-H]a(\mathbf{p})\frac{1}{E+i\eta-H}=j(\mathbf{p})\frac{1}{E+i\eta-H}+a(\mathbf{p}),$$

which is the content of Eq. (15).

Because of the energy-conserving delta function in (12), we need only concern ourselves with ω's on the energy shell, $\sum\omega(\mathbf{p}_i)=\sum\omega(\mathbf{k}_j)$. In this case the expression (13) for $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|Q|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ can ultimately be rearranged in a form that is described as follows.

(*a*) *Single-vertex terms.*—There are $(n+m)!$ of these terms, each corresponding to a distinct sequence of annihilation of the mesons $\mathbf{k}_1$ to $\mathbf{k}_m$ and creation of the mesons $\mathbf{p}_1$ to $\mathbf{p}_n$. A typical term of this type,

$$\left(\Psi_0,j(\mathbf{p}_4)\frac{1}{\omega(\mathbf{p}_4)+i\eta-H}\cdots\frac{1}{\omega(\mathbf{k}_3)-\omega(\mathbf{p}_5)+\omega(\mathbf{k}_2)+i\eta-H}j^*(\mathbf{k}_2)\times\frac{1}{\omega(\mathbf{k}_3)-\omega(\mathbf{p}_5)+i\eta-H}j(\mathbf{p}_5)\frac{1}{\omega(\mathbf{k}_3)+i\eta-H}j^*(\mathbf{k}_3)\Psi_0\right), \quad (17)$$

can be represented diagrammatically by Fig. 1, and corresponds to the sequence: absorption of $\mathbf{k}_3$, emission of $\mathbf{p}_5$, absorption of $\mathbf{k}_2$, $\cdots$, emission of $\mathbf{p}_4$. It should be noted that the *structure* of expression (17), as well as those terms described below, is formally identical with the corresponding expressions for the given order in perturbation theory.

(*b*) *Multiple-vertex terms.*—In addition to terms of type (a) where the vertex operators for single destruction of $\mathbf{k}$ and for single creation of $\mathbf{p}$ are $j^*(\mathbf{k})$ and $j(\mathbf{p})$ respectively, there are terms which contain multiple creation and destruction vertices. In analogy to $j^*(\mathbf{k})$ and $j(\mathbf{p})$, the vertex operator for multiple destruction of $\mathbf{k}_1$, $\mathbf{k}_2$, $\cdots$, $\mathbf{k}_s$ and creation of $\mathbf{p}_1$, $\mathbf{p}_2$, $\cdots$, $\mathbf{p}_t$ is

$$[a(\mathbf{p}_1),\cdots,[[[a(\mathbf{p}_t),[j^*(\mathbf{k}_1),a^*(\mathbf{k}_2)]],a^*(\mathbf{k}_3)],\cdots,a^*(\mathbf{k}_s)]]=[[a(\mathbf{p}_1),\cdots,[[a(\mathbf{p}_{t-1}),j(\mathbf{p}_t)],a^*(\mathbf{k}_1)]],\cdots,a^*(\mathbf{k}_s)] \quad (18)$$

where the order of the bracketing is immaterial as long as the innermost bracket contains the j or j^*.

A typical term of type (*b*),

$$\left(\Psi_0,[a(\mathbf{p}_6),[a(\mathbf{p}_7),j(\mathbf{p}_8)]]\frac{1}{\omega(\mathbf{p}_6)+\omega(\mathbf{p}_7)+\omega(\mathbf{p}_8)+i\eta-H}\cdots\frac{1}{\omega(\mathbf{k}_2)-\omega(\mathbf{p}_3)-\omega(\mathbf{p}_1)+i\eta-H}j(\mathbf{p}_1)\times\frac{1}{\omega(\mathbf{k}_2)-\omega(\mathbf{p}_3)+i\eta-H}[a(\mathbf{p}_3),j^*(\mathbf{k}_2)]\Psi_0\right), \quad (19)$$

can be represented by the diagram shown in Fig. 2. There are as many multiple-vertex terms as there are distinct ways of placing the $\mathbf{p}$'s and $\mathbf{k}$'s in the vertices and of arranging the vertices in a definite sequence. Two terms, however, which differ only by a permutation of the $\mathbf{p}$'s and $\mathbf{k}$'s *within* the vertices are not distinct. Also we note from Eq. (3) that there are no terms with a vertex containing more than N $\mathbf{k}$'s and $\mathbf{p}$'s.

(*c*) *Delta function terms.*—A typical term of this type is

$$\delta(\mathbf{p}_1-\mathbf{k}_2)\delta(\mathbf{p}_3-\mathbf{k}_1)\left(\Psi_0,j(\mathbf{p}_5)\frac{1}{\omega(\mathbf{p}_5)+i\eta-H}\cdots\frac{1}{\omega(\mathbf{k}_3)-\omega(\mathbf{p}_2)-\omega(\mathbf{p}_4)+i\eta-H}\times[a(\mathbf{p}_2),j(\mathbf{p}_4)]\frac{1}{\omega(\mathbf{k}_3)+i\eta-H}j^*(\mathbf{k}_3)\Psi_0\right) \quad (20)$$

which is represented by the diagram in Fig. 3. There is a term of type (*c*) corresponding to every distinct way such a term can be constructed in which some, but not all, of the $\mathbf{k}_i$ are contained in the delta functions (i.e., pass by the source without deflection). The term in which all of the $\mathbf{k}_i$ are in delta functions is the first term of (12) and is not contained in the matrix element of Q.

The matrix element $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|Q|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ is the sum of all the terms of types (*a*), (*b*), and (*c*). Therefore, if we define $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|R|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ to be the sum of all the terms of types (*a*) and (*b*), it is clear that Eq. (12)

can be rewritten in the form

$$(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|S|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)=(m!n!)^{-\frac{1}{2}}\{\sum_{P(\mathbf{k})}\delta(m,n)\delta(\mathbf{p}_1-\mathbf{k}_1)\delta(\mathbf{p}_2-\mathbf{k}_2)\cdots\delta(\mathbf{p}_n-\mathbf{k}_m)$$

$$-2\pi i\delta(\sum_{i=1}^{n}\omega(\mathbf{p}_i)-\sum_{j=1}^{m}\omega(\mathbf{k}_j))[(\mathbf{p}_{12}\mathbf{p}\cdots\mathbf{p}_n|R|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)+\sum_{i,j}\delta(\mathbf{p}_i-\mathbf{k}_j)(\mathbf{p}_1\cdots\mathbf{p}_n\mathbf{p}_i^{-1}|R|\mathbf{k}_1\cdots\mathbf{k}_m\mathbf{k}_j^{-1})$$

$$+\tfrac{1}{2}\sum_{i,j,r,l}\delta(\mathbf{p}_i-\mathbf{k}_j)\delta(\mathbf{p}_r-\mathbf{k}_l)(\mathbf{p}_1\cdots\mathbf{p}_n\mathbf{p}_i^{-1}\mathbf{p}_r^{-1}|R|\mathbf{k}_1\cdots\mathbf{k}_m\mathbf{k}_j^{-1}\mathbf{k}_l^{-1})+\cdots$$

$$+[(m-1)!]^{-1}\sum_{P(\mathbf{p})}\sum_{P(\mathbf{k})}\delta(\mathbf{p}_1-\mathbf{k}_1)\delta(\mathbf{p}_2-\mathbf{k}_2)\cdots\delta(\mathbf{p}_{m-1}-\mathbf{k}_{m-1})(\mathbf{p}_m\mathbf{p}_{m+1}\cdots\mathbf{p}_n|R|\mathbf{k}_m)]\},\quad(21)$$

where the notation $\mathbf{p}^{-1}$ indicates that the meson with this momentum is omitted from the designation of the state. Also, we have assumed for simplicity that $m\leq n$ in writing down the explicit form of the last term in (21).

In Eq. (21) we have explicitly exhibited the delta-function singularities of the S matrix which arise from processes in which some of the mesons do not interact with the source. However, since the Hamiltonian H has an eigenvalue zero it is clear from the description of the terms of types (*a*) and (*b*) that $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|R|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ has a delta-function singularity when a *partial* sum over the energies of the initial mesons equals a *partial* sum over the energies of the final mesons. Physically this type of singularity arises from processes in which the transition from the state $|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ to the state $|\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n)$ is not a single transition of order $m+n$ but is the product of two or more lower order energy-conserving processes. In order to demonstrate the S matrix in a form in which all its delta-function singularities are explicit, we must therefore subtract these singularities from the R matrix. This subtraction proceeds as follows.

First we look at a typical term of $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|R|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$, the expression (17) as an example, rewritten in the form

$$\sum_{E(\tau_1),E(\tau_2),\cdots,E(\tau_{n+m-1})}\Big[(\Psi_0,j(\mathbf{p}_4)\Psi_{\tau_{n+m-1}})\frac{1}{\omega(\mathbf{p}_4)+i\eta-E(\tau_{n+m-1})}\cdots\frac{1}{\omega(\mathbf{k}_3)-\omega(\mathbf{p}_5)+\omega(\mathbf{k}_2)+i\eta-E(\tau_3)}(\Psi_{\tau_3},j^*(\mathbf{k}_2)\Psi_{\tau_2})$$

$$\times\frac{1}{\omega(\mathbf{k}_3)-\omega(\mathbf{p}_5)+i\eta-E(\tau_2)}(\Psi_{\tau_2},j(\mathbf{p}_5)\Psi_{\tau_1})\frac{1}{\omega(\mathbf{k}_3)+i\eta-E(\tau_1)}(\Psi_{\tau_1},j^*(\mathbf{k}_3)\Psi_0)\Big],\quad(17')$$

where Ψ_{τ_i} is an eigenstate of H belonging to the eigenvalue $E(\tau_i)$. With the assumption that the Hamiltonian contains no bound states other than the ground state of the source, the sum over each $E(\tau)$ includes the discrete state $E(\tau)=0$ and a continuum from μ to infinity. When an $E(\tau)$ assumes its zero value in the sum, the energy factor containing this $E(\tau)$ is of the form

$$\frac{1}{\sum\omega(\mathbf{k}_i)-\sum\omega(\mathbf{p}_j)+i\eta}=\mathcal{P}\left(\frac{1}{\sum\omega(\mathbf{k}_i)-\sum\omega(\mathbf{p}_j)}\right)-i\pi\delta[\sum\omega(\mathbf{k}_i)-\sum\omega(\mathbf{p}_j)],\quad(22)$$

where $\mathcal{P}$ means the principal value; and it is therefore possible to decompose (17′) into two parts. The first part is the result of replacing all such *ground-state* expectation values of energy factors in (17′) by their principle values, and the second part is the remainder of (17′). We observe that each term in the second part is a product of delta functions and expressions which are "first parts" of processes of lower order; that is, of processes with less than m initial mesons and less than n final mesons.

If all the terms of types (*a*) and (*b*) are decomposed in this same manner, a corresponding decomposition of $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|R|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ is induced. In particular, if we define $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|T|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ to be the sum of the "first parts" of all the terms contributing to $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|R|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$, the R matrix can be expressed in terms of this T matrix by an equation of a rather simple structure. The relation for the case $n=m=2$ is

$$\delta(\sum_{i=1}^{2}\omega(\mathbf{p}_i)-\sum_{j=1}^{2}\omega(\mathbf{k}_j))(\mathbf{p}_1\mathbf{p}_2|R|\mathbf{k}_1\mathbf{k}_2)=\delta(\sum_{i=1}^{2}\omega(\mathbf{p}_i)-\sum_{j=1}^{2}\omega(\mathbf{k}_j))(\mathbf{p}_1\mathbf{p}_2|T|\mathbf{k}_1\mathbf{k}_2)$$

$$-i\pi\sum_{P(\mathbf{p})}\sum_{P(\mathbf{k})}\delta(\omega(\mathbf{p}_1)-\omega(\mathbf{k}_1))\delta(\omega(\mathbf{p}_2)-\omega(\mathbf{k}_2))(\mathbf{p}_1|T|\mathbf{k}_1)(\mathbf{p}_2|T|\mathbf{k}_2),\quad(23)$$

and the equivalent relation for the general case, although somewhat lengthy, is of the same form and can readily be constructed. If the general relation corresponding to Eq. (23) is substituted into Eq. (21), an expression for the S matrix is obtained in which all the delta function singularities are explicitly exhibited. However, it is possible to state the relationship between the S matrix and the T matrix in a more concise manner. Defining Φ_0 to be the state

of the *bare* source such that $a(\mathbf{q})\Phi_0=0$, we can write[8]

$$(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|S|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$$
$$=(n!m!)^{-\frac{1}{2}}\left(a^*(\mathbf{p}_1)a^*(\mathbf{p}_2)\cdots a^*(\mathbf{p}_n)\Phi_0\left|\text{Normal Product}\frac{1-i\pi K}{1+i\pi K}\right|a^*(\mathbf{k}_1)a^*(\mathbf{k}_2)\cdots a^*(\mathbf{k}_m)\Phi_0\right), \quad (24)$$

where

$$K=\sum_{i,j=1}^{\infty}\frac{1}{i!j!}\int d\mathbf{q}_1\cdots d\mathbf{q}_i d\mathbf{s}_1\cdots d\mathbf{s}_j a^*(\mathbf{q}_1)\cdots a^*(\mathbf{q}_i)(\mathbf{q}_1\mathbf{q}_2\cdots\mathbf{q}_i|T|\mathbf{s}_1\mathbf{s}_2\cdots\mathbf{s}_j)\delta(\sum_{\mu=1}^{i}\omega(q_\mu)-\sum_{\nu=1}^{j}\omega(s_\nu))a(\mathbf{s}_1)\cdots a(\mathbf{s}_j). \quad (25)$$

The reader can readily verify that the relations (24) and (25) are equivalent to the expression for $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|S|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ which would be obtained by the direct substitution indicated above.

The transition matrix $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|T|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ which we have defined can be thought of physically as representing the actual m meson to n meson transition process. It has no delta function singularity when a partial sum of the initial energies equals a partial sum of the final energies, since the principle value integral is to be taken about such points in the energy space. The fact that for these energies the state $|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ can propagate to the state $|\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n)$ by means of two or more processes of order lower than $m+n$ is represented by the explicit dependence of $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|S|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ upon the products of lower order T matrices.

It is evident from the definitions of the T matrix and the R matrix that they are identical for values of the energies in which no partial sum of the initial meson energies equals a partial sum of the final meson energies. Also, $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|T|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ is obtained from $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|R|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ by taking the principle value of all the terms in $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|R|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ which are singular because of the vanishing of energy denominators when partial sums of the $\omega(\mathbf{k}_i)$ equal partial sums of the $\omega(\mathbf{p}_j)$. Realizing that all the other terms of $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|R|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ are finite at these energies, we can write

$$(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|T|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)=\mathcal{P}(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|R|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m) \quad (26)$$

as an alternative definition of the T matrix. The "$\mathcal{P}$" indicates that the principal value of $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|R|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ is to be taken wherever an integration of the $\omega(\mathbf{p}_i)$ and $\omega(\mathbf{k}_j)$ is performed through points for which any sum of the initial energies $\omega(\mathbf{k}_j)$ equals any sum of the final energies $\omega(\mathbf{p}_i)$. The definition of the T matrix given in Eq. (26) will become useful in Sec. III where we extend $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|T|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$ into the complex plane.

The preceding description of the structure of the S matrix is independent of the interaction Hamiltonian. However, in the remainder of this paper we will make use of the explicit form of H_I as exhibited in Eq. (3).

With the H_I shown in Eq. (3), a typical vertex operator which occurs in the terms of the T matrix can be written as

$$[a(\mathbf{p}_1),(\cdots,\{a(\mathbf{p}_t),[j^*(\mathbf{k}_1),a^*(\mathbf{k}_2)]\},\cdots,a^*(\mathbf{k}_s))]$$
$$=\sum_{\mu=1}^{N}\lambda_\mu\mu(\mu-1)\cdots[\mu-(t+s)+1]\left(\int\rho(x)\phi(\mathbf{x})d\mathbf{x}\right)^{\mu-(t+s)}\frac{\rho(k_1)\cdots\rho(k_s)\rho(p_1)\cdots\rho(p_t)}{[2^{s+t}\omega(\mathbf{k}_1)\cdots\omega(\mathbf{k}_s)\omega(\mathbf{p}_1)\cdots\omega(\mathbf{p}_t)]^{\frac{1}{2}}}, \quad (27)$$

which follows directly from the commutation rules $[a(\mathbf{k}),a^*(\mathbf{k}')]=\delta(\mathbf{k}-\mathbf{k}')$ and the expression for $\phi(\mathbf{x})$,

$$\phi(\mathbf{x})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{d\mathbf{k}}{(2\omega)^{\frac{1}{2}}}(a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}+a^*(\mathbf{k})e^{-\mathbf{k}\cdot\mathbf{x}}). \quad (28)$$

Consequently, if we define

$$J_r\equiv\sum_{\mu=1}^{N}\lambda_\mu\mu(\mu-1)\cdots(\mu-r+1)\left(\int\rho(x)\phi(\mathbf{x})d\mathbf{x}\right)^{\mu-r}, \quad (29)$$

a typical term of $(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|T|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m)$, the term (19) for example, can be rewritten in the form

$$\frac{\rho(k_1)\cdots\rho(k_m)\rho(p_1)\cdots\rho(p_n)}{[2^{m+n}\omega(\mathbf{k}_1)\cdots\omega(\mathbf{k}_m)\omega(\mathbf{p}_1)\cdots\omega(\mathbf{p}_n)]^{\frac{1}{2}}}$$
$$\times\left(\Psi_0,J_3\frac{1}{\omega(\mathbf{p}_6)+\omega(\mathbf{p}_7)+\omega(\mathbf{p}_8)+i\eta-H}\cdots\frac{1}{\omega(\mathbf{k}_2)-\omega(\mathbf{p}_3)-\omega(\mathbf{p}_1)+i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_2)-\omega(\mathbf{p}_3)+i\eta-H}J_2\Psi_0\right). \quad (19')$$

[8] We caution the reader that K in Eq. (25) is not the usual reaction matrix, defined, for example, in reference 5, since only a selective removal of energy-conserving intermediate states has been effected.

We are led, therefore, to define a G matrix,

$$(\omega(\mathbf{p}_1)\cdots\omega(\mathbf{p}_n)|G|\omega(\mathbf{k}_1)\cdots\omega(\mathbf{k}_m))\equiv\frac{[2^{m+n}\omega(\mathbf{k}_1)\cdots\omega(\mathbf{k}_m)\omega(\mathbf{p}_1)\cdots\omega(\mathbf{p}_n)]^{\frac{1}{2}}}{\rho(k_1)\cdots\rho(k_m)\rho(p_1)\cdots\rho(p_n)}(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n|T|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_m), \quad (30)$$

which depends only upon the energies of the incident and outgoing mesons in denominators typical of a usual perturbation expansion. The structure of the G matrix is seen to be the same as that of the T matrix except that the energy dependent vertex operators in the terms of the T matrix are replaced by the appropriate J_i operators.

Consider now the typical term of $(\omega(\mathbf{p}_1)\cdots\omega(\mathbf{p}_n)|G|\omega(\mathbf{k}_1)\cdots\omega(\mathbf{k}_m))$ which corresponds to Eq. (17),

$$\left(\Psi_0,J_1\frac{1}{\omega(\mathbf{p}_4)+i\eta-H}\cdots\frac{1}{\omega(\mathbf{k}_3)-\omega(\mathbf{p}_5)+\omega(\mathbf{k}_2)+i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_3)-\omega(\mathbf{p}_5)+i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_3)+i\eta-H}J_1\Psi_0\right). \quad (31)$$

The complex conjugate of (31) is

$$\left(\Psi_0,J_1\frac{1}{\omega(\mathbf{k}_3)-i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_3)-\omega(\mathbf{p}_5)-i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_3)-\omega(\mathbf{p}_5)+\omega(\mathbf{k}_2)-i\eta-H}\cdots\frac{1}{\omega(\mathbf{p}_4)-i\eta-H}J_1\Psi_0\right), \quad (32)$$

since the J_i defined in Eq. (29) are Hermitian. Except for the replacement of $i\eta$ by $-i\eta$ this term is the same as the term in $(\omega(\mathbf{k}_1)\cdots\omega(\mathbf{k}_m)|G|\omega(\mathbf{p}_1)\cdots\omega(\mathbf{p}_n))$ which corresponds to the sequence: absorption of $\mathbf{p}_4$, $\cdots$, emission of $\mathbf{k}_2$, absorption of $\mathbf{p}_5$, and emission of $\mathbf{k}_3$. However, since the interaction H_I contains only S waves, the time reversal invariance of the theory forces all the matrices, including the G matrix, to be symmetric. Consequently it follows that $(\omega(\mathbf{p}_1)\cdots\omega(\mathbf{p}_n)|G|\omega(\mathbf{k}_1)\cdots\omega(\mathbf{k}_m)^*$ is obtained from $(\omega(\mathbf{p}_1)\cdots\omega(\mathbf{p}_n)|G|\omega(\mathbf{k}_1)\cdots\omega(\mathbf{k}_m))$ by replacing every $i\eta$ by its negative in all the terms which contribute to the matrix element of G.

In Appendix B we describe how this property of the G matrix is used to derive integral equations for the G matrix elements which are equivalent to the unitarity condition on the S matrix. For illustrative purposes we again choose the case $n=m=2$. In this case, the unitarity relation is

$$\begin{aligned}
\mathrm{Im}(\omega(\mathbf{p}_1)\omega(\mathbf{p}_2)|G|\omega(\mathbf{k}_1)\omega(\mathbf{k}_2))=&-\frac{\pi}{2}\sum_{P(\mathbf{p})}\sum_{P(\mathbf{k})}\sum_{r=1}^{\infty}\int d\mathbf{q}_1\cdots d\mathbf{q}_r\frac{(\rho(q_1)\cdots\rho(q_r))^2}{2^r\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_r)}\\
\times\Bigg\{&\left[\frac{(\omega(\mathbf{p}_1)\omega(\mathbf{p}_2)|G|\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_r))^*}{2!r!}+i\pi\sum_{l=1}^{r-1}\frac{(\omega(\mathbf{p}_1)|G|\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_l))^*(\omega(\mathbf{p}_2)|G\delta|\omega(\mathbf{q}_{l+1})\cdots\omega(\mathbf{q}_r))^*}{l!(r-l)!}\right]\\
\times&\sum_{P(q)}\left[\frac{(\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_r)|G\delta|\omega(\mathbf{k}_1)\omega(\mathbf{k}_2))}{r!2!}-i\pi\sum_{l'=1}^{r-1}\frac{(\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_{l'})|G\delta|\omega(\mathbf{k}_1))(\omega(\mathbf{q}_{l'+1})\cdots\omega(\mathbf{q}_r)|G\delta|\omega(\mathbf{k}_2))}{l'!(r-l')!}\right]\\
+&(\omega(\mathbf{p}_2)|G|\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_r))^*\left[\frac{(\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_r)\omega(\mathbf{p}_1)|G\delta|\omega(\mathbf{k}_1)\omega(\mathbf{k}_2))}{r!2!}\right.\\
&\left.-i\pi\sum_{l=1}^{r-1}\frac{(\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_l)\omega(\mathbf{p}_1)|G\delta|\omega(\mathbf{k}_1))(\omega(\mathbf{q}_{l+1})\cdots\omega(\mathbf{q}_r)|G\delta|\omega(\mathbf{k}_2))}{l!(r-l)!}\right]+\left[\frac{(\omega(\mathbf{p}_1)\omega(\mathbf{p}_2)|G|\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_r)\omega(\mathbf{k}_1))^*}{2!r!}\right.\\
&\left.+i\pi\sum_{l=1}^{r-1}\frac{(\omega(\mathbf{p}_1)|G|\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_l)\omega(\mathbf{k}_1))^*(\omega(\mathbf{p}_2)|G\delta|\omega(\mathbf{q}_{l+1})\cdots\omega(\mathbf{q}_r))^*}{l!(r-l)!}\right](\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_r)|G\delta|\omega(\mathbf{k}_2))\Bigg\}\\
&+\text{complex conjugate}, \quad (33)
\end{aligned}$$

where the symbol $(\omega(\mathbf{p}_1)\cdots\omega(\mathbf{p}_n)|G\delta|\omega(\mathbf{k}_1)\cdots\omega(\mathbf{k}_m))$ is used to represent

$$(\omega(\mathbf{p}_1)\cdots\omega(\mathbf{p}_n)|G|\omega(\mathbf{k}_1)\cdots\omega(\mathbf{k}_m))\delta(\sum_{i=1}^{n}\omega(\mathbf{p}_i)-\sum_{j=1}^{n}\omega(\mathbf{k}_j)).$$

The general case of an arbitrary number of incident and emergent mesons has the same general structure.

III. DERIVATION OF THE DISPERSION RELATIONS

As previously noted, the matrix element $(\omega(\mathbf{p}_1)\cdots\omega(\mathbf{p}_n)|G|\omega(\mathbf{k}_1)\cdots\omega(\mathbf{k}_m))$ depends upon the initial and final meson energies only through energy factors of the form $(\sum\omega(\mathbf{k}_i)-\sum\omega(\mathbf{p}_j)+i\eta-H)^{-1}$. If we define

$$\begin{aligned}\mu_i&\equiv\omega(\mathbf{p}_i)/\omega,\\ \nu_i&\equiv\omega(\mathbf{k}_i)/\omega,\\ \omega&\equiv\sum_{i=1}^{n}\omega(\mathbf{p}_i)=\sum_{j=1}^{m}\omega(\mathbf{k}_j),\end{aligned}\tag{34}$$

we can write $(\omega(\mathbf{p}_1)\cdots\omega(\mathbf{p}_n)|G|\omega(\mathbf{k}_1)\cdots\omega(\mathbf{k}_m))$ as a function of the μ_i, ν_j, and ω by replacing all energy factors $[\sum\omega(\mathbf{k}_i)-\sum\omega(\mathbf{p}_j)+i\eta-H]^{-1}$ by $[(\sum\nu_i-\sum\mu_j)\omega+i\eta-H]^{-1}$. We now denote the G matrix by $(\mu_1\cdots\mu_n|G(\omega)|\nu_1\cdots\nu_m)$. The next step is to eliminate the explicit appearance of the $i\eta$ in the denominators by giving ω an infinitesimal, positive imaginary part. Because of the principal value definition of the G matrix we need only justify this replacement for nonzero values of the coefficients $(\sum\nu_i-\sum\mu_j)$. However, this procedure is obviously correct if the coefficients are positive while the non-negative property of the spectrum of H insures that there is no error made if $(\sum\nu_i-\sum\mu_j)$ is negative. Consequently, we can extend $(\mu_1\cdots\mu_n|G|\nu_1\cdots\nu_m)$ into the complex plane such that the extended function $(\mu_1\cdots\mu_n|G(z)|\nu_1\cdots\nu_m)$ [occasionally abbreviated to $G_{nm}(z)$] is obtained from $(\mu_1\cdots\mu_n|G(\omega)|\nu_1\cdots\nu_m)$ by writing all energy factors in the form $[(\sum\nu_i-\sum\mu_j)z-H]^{-1}$ and such that the equation

$$G_{nm}(\omega)=\lim_{\eta\to 0_+}G_{nm}(\omega+i\eta),\tag{35}$$

is satisfied.

We now list the properties of $G_{nm}(z)$ which are relevant to the derivation of the dispersion relations.

(a) $G_{nm}(z)$ is analytic in the upper half-plane. This property follows immediately from the reality of the spectrum of H.

(b) $G_{nm}(z)$ has no poles on the real axis. The origin of the z plane is the only point on the real axis for which this statement is not obvious. In Appendix C we prove that there is no pole at $z=0$.

(c) $G_{nm}(z)$ has branch lines extending from μ to ∞ and from $-\mu$ to $-\infty$ along the real axis. This property follows immediately from the form of the energy denominators in the terms of $G_{nm}(z)$ and from the assumption that H has a continuum spectrum extending from μ to ∞.

(d) $G_{nm}{}^*(\omega+i\eta)=G_{nm}(\omega-i\eta)$. This fact has been mentioned in the discussion following Eq. (32).

(e) $G_{nm}(\omega+i\eta)=G_{nm}(-\omega-i\eta)$. Under the replacement of $\omega+i\eta$ by its negative, every energy factor of the form $[(\sum\nu_i-\sum\mu_j)(\omega+i\eta)-H]^{-1}$ is replaced by $[(\sum\mu_j-\sum\nu_i)(\omega+i\eta)-H]^{-1}$. However, such a replacement interchanges the roles of the initial and final states so that the validity of property (e) is guaranteed by the symmetry of the G matrix.

(f) An immediate consequence of (d) and (e) is that the real (imaginary) part of $G_{nm}(\omega+i\eta)$ is an even (odd) function of ω.

(g) For sufficiently large $|z|$, $G_{nm}(z)\sim C_{nm}z^{-2\lambda_{nm}}$ where C_{nm} is a finite constant and λ_{nm} is the non-negative integer which satisfies the inequality $(2\lambda_{nm}-1)N<n+m\leq(2\lambda_{nm}+1)N$. The use of "$N$" here is the same as in Eq. (3). One verifies this property from a simple examination of the structure of the terms in $G_{nm}(z)$.

The properties (a), (b), and (g), together with Cauchy's theorem, indicate that

$$\oint\frac{G_{nm}(z)(z-\omega_0)^{2\lambda_{nm}-1}}{z-\omega}dz=0,\tag{36}$$

where C is the contour shown in Fig. 4. If we now use property (g) to neglect the integral over the infinite arc, we obtain the expression

$$G_{nm}(\omega)=\delta(\lambda_{nm},0)G_{nm}(\omega_0)+\frac{(\omega-\omega_0)^{1-2\lambda_{nm}}}{\pi i}\mathcal{P}\int_{-\infty}^{\infty}\frac{G_{nm}(\omega')(\omega'-\omega_0)^{2\lambda_{nm}-1}}{\omega'-\omega}d\omega',\tag{37}$$

whose real part can be written as

$$\begin{aligned}\mathrm{Re}(\mu_1\cdots\mu_n|G(\omega)|\nu_1\cdots\nu_m)&=\delta(\lambda_{nm},0)\,\mathrm{Re}(\mu_1\cdots\mu_n|G(\omega_0)|\nu_1\cdots\nu_m)\\&+\frac{(\omega-\omega_0)^{1-2\lambda_{nm}}}{\pi}\mathcal{P}\int_{\mu}^{\infty}d\omega'\left[\frac{(\omega'+\omega)(\omega'-\omega_0)^{2\lambda_{nm}-1}-(\omega'-\omega)(\omega'+\omega_0)^{2\lambda_{nm}-1}}{\omega'^2-\omega^2}\right]\mathrm{Im}(\mu_1\cdots\mu_n|G(\omega)|\nu_1\cdots\nu_m),\end{aligned}\tag{38}$$

by employing the properties of $G_{nm}(z)$ listed in (d) and (f).

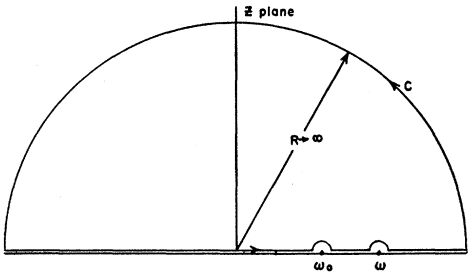

FIG. 4. Contour C for the application of Cauchy's theorem to the transition amplitudes G_{nm}.

The Eqs. (38) constitute an infinite set of dispersion relations for the transition matrix G. The relations which correspond to processes in which the total number of mesons involved, $n+m$, is less than or equal to N, have as parameters the real parts of the corresponding G matrices at the arbitrarily chosen energies ω_0. However, all the other dispersion relations contain no arbitrary parameters. In fact, the relations which correspond to process where $n+m$ is greater than N (i.e., $\lambda_{nm}\geq 1$) are of the form

$$\mathrm{Re}G_{nm}(\omega)=\frac{(\omega-\omega_0)^{1-2\lambda_{nm}}}{\pi}\mathcal{P}\int_{\mu}^{\infty}d\omega'\left[\frac{(\omega'+\omega)(\omega'-\omega_0)^{2\lambda_{nm}-1}-(\omega'-\omega)(\omega'+\omega_0)^{2\lambda_{nm}-1}}{\omega'^2-\omega^2}\right]\mathrm{Im}G_{nm}(\omega'), \tag{39}$$

and for $\lambda_{nm}\geq 2$ the right side of this expression is not manifestly independent of ω_0. Therefore, if we choose particular values of ω_0 for each dispersion relation, and from this choice obtain a solution of the coupled integral equations consisting of (38) and the generalization of (33), this solution can be considered of physical interest only if it is independent of every ω_0 chosen in the dispersion relations for $n+m>3N$.

Suppose now that we pick the arbitrary parameters which appear in the dispersion relations for $n+m\leq N$ to be all evaluated at zero or at infinite energy. In either of these cases it is easy to see from the energy dependence of the G matrices that these parameters depend only upon $n+m$ and not separately upon n and m nor upon the μ_i and ν_j. Consequently the set of equations arising from either of these choices for the ω_0 ($n+m\leq N$) have only $N-1$ independent parameters, one corresponding to each value of $n+m$ in the range $2\leq n+m\leq N$. The fact that the number of independent parameters cannot depend upon our choice of the ω_0, insures us that this is a general property of the dispersion relations (38). This result is not surprising. It undoubtedly is a reflection of the existence of N arbitrary constants λ_i in the Hamiltonian and of the well-known result that for $N=1$ the interaction (3) has no dynamical consequences.

IV. QUALITATIVE DISCUSSION OF SOLUTIONS

We now turn our attention to the problem of solving the coupled integral Eqs. (33) and (38). We consider first the possibility of choosing the coupling constants λ_i to be of an appropriate order in a small parameter ϵ and then of solving the equations by iteration. Neglecting questions of convergence, such a procedure is easily found. In particular, if we choose each λ_i to be of order ϵ^i, we obtain immediately from the form of the equations that to order ϵ^2, $\mathrm{Re}G_{11}(\omega)=\mathrm{Re}G_{11}(\omega_0)$ and that all other matrix elements of G are zero. To order ϵ^3, $\mathrm{Re}G_{11}(\omega)=\mathrm{Re}G_{11}(\omega_0)$, $\mathrm{Re}G_{12}(\omega)=\mathrm{Re}G_{21}(\omega)=\mathrm{Re}G_{12}(\omega_0)$, and all the other matrix elements are zero. If this approximation to $\mathrm{Re}G_{11}(\omega)$ is inserted into the generalization of the unitarity expression (33) for $\mathrm{Im}G_{11}(\omega)$, we obtain $\mathrm{Im}G_{11}(\omega)$ to order ϵ^4. Substitution of this quantity back into the integrand of the scattering dispersion relation then yields $\mathrm{Re}G_{11}(\omega)$ to order ϵ^4, etc. This procedure can be continued indefinitely to obtain all the matrix elements of G up to any order in ϵ, and, in particular, is not hindered by the nonexistence of arbitrary constants in the dispersion relations for the higher order processes. The reason for this latter statement is that for $n+m>3$ the unitarity condition (33) relates such processes in leading nonvanishing order to a product of lower order matrix elements already determined in the sequence. Since the two cases $n+m=2$ and $n+m=3$ are exceptions to this rule, the corresponding matrix elements of G are forced to be zero by the iterative procedure if the dispersion relations for these matrix elements contain no arbitrary constants. It follows, therefore, that for $N=1$ all the G matrix elements are zero, and for $N=2$ only the scattering amplitude is nonvanishing. We should remark also that this method is only possible because the unitarity condition (33) is valid for all energies from μ to ∞. Since this energy range contains subregions which are nonphysical for processes other than scattering, it was necessary to derive (33) from the explicit structure of the G matrix. To simply assert the unitarity of the S matrix would not have been sufficient unless the implied relations could somehow be extended to energies below the production thresholds.

By virtue of the dispersion relations, the real part of each matrix element at one energy is related to an integral of its imaginary part over all energies from μ to ∞. Consequently, for this iterative procedure to yield a solution,

it is necessary that the expansions of the matrix elements in powers of ϵ converge uniformly over this full energy domain. It is reasonable that this type of convergence occurs if the solution does not exhibit a resonant behavior. Put another way, we conjecture that the iterative procedure picks out the solution (if there is one and only one) which does not lead to resonance processes.

It must be admitted that our Eqs. (33) and (38), almost certainly do not possess a unique solution. This has been shown rigorously for the simplest cases of linear[9] or quadratic[10] coupling of the meson field to its source. Indeed we shall here demonstrate the nonuniqueness in a more physical manner by exhibiting an extended class of Hamiltonians,[11] obviously different in their physical consequences, which nevertheless satisfy the same sequence of dispersion relations which we have derived from the Hamiltonian in Eq. (3). We may consider, for example, the Hamiltonian $\mathfrak{IC}=\mathfrak{IC}_0+\mathfrak{IC}_I$, where

$$\mathfrak{IC}_0=\sum_{\mathbf{k}} a^*(\mathbf{k})a(\mathbf{k})\omega(\mathbf{k})+(\Psi_A^*\Psi_B^*)\begin{pmatrix}\omega_A & 0\\ 0 & \omega_B\end{pmatrix}\begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix}, \tag{40}$$

$$\mathfrak{IC}_I=\sum_{\mu=1}^{N}(\Psi_A^*\Psi_B^*)\begin{pmatrix}g_{AA^\mu} & g_{AB^\mu}\\ g_{AB^\mu} & g_{BB^\mu}\end{pmatrix}\begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix}\left[\int\rho(x)\phi(\mathbf{x})d\mathbf{x}\right]^\mu. \tag{41}$$

The free field Hamiltonian, $\mathfrak{IC}_0$, differs from the one in Eq. (2) in that it describes a source capable, in the absence of interaction, of existing in two states, the ground state of energy ω_A in which $\Psi_A^*\Psi_A=1$ and $\Psi_B^*\Psi_B=0$, and an excited state of energy ω_B in which $\Psi_B^*\Psi_B=1$ and $\Psi_A^*\Psi_A=0$. However, if $\omega_B-\omega_A>\mu$, the excited state is unstable against meson emission when the interaction H_I is included, and, in this case the total Hamiltonian has the same spectrum as the one in Eq. (1). Similarly, by further complicating the structure of the source, we can generate an infinite class of Hamiltonians of this form, all having the same spectrum as the Hamiltonian in Eq. (1). From the spectral identity and from the obvious structural similarity, it follows that all these Hamiltonians are described by the same set of dispersion relations (39) and by the same equations of unitarity (33). However, physically it is clear that the solutions for each of these Hamiltonians must be different, since the transition amplitudes will exhibit resonances at energies near the excited states of the unperturbed source. This situation at least suggests that the iterative solution, if it exists, does describe the original theory represented by the Hamiltonian (1). We should remark also that this is the only Hamiltonian of the infinite class which contains essentially (see Sec. III) the same number of coupling constants as there are arbitrary constants in the dispersion relations. The extended class of Hamiltonians will be studied further in a subsequent publication.

APPENDIX A

In this appendix we prove the orthogonality relation (11).[12] We shall derive the relation explicitly for the $\Psi^{(+)}$, but it will be clear from the proof that a completely equivalent derivation holds for the $\Psi^{(-)}$. For conciseness we use abbreviations exemplified by the following:

$$\begin{aligned}\Psi^{(+)}(\mathbf{k}_m)&=\Psi^{(+)}(\mathbf{k}_1\cdots\mathbf{k}_m),\\ \mathbf{a}(\mathbf{k}_m)&=a(\mathbf{k}_1)a(\mathbf{k}_2)\cdots a(\mathbf{k}_m),\\ \boldsymbol{\omega}(\mathbf{k}_m)&=\sum_{i=1}^{m}\omega(\mathbf{k}_i),\end{aligned} \tag{A.1}$$

with a similar convention for the primed variables below.

Neglecting the normalization constants $N(k)$ and $N(k')$, we have from (10) that

$$(\Psi^{(+)}(\mathbf{k}_m),\Psi^{(+)}(\mathbf{k}_n'))=(\Psi_0,\mathbf{a}(\mathbf{k}_m)\Psi^{(+)}(\mathbf{k}_n'))+(\boldsymbol{\omega}(\mathbf{k}_m)-\boldsymbol{\omega}(\mathbf{k}_n')-i\eta)^{-1}(\Psi_0,[\mathbf{a}(\mathbf{k}_m),H_I]\Psi^{(+)}(\mathbf{k}_n')), \tag{A.2}$$

and using (10) to expand $\Psi^{(+)}(\mathbf{k}_n')$ in the second term of (A.2), there results after slight rearrangement

$$\begin{aligned}(\Psi_0,[\mathbf{a}(\mathbf{k}_n),H_I]\Psi^{(+)}(\mathbf{k}_n'))&=(\Psi_0,[[\mathbf{a}(\mathbf{k}_m),H_I],\mathbf{a}^*(\mathbf{k}_n')]\Psi_0)\\ &+(\mathbf{a}(\mathbf{k}_n')\Psi_0,[\mathbf{a}(\mathbf{k}_m),H_I]\Psi_0)+(\Psi_0,[\mathbf{a}(\mathbf{k}_m),H_I](\boldsymbol{\omega}(\mathbf{k}_n')+i\eta-H)^{-1}[H_I,\mathbf{a}^*(\mathbf{k}_n^1)]\Psi_0).\end{aligned} \tag{A.3}$$

[9] Castillejo, Dalitz, and Dyson, Phys. Rev. **101**, 453 (1956).
[10] A. Klein, Phys. Rev. **104**, 1136 (1956).
[11] The model which we consider here is similar to the one which was used for a similar purpose by F. J. Dyson, Phys. Rev. **106**, 157 (1957).
[12] This proof is a generalization of the one given by Wick (reference 6, Appendix C) for single-meson states and the P-wave interaction.

Now, applying the relation

$$\mathbf{a}(\mathbf{k}_n')\Psi_0=-(\omega(\mathbf{k}_n')+H)^{-1}[\mathbf{a}(\mathbf{k}_n'),H_I]\Psi_0, \tag{A.4}$$

to the second term in (A.3) and substituting the result into (A.2), we obtain

$$(\Psi^{(+)}(\mathbf{k}_m),\Psi^{(+)}(\mathbf{k}_n'))=(\Psi_0,\mathbf{a}(\mathbf{k}_m)\Psi^{(+)}(\mathbf{k}_n'))+(\omega(\mathbf{k}_m)-\omega(\mathbf{k}_n')-i\eta)^{-1}[(\Psi_0,[[\mathbf{a}(\mathbf{k}_m),H_I],\mathbf{a}^*(\mathbf{k}_n')]\Psi_0)$$
$$-(\Psi_0,[H_I,\mathbf{a}^*(\mathbf{k}_r')](\omega(\mathbf{k}_n')+H)^{-1}[\mathbf{a}(\mathbf{k}_m),H_I]\Psi_0)+(\Psi_0,[\mathbf{a}(\mathbf{k}_m),H_I](\omega(\mathbf{k}_n')+i\eta-H)^{-1}[H_I,\mathbf{a}^*(\mathbf{k}_n')]\Psi_0)]. \tag{A.5}$$

If (10) is used to expand $\Psi^{(+)}(k_n')$ in the first term of (A.5) and if the result is re-expressed with the use of (A.4) and substituted back into (A.5), we have

$$(\Psi^{(+)}(\mathbf{k}_m),\Psi^{(+)}(\mathbf{k}_n'))=(\Psi_0,[\mathbf{a}(\mathbf{k}_m),\mathbf{a}^*(\mathbf{k}_n')]\Psi_0)+(\Psi_0,[H_I,\mathbf{a}^*(\mathbf{k}'_n)](\omega(\mathbf{k}_n')+H)^{-1}(\omega(\mathbf{k}_m)+H)^{-1}[\mathbf{a}(\mathbf{k}_m),H_I]\Psi_0)$$
$$+(\Psi_0,\mathbf{a}(\mathbf{k}_m)(\omega(\mathbf{k}_n')+i\eta-H)^{-1}[H_I,\mathbf{a}^*(\mathbf{k}_n')]\Psi_0)+(\omega(\mathbf{k}_m)-\omega(\mathbf{k}_n')-i\eta)^{-1}[(\Psi_0,[[\mathbf{a}(\mathbf{k}_m),H_I],\mathbf{a}^*(\mathbf{k}_n')]\Psi_0)$$
$$-(\Psi_0,[H_I,\mathbf{a}^*(\mathbf{k}_n')](\omega(\mathbf{k}_n')+H)^{-1}[\mathbf{a}(\mathbf{k}_m),H_I]\Psi_0)+(\Psi_0,[\mathbf{a}(\mathbf{k}_m),H_I](\omega(\mathbf{k}_n')+i\eta-H)^{-1}[H_I,\mathbf{a}^*(\mathbf{k}_n')]\Psi_0)]. \tag{A.6}$$

Equation (15) can now be used to transform the third term in (A.6). The result is

$$(\Psi^{(+)}(\mathbf{k}_m),\Psi^{(+)}(\mathbf{k}_n'))=(\Psi_0,[\mathbf{a}(\mathbf{k}_n),\mathbf{a}^*(\mathbf{k}_n')]\Psi_0)+(\Psi_0,[H_I,\mathbf{a}^*(\mathbf{k}_n')](\omega(\mathbf{k}_n')+H)^{-1}(\omega(\mathbf{k}_n)+H)^{-1}[\mathbf{a}(\mathbf{k}_m),H_I]\Psi_0)$$
$$+(\omega(\mathbf{k}_m)-\omega(\mathbf{k}_n')-i\eta)^{-1}[(\Psi_0,[[\mathbf{a}(\mathbf{k}_m),H_I],\mathbf{a}^*(\mathbf{k}_n')]\Psi_0)-(\Psi_0,\mathbf{a}(\mathbf{k}_m)[H_I,\mathbf{a}^*(\mathbf{k}_n')]\Psi_0)$$
$$-(\Psi_0,[H_I,\mathbf{a}^*(\mathbf{k}_n')](\omega(\mathbf{k}_n')+H)^{-1}[\mathbf{a}(\mathbf{k}_m),H_I]\Psi_0)]. \tag{A.7}$$

Collecting the first two terms in the square bracket and employing the equality (A.4), we obtain

$$(\Psi^{(+)}(\mathbf{k}_m),\Psi^{(+)}(\mathbf{k}_n'))=(\Psi_0,[\mathbf{a}(\mathbf{k}_m),\mathbf{a}^*(\mathbf{k}_n')]\Psi_0)+(\omega(\mathbf{k}_n')-\omega(\mathbf{k}_m)-i\eta)^{-1})\Psi_0,[H_I,[\mathbf{a}(\mathbf{k}_m),\mathbf{a}^*(\mathbf{k}_n')]]\Psi_0)$$
$$+(\Psi_0,[H_I,\mathbf{a}^*(\mathbf{k}_n')]L[\mathbf{a}(\mathbf{k}_m),H_I]\Psi_0), \tag{A.8}$$

where

$$L=\frac{1}{(\omega(\mathbf{k}_n')+H)(\omega(\mathbf{k}_m)+H)}+\frac{1}{\omega(\mathbf{k}_m)-\omega(\mathbf{k}_n')-i\eta}\left(\frac{1}{\omega(\mathbf{k}_m)+H}-\frac{1}{\omega(\mathbf{k}_n')+H}\right)=0, \tag{A.9}$$

and therefore

$$(\Psi^{(+)}(\mathbf{k}_m),\Psi^{(+)}(\mathbf{k}_n'))=(\Psi_0,[\mathbf{a}(\mathbf{k}_m),\mathbf{a}^*(\mathbf{k}_n')]\Psi_0)+(\omega(\mathbf{k}_n')-\omega(\mathbf{k}_m)-i\eta)^{-1}(\Psi_0,[H_I,[\mathbf{a}(\mathbf{k}_m),\mathbf{a}^*(\mathbf{k}_n')]]\Psi_0). \tag{A.10}$$

In order to show that (A.10) is equivalent to (11), we note that

$$[a(\mathbf{k}_m),a^*(\mathbf{k}_n')]=\delta(m,n)\sum_{P(\mathbf{k}')}\boldsymbol{\delta}(\mathbf{k}_m-\mathbf{k}_m')+\sum\boldsymbol{\delta}(\mathbf{k}_l-\mathbf{k}_l')\mathbf{a}^*(\mathbf{k}_{n-l}')\mathbf{a}(\mathbf{k}_{m-l}), \tag{A.11}$$

where

$$\begin{aligned}\boldsymbol{\delta}(\mathbf{k}_m-\mathbf{k}_m')&=\delta(\mathbf{k}_1-\mathbf{k}_1')\delta(\mathbf{k}_2-\mathbf{k}_2')\cdots\delta(\mathbf{k}_m-\mathbf{k}_m'),\\ \mathbf{a}^*(\mathbf{k}_{n-l}')&=a^*(\mathbf{k}_{l+1}')a^*(\mathbf{k}_{l+2}')\cdots a^*(\mathbf{k}_n'),\\ \mathbf{a}(\mathbf{k}_{m-l})&=a(\mathbf{k}_{l+1})a(\mathbf{k}_{l+2})\cdots a(\mathbf{k}_m),\end{aligned} \tag{A.12}$$

and where $\sum$ indicates a sum over all such possible terms.

Denoting

$$\sum_{i=l+1}^{n}\omega(k_i') \quad \text{and} \quad \sum_{i=l+1}^{m}\omega(k_i)$$

by $\omega(\mathbf{k}_{n-l})$ and $\omega(\mathbf{k}_{m-l})$, respectively, we obtain from substituting (A.11) into (A.10) that

$$(\Psi^{(+)}(\mathbf{k}_m),\Psi^{(+)}(\mathbf{k}_n'))=(\Psi_0,\Psi_0)\delta(m,n)\sum_{P(\mathbf{k}')}\boldsymbol{\delta}(\mathbf{k}_m-\mathbf{k}_m')$$
$$+\sum\boldsymbol{\delta}(\mathbf{k}_l-\mathbf{k}_l')(\Psi_0,[\mathbf{a}^*(\mathbf{k}_{n-l}')\mathbf{a}(\mathbf{k}_{m-l})+(\omega(\mathbf{k}_{n-l}')-\omega(\mathbf{k}_{m-l})-i\eta)^{-1}[H_I,\mathbf{a}^*(\mathbf{k}_{n-l}')\mathbf{a}(\mathbf{k}_{m-l})]]\Psi_0), \tag{A.13}$$

and since by (A.4) the operator in the square bracket can be put into the form,

$$[H_I,\mathbf{a}^*(\mathbf{k}_{n-l}')]\left\{\frac{1}{(\omega(\mathbf{k}_{n-l}')+H)(\omega(\mathbf{k}_{m-l})+H)}+\frac{1}{\omega(\mathbf{k}_{n-l}')-\omega(\mathbf{k}_{m-l})-i\eta}\left[\frac{1}{\omega(\mathbf{k}_{n-l}')+H}-\frac{1}{\omega(\mathbf{k}_{m-l})+H}\right]\right\}$$
$$\times[\mathbf{a}(\mathbf{k}_{m-l}),H_I]=0,$$

it follows that

$$(\Psi^{(+)}(\mathbf{k}_m),\Psi^{(+)}(\mathbf{k}_n'))=\sum_{P(\mathbf{k}')}\delta(m,n)\delta(\mathbf{k}_m-\mathbf{k}_m'). \tag{A.14}$$

The replacement of the normalization constants and the elimination of the abbreviations gives directly Eq. (11).

APPENDIX B

We now indicate how to derive the expression for the imaginary part of the matrix element of G, choosing for illustrative purposes, the example of two mesons in both the initial and final state. In the domain of physical energies this relation is equivalent to the unitarity condition on the S matrix. The treatment of this case is sufficient to indicate how the corresponding relation for an arbitrary matrix element can be obtained.

From the discussion following Eq. (32) in Sec. II, we can write

$$\begin{aligned}&\mathrm{Im}(\omega(\mathbf{p}_1)\omega(\mathbf{p}_2)\,|G|\,\omega(\mathbf{k}_1)\omega(\mathbf{k}_2))\\&=(2i)^{-1}\sum\Bigg[\Bigg(\Psi_0,J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)+i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)+i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_1)+i\eta-H}J_1\Psi_0\Bigg)\\&\quad-\Bigg(\Psi_0,J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)-i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_1)-i\eta-H}J_1\Psi_0\Bigg)\Bigg],\end{aligned} \tag{B.1}$$

where the symbol $\sum$ indicates a summation over all the terms which contribute to $(\omega(\mathbf{p}_1)\omega(\mathbf{p}_2)\,|G|\,\omega(\mathbf{k}_1)\omega(\mathbf{k}_2))$. According to the principal-value definition of the G matrix, it is assumed that $\omega(\mathbf{k}_2)\neq\omega(\mathbf{p}_1)\neq\omega(\mathbf{k}_1)\neq\omega(\mathbf{p}_2)\neq\omega(\mathbf{k}_2)$ and of course that $\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)=\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)$.

If we now add zero, in the form

$$\begin{aligned}&\sum(1-1)\Bigg[\Bigg(\Psi_0,J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)-i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)+i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_1)+i\eta-H}J_1\Psi_0\Bigg)\\&\quad+\Bigg(\Psi_0,J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(p_1)-i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_1)+i\eta-H}J_1\Psi_0\Bigg)\Bigg]\end{aligned} \tag{B.2}$$

to the right side of (B.1), there results

$$\begin{aligned}&\mathrm{Im}(\omega(\mathbf{p}_1)\omega(\mathbf{p}_2)\,|G|\,\omega(\mathbf{k}_1)\omega(\mathbf{k}_2))\\&=-\pi\sum\Bigg[\Bigg(\Psi_0,J_1\delta(\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)-H)J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)+i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_1)+i\eta-H}J_1\Psi_0\Bigg)\\&+\Bigg(\Psi_0,J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)-i\eta-H}J_1\delta(\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-H)J_1\frac{1}{\omega(\mathbf{k}_1)+i\eta-H}J_1\Psi_0\Bigg)\\&\quad+\Bigg(\Psi_0,J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)-i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-i\eta-H}J_1\delta(\omega(\mathbf{k}_1)-H)J_1\Psi_0\Bigg)\Bigg].\end{aligned} \tag{B.3}$$

Considering the first term on the right of (B.3), we note that because of the delta function the only terms in the sum $\sum$ which contribute are the ones for which the vertex operator J_1 at the extreme left corresponds to single emission of p_2 or to single emission of p_1. If we denote by $\sum_{p_2}$ the sum over all the terms for which the emission of p_2 occurs last, this contribution to the first matrix element on the right of (B.3) can be written as

$$\begin{aligned}&-\pi\sum_{p_2}\sum_{r=1}^{\infty}\int d\mathbf{q}_1\cdots d\mathbf{q}_r(\Psi_0,J_1\Psi^{(-)}(\mathbf{q}_1\cdots\mathbf{q}_r))\delta(\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)-\sum_{i=1}^{r}\omega(\mathbf{q}_i))\\&\quad\times\Bigg(\Psi^{(-)}(\mathbf{q}_1\cdots\mathbf{q}_r),J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)+i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_1)+i\eta-H}J_1\Psi_0\Bigg).\end{aligned} \tag{B.4}$$

But

$$(\Psi_0,J_1\Psi^{(-)}(\mathbf{q}_1\cdots\mathbf{q}_r))\delta(\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)-\sum_{i=1}^{r}\omega(\mathbf{q}_i))=(\Psi^{(-)}(\mathbf{q}_1\cdots\mathbf{q}_r),J_1\Psi_0)^*\delta(\omega(\mathbf{p}_2)-\sum_{i=1}^{r}\omega(\mathbf{q}_i)) \quad (B.5)$$

$$=\left(\frac{2\omega(\mathbf{p}_2)}{r!}\right)^{\frac{1}{2}}\frac{(\mathbf{q}_1\cdots\mathbf{q}_r|R|\mathbf{p}_2)^*}{\rho(p_2)}\delta(\omega(\mathbf{p}_2)-\sum_{i=1}^{r}\omega(\mathbf{q}_i)) \quad (B.6)$$

$$=\frac{\rho(q_1)\cdots\rho(q_r)}{[r!2^r\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_r)]^{\frac{1}{2}}}(\omega(\mathbf{p}_2)|G|\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_r))^*\delta(\omega(\mathbf{p}_2)-\sum_{i=1}^{r}\omega(\mathbf{q}_i)) \quad (B.7)$$

where (B.6) follows from (B.5) by comparing the structure of the R matrix defined in Sec. II to the result of employing the relations (14), (15), and (16) to $(\Psi^{(-)}(\mathbf{q}_1\cdots\mathbf{q}_r),J_1\Psi_0)$ after $\Psi^{(-)}(\mathbf{q}_1\cdots\mathbf{q}_r)$ is expanded with the use of Eq. (10). Equation (B.7) is obtained from (B.6) simply by utilizing the definition of the T and G matrix.

In a completely analogous manner, it follows that

$$\sum_{p_2}\delta(\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)-\sum_{i=1}^{r}\omega(\mathbf{q}_i))\left(\Psi^{(-)}(\mathbf{q}_1\cdots\mathbf{q}_r),J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)+i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_1)+i\eta-H}J_1\Psi_0\right)$$

$$=\frac{[2^3\omega(\mathbf{p}_1)\omega(\mathbf{k}_1)\omega(\mathbf{k}_2)]^{\frac{1}{2}}}{(r!)^{\frac{1}{2}}\rho(k_1)\rho(k_2)\rho(p_1)}(\mathbf{q}_1\cdots\mathbf{q}_r\mathbf{p}_1|R|\mathbf{k}_1\mathbf{k}_2)\delta(\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)-\sum_{i=1}^{r}\omega(\mathbf{q}_i)), \quad (B.8)$$

where in this case there is a distinction between the R and T matrix because there exists more than one meson in both the initial and final state. In particular, we have from the appropriate generalization of Eq. (23) that

$$\delta(\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)-\sum_{i=1}^{r}\omega(\mathbf{q}_i))(\mathbf{q}_1\cdots\mathbf{q}_r\mathbf{p}_1|R|\mathbf{k}_1\mathbf{k}_2)=(\mathbf{q}_1\cdots\mathbf{q}_r\mathbf{p}_1|T|\mathbf{k}_1\mathbf{k}_2)\delta(\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)-\sum_{i=1}^{r}\omega(\mathbf{q}_i))$$

$$-i\pi\sum_{P(q)}\sum_{P(k)}\sum_{l=1}^{r-1}\frac{1}{l!(r-l)!}(\mathbf{q}_1\cdots\mathbf{q}_l\mathbf{p}_1|T|\mathbf{k}_1)(\mathbf{q}_{l+1}\cdots\mathbf{q}_r|T|\mathbf{k}_2)$$

$$\times\delta(\omega(\mathbf{k}_1)-\omega(\mathbf{p}_1)-\sum_{i=1}^{l}\omega(\mathbf{q}_i))\delta(\omega(\mathbf{k}_2)-\sum_{i=l+1}^{r}\omega(\mathbf{q}_i)). \quad (B.9)$$

We can therefore rewrite (B.8) in terms of the G matrices as follows:

$$\sum_{p_2}\delta(\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)-\sum_{i=1}^{r}\omega(\mathbf{q}_i))\left(\Psi^{(-)}(\mathbf{q}_1\cdots\mathbf{q}_r),J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)+i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_1)+i\eta-H}J_1\Psi_0\right)$$

$$=\frac{\rho(q_1)\cdots\rho(q_r)}{[2^r r!\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_r)]^{\frac{1}{2}}}\Big[(\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_r)\omega(\mathbf{p}_1)|G|\omega(\mathbf{k}_1)\omega(\mathbf{k}_2))\delta(\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)-\sum_{i=1}^{r}\omega(\mathbf{q}_i))$$

$$-i\pi\sum_{P(q)}\sum_{P(k)}\sum_{l=1}^{r-1}\frac{1}{l!(r-l)!}(\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_l)\omega(\mathbf{p}_1)|G|\omega(\mathbf{k}_1))(\omega(\mathbf{q}_{l+1})\cdots\omega(\mathbf{q}_r)|G|\omega(\mathbf{k}_2))$$

$$\times\delta(\omega(\mathbf{k}_1)-\omega(\mathbf{p}_1)-\sum_{i=1}^{l}\omega(\mathbf{q}_i))\delta(\omega(\mathbf{k}_2)-\sum_{i=l+1}^{r}\omega(\mathbf{q}_i))\Big]. \quad (B.10)$$

The first term on the right of (B.3) can now be obtained by substituting Eqs. (B.7) and (B.10) into (B.4) and summing the result over both permutations of p_1 and p_2. This latter step gives the contribution from both sums $\sum_{p1}$ and $\sum_{p2}$. The result of this manipulation is that

$$-\pi\sum\left(\Psi_0,J_1\delta(\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{p}_1)-H)J_1\frac{1}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)+i\eta-H}J_1\frac{1}{\omega(\mathbf{k}_1)+i\eta-H}J_1\Psi_0\right)$$

$$=-\pi\sum_{P(p)}\sum_{P(k)}\sum_{r=1}^{\infty}\int d\mathbf{q}_1\cdots d\mathbf{q}_r\frac{(\rho(q_1)\cdots\rho(q_r))^2}{2^r\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_r)}\Big\{\frac{(\omega(\mathbf{p}_2)|G|\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_r))^*}{r!}$$

$$\times\sum_{P(q)}\Big[\frac{(\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_r)\omega(\mathbf{p}_1)|G\delta|\omega(\mathbf{k}_1)\omega(\mathbf{k}_2))}{2!r!}-i\pi\sum_{l=1}^{r-1}\frac{(\omega(\mathbf{q}_1)\cdots\omega(\mathbf{q}_l)\omega(\mathbf{p}_1)|G\delta|\omega(\mathbf{k}_1))}{l!}$$

$$\times\frac{(\omega(\mathbf{q}_{l+1})\cdots\omega(\mathbf{q}_r)|G\delta|\omega(\mathbf{k}_2))}{(r-l)!}\Big]\Big\}, \quad (B.11)$$

where the symbol $G\delta$ is defined following Eq. (33) in Sec. II.

We can now do the same thing for the other two terms in (B.3) and finish the derivation by adding these results to (B.11). However, in order to obtain an expression for the imaginary part of $(\omega(\mathbf{p}_1)\omega(\mathbf{p}_2)|G|\omega(\mathbf{k}_1)\omega(\mathbf{k}_2))$ which is manifestly real, we formally average the result of this derivation with the result obtained by an identical procedure except that the expression (B.2) is altered by replacing each $i\eta$ by its negative. If we perform this averaging, we obtain Eq. (33).

APPENDIX C

In this appendix we prove that the function $G_{nm}(z)$ has no pole at $z=0$. Eliminating the subscripts n and m, we define $G_1(z)$ as that part of $G(z)$ which arises from single vertex terms [i.e., terms of type (a) in Sec. II]. The major part of this appendix consists of the proof that $G_1(z)$ has no pole at the origin while the final paragraph indicates how the proof can be extended to all the $G(z)$. We should also note the following changes in notation from those used in the text: the energy ratios $\nu_1, \nu_2, \cdots, \nu_m$ and $\mu_1, \mu_2, \cdots, \mu_n$ are denoted here by $f(1), f(2), \cdots, f(m)$ and $-f(m+1), -f(m+2), \cdots, -f(m+n)=-f(u)$ respectively.

We now prove the following theorem: The function

$$(-f(m+1), -f(m+2), \cdots -f(m+n)|G(z)|f(1), f(2), \cdots f(m))$$
$$\equiv \sum_{P(f)} \sum_{E[\tau(1)],E[\tau(2)],\cdots,E[\tau(u-1)]} \frac{J_1[0\tau(u-1)]J_1[\tau(u-1)\tau(u-2)]\cdots J_1[\tau(2)\tau(1)]J_1[\tau(1)0]}{[(f(1)+f(2)+\cdots+f(u-1))z-E[\tau(u-1)]]\cdots[(f(1)+f(2))z-E[\tau(2)]]}$$
$$\times[f(1)z-E[\tau(1)]], \quad \text{(C.1)}$$

has no pole at $z=0$. The symbol $P(f)$ indicates a sum over all permutations of $f(1)$ to $f(u)$ and $J_1[\tau(j)\tau(j-1)] \equiv (\tau(j)|J_1|\tau(j-1))$, where $|\tau(j))$ is an eigenstate of H with eigenvalue $E[\tau(j)]$.

Poles at $z=0$ could only arise from terms in the sum over the $E[\tau]$'s in which some (or all) of the $E[\tau]$'s take on the value zero. In the terms where r of the $E[\tau]$'s are zero, z^{-r} is a common factor. Therefore, terms of this type could conceivably lead to poles of order $r, r-1, r-2, \cdots$ or 1. To show that none of these possibilities actually occur, it is sufficient to prove that the coefficient of z^{-r} and the first $r-1$ derivatives of this coefficient all vanish at $z=0$. That is, defining $C_r(z)$ to be the coefficient of z^{-r} in the terms of (C.1) in which r of the $E[\tau]$'s are zero, we must show that

$$\lim_{z\to 0} \frac{d^l}{dz^l} C_r(z)=0 \quad \text{for} \quad 0\leq l\leq r-1. \quad \text{(C.2)}$$

We now define $C_r(S_{r+1},S_r,\cdots S_2,S_1;z)$ (where $S_{r+1}=u-\sum_{i=1}^{r}S_i$), as that part of $C_r(z)$ which arises from terms in which the S_1th, (S_1+S_2)th, $\cdots$, and $(S_1+S_2+\cdots+S_r)$th $E[\tau]$'s from the right in (C.1) are zero, and remark that it is sufficient to prove

$$\lim_{z\to 0} \frac{d^l}{dz^l} \sum_{P(S)} C_r(S_{r+1},S_r,\cdots,S_2,S_1;z) \quad \text{for} \quad 0\leq l\leq r-1, \quad \text{(C.3)}$$

for all possible choices of the S_1 to S_{r+1}.

From (C.1), we have that

$$C_r(S_{r+1},\cdots,S_1;z)$$
$$=\sum_{P(f)} \sum_{E[\tau(1)],\cdots,E[\tau(u-1)]}{}' \Bigg\{ \frac{J_1[0\tau(u-1)]\cdots J_1[\tau(\Sigma_r+1)0]}{[(f(1)+\cdots+f(u-1)z-E[\tau(u-1)]]\cdots[(f(1)+\cdots+f(\Sigma_r+1))z-E[\tau(\Sigma_r+1)]]}$$
$$\times\frac{1}{[f(1)+\cdots+f(\Sigma_r)]}\times\frac{J_1[0\tau(\Sigma_r-1)\cdots J_1[\tau(\Sigma_{r-1})0]}{[(f(1)+\cdots+f(\Sigma_r-1))z-E[\tau(\Sigma_r-1)]]\cdots[(f(1)+\cdots+f(\Sigma_{r-1}+1))z-E[\tau(\Sigma_{r-1}+1)]]}$$
$$\times\frac{1}{[f(1)+\cdots+f(\Sigma_{r-1})]}\times\cdots\times\frac{J_1[0\tau(\Sigma_2-1)]\cdots J_1[\tau(\Sigma_1+1)0]}{[(f(1)+\cdots f(\Sigma_2-1))z-E[\tau(\Sigma_2-1)]]\cdots[(f(1)+\cdots f(\Sigma_1+1))z-E[\tau(\Sigma_1+1)]]}$$
$$\times\frac{1}{[f(1)+\cdots+f(\Sigma_1)]}\times\frac{J_1[0\tau(\Sigma_1-1)]\cdots J_1[\tau(1)0]}{[(f(1)+\cdots+f(\Sigma_1-1))z-E[\tau(\Sigma_1-1)]\cdots[f(1)z-E[\tau(1)]]}\Bigg\}, \quad \text{(C.4)}$$

where $\sum_l \equiv \sum_{i=1}^{l} S_i$ and the prime on the energy sum indicates that $E[\tau(\Sigma_1)]$, $E[\tau(\Sigma_2)]$, $\cdots$, $E[\tau(\Sigma_r)]$ are all zero and therefore not summed. For each S_i there is in (C.4) a corresponding product of $S_i J_1$ matrix elements and $S_i - 1$ energy denominators. Also, in the chain rule expansion of the lth derivative of (C.4) each term can be labeled by indicating which factors are differentiated. Therefore, we let

$$C_r[S_{r+1}(s_{r+1}^1, s_{r+1}^2, \cdots s_{r+1}^{p_{r+1}}), S_r(s_r^1, s_r^2, \cdots, s_r^{p_r}), \cdots, S_1(s_1^1, s_1^2, \cdots, s_1^{p_r})],$$

denote the contribution to

$$\lim_{z\to 0}(d^l/dz^l) C_r(S_{r+1}, S_r, \cdots, S_1; z),$$

which comes from the term in the chain rule expansion of this derivative, where for each S_j, the factors differentiated are the s_j^1, s_j^2, $\cdots$, and $s_j^{p_j}$ from the right (in (C.4)) of the product of energy factors corresponding to S_j. It is clear that a number of the s_j^k's can be equal for different k's if the same factor is differentiated more than once. Also, since we are interested in derivatives only up to order $r-1$, it follows that

$$\sum_{i=0}^{r+1} p_i = l \leq r-1. \tag{C.5}$$

In order to prove (C.3), we need only show that

$$\sum_{P(S)} C_r[S_{r+1}(s_{r+1}^1, \cdots s_{r+1}^{p_{r+1}}), S_r(s_r^1, \cdots, s_r^{p_r}), \cdots S_1(s_1^1, \cdots s_1^{p_1})] = 0, \tag{C.6}$$

for all p_i which satisfy (C.5) and all possible choices of the S_j and s_k^l. First, however, we remark that in the expansion of (C.6), the sum over the energies and the product of J matrix elements are a common factor to the whole expression. Therefore, calling this factor $\mathcal{J}$, we define D_r,

$$D_r[S_{r+1}(s_{r+1}^1, \cdots s_{r+1}^{p_{r+1}}), S_r(s_r^1, \cdots s_r^{p_r}), \cdots, S_1(s_1^1, \cdots s_1^{p_1})]$$
$$\equiv (\mathcal{J})^{-1} \sum_{P(S)} C_r[S_{r+1}(s_{r+1}^1, \cdots s_{r+1}^{p_{r+1}}), S_r(s_r^1, \cdots s_r^{p_r}), \cdots, S_1(s_1^1, \cdots s_1^{p_1})], \tag{C.7}$$

so that the proof of the theorem is completed once it is demonstrated that D_r is zero.

Noting that

$$\lim_{z\to 0} \frac{d^l}{dz^l}(az - E)^{-1} = -l! E^{-(l+1)} a^l,$$

we obtain from the structure of (C.4) that

$$D_r[S_{r+1}(s_{r+1}^1, \cdots s_{r+1}^{p_{r+1}}), S_r(s_r^1, \cdots s_r^{p_r}), \cdots, S_1(s_1^1 \cdots s_1^{p_1})]$$
$$= A \sum_{P(f)} \sum_{P(S)} \{[f(\Sigma_r + s_{r+1}^{p_{r+1}}) + \cdots + f(1)] \cdots [f(\Sigma_r + s_{r+1}^1) + \cdots f(1)]$$
$$\times \frac{1}{[f(\Sigma_r) + \cdots + f(1)]} \times [f(\Sigma_{r-1} + s_r^{p_r}) + \cdots + f(1)] \cdots [f(\Sigma_{r-1} + s_r^1) + \cdots + f(1)]$$
$$\times \frac{1}{[f(\Sigma_{r-1}) + \cdots + f(1)]} \times \cdots \times \frac{1}{[f(\Sigma_1) + \cdots + f(1)]} \times [f(s_1^{p_1}) + \cdots + f(1)] \times [f(s_1^1) + \cdots + f_1]\}, \tag{C.8}$$

where A is an unimportant constant factor. In each term of the double sum indicated in (C.8) (except those terms which correspond to permutations of the S's in which S_1 is replaced by S_i), the product of p_i numerator factors corresponding to S_i have a denominator factor to their immediate right. This denominator can be written as $[f(R)' + \cdots + f(1)']$, where, in the permutation of the f's which characterize the term, f_j is replaced by f_j' ($j = 1$ to u). With this notation we define the meaning of *contain* by saying that in such a term of (C.8), S_i *contains* f_{R+1}', f_{R+2}', $\cdots$ and f_{R+s_i}'; and if the term is one in which S_1 is replaced by S_i, then S_i *contains* f_1', $f_2' \cdots$, and f_{s_i}'.

We now define the function D_r^v,

$$D_r^v[S_{r+1}(s_{r+1}^1,\cdots s_{r+1}^{p_{r+1}}),S_r(s_r^1,\cdots s_r^{p_r}),\cdots,S_1(s_1^1,\cdots s_1^{p_1})]$$

$$\equiv \sum_{P(f)}\sum_{P(S)} v[S_{r+1};|f(S_{r+1})|]\cdots v[S_1;|f(S_1)|]$$

$$\times\Big\{[f(\Sigma_r+s_{r+1}^{p_{r+1}})+\cdots+f(1)]\cdots[f(\Sigma_r+s_{r+1}^1)+\cdots+f(1)]$$

$$\times\frac{1}{[f(\Sigma_r)+\cdots+f(1)]}[f(\Sigma_{r-1}+s_r^{p_r})+\cdots+f(1)][f(\Sigma_{r-1}+s_r^1)+\cdots+f(1)]$$

$$\times\frac{1}{[f(\Sigma_{r-1})+\cdots+f(1)]}\times\cdots\times\frac{1}{[f(\Sigma_1)+\cdots+f(1)]}[f(s_1^{p_1})+\cdots+f(1)][f(s_1^1)+\cdots+f(1)]\Big\}, \quad (C.9)$$

where $v[S_i;|f(S_i)|]$ is an arbitrary function of the f's *contained* in S_i.

By induction we shall now prove the following lemma: The function D_r^v defined in (C.9) is zero independently of the functional form of the functions $v[S_i;|f(S_i)|]$ as long as there is at least one more f sum factor in the denominator than there is in the numerator (i.e., with the restriction (C.5)). Since D_r is the special case of D_r^v when all the v's are unity, the proof of the lemma constitutes a proof of the theorem [see remark following Eq. (C.7)].

If l is the number of f sum factors in the numerator, we first assume that the lemma is true for $l\leq l_0-1$ and show that this implies its validity for $l=l_0$. Finally we complete the proof by proving the lemma for $l=0$. For the case $l=l_0\neq 0$, the numerator contains an f sum factor which is of the form $[f(\Sigma_{k-1}+s_k^1)+\cdots+f(1)]$ in the term of (C.9) written out explicitly in the square bracket. Therefore, if we write

$$[f(\Sigma_{k-1}+s_k^1)+\cdots+f(1)]=[f(\Sigma_{k-1}+s_k^1)+\cdots+f(\Sigma_{k-1}+1)]+[f(\Sigma_{k-1})+\cdots+f(1)], \quad (C.10)$$

the function D_r^v can be thought of as the sum of two functions, one arising from each of the two terms in (C.10). Since all the f's in the first term of (C.10) are *contained* in S_k, it is possible to incorporate this term into the definition of $v[S_k;|f(S_k)|]$ with the result that the function arising from this is a D_r^v with $l=l_0-1$ and is zero by the inductive hypothesis. The remainder of D_r^v can be thought of as the sum of $r+1$ functions such that the first of these is the sum of all the terms in which S_k lies to the far right in the expansion of (C.9), and each of the r other functions are distinguished by specifying which S_i lies to the immediate right of S_k. The first of these $r+1$ functions is zero since when S_k lies to the far right there is no second term in the decomposition (C.10). For each of the remaining r functions a numerator factor cancels against an identical denominator factor and the result is a D function with one less l and one less r. Consequently, each of these r functions are also zero by the inductive hypothesis.

To complete the proof we must show that $D_r^v=0$ for the case of $l=0$. Hence, we must show that

$$\sum_{P(f)}\sum_{P(S)}\left\{\frac{v[S_{r+1};f(u),f(u-1),\cdots,f(\Sigma_r+1)]v[S_r;f(\Sigma_r),\cdots,f(\Sigma_{r-1}+1)]\cdots v[S_1;f(\Sigma_1),\cdots,f(1)]}{[f(\Sigma_r)+\cdots f(1)][f(\Sigma_{r-1})+\cdots+f(1)]\cdots[f(\Sigma_1)+\cdots+f(1)]}\right\}=0. \quad (C.11)$$

Denoting the expression in the square bracket of (C.11) by $\{S_{r+1},S_r,\cdots,S_1\}$, we rewrite (C.11) in the form

$$\sum_{P'(S)}\sum_{P(f)}[\{S_{r+1},S_r,\cdots,S_1\}+\{S_r,S_{r+1},\cdots,S_1\}+\cdots+\{S_r,\cdots,S_2,S_{r+1},S_1\}+\{S_r,\cdots,S_1,S_{r+1}\}]=0, \quad (C.12)$$

where $P'(S)$ indicates the sum over all permutations of S_1 to S_r.

We have

$$\sum_{P(f)}\{S_r,S_{r+1},\cdots,S_1\}$$

$$=\sum_{P(f)}\frac{v[S_r;f(u),\cdots,f(\Sigma_{r-1}+Sr_{+1}+1)]v[S_{r+1};f(\Sigma_{r-1}+S_{r+1}),\cdots,f(\Sigma_{r-1}+1)]\cdots v[S_1;f(\Sigma_1),\cdots f(1)]}{[f(\Sigma_{r-1}+S_{r+1})+\cdots+f(1)][f(\Sigma_{r-1})+\cdots+f(1)]\cdots[f(\Sigma_1)+\cdots+f(1)]} \quad (C.13)$$

and using the fact that $\sum_{i=1}^{u} f_i = 0$ on the denominator factor at the left, there results

$$\sum_{P(f)} \{S_r, S_{r+1}, \cdots, S_1\} = -\sum_{P(f)} \frac{v[S_r; f(u), \cdots f(\Sigma_{r-1}+S_{r+1}+1)]v[S_{r+1}; f(\Sigma_{r-1}+S_{r+1}), \cdots f(\Sigma_{r-1}+1)]\cdots v[S_1; f(\Sigma_1), \cdots, f(1)]}{[f(u)+f(u-1)+\cdots+f(\Sigma_{r-1}+Sr_{+1}+1)][f(\Sigma_{r-1})+\cdots+f(1)]\cdots[f(\Sigma_1)+\cdots+f(1)]}. \quad \text{(C.14)}$$

Now, since we are free to permute the f's within the summation sign without affecting the result, we perform the following rearrangement which transforms the numerator of (C.14) into the same form as the one in (C.11):

$$\begin{aligned} f(u) &\to f(u-S_{r+1}) = f(\Sigma_r), & f(\Sigma_{r-1}+S_{r+1}) &\to f(u), \\ f(u-1) &\to f(u-S_{r+1}-1) = f(\Sigma_r - 1), & f(\Sigma_{r-1}+S_{r+1}-1) &\to f(u-1), \\ &\vdots & &\vdots \\ f(\Sigma_{r-1}+S_{r+1}+1) &\to f(\Sigma_{r-1}+1), & f(\Sigma_{r-1}+1) &\to f(u-S_{r+1}+1) = f(\Sigma_r+1), \end{aligned} \quad \text{(C.15)}$$

with all the other f's remaining unchanged. Equation (C.14) can now be written as

$$\sum_{P(f)} \{S_r, S_{r+1}, \cdots, S_1\} = -\sum_{P(f)} \frac{v[S_{r+1}; f(u), f(u-1), \cdots f(\Sigma_{r+1})]v[S_r; f(\Sigma_r), \cdots, f(\Sigma_{r-1}+1)]\cdots v[S_1; f(\Sigma_1), \cdots, f(1)]}{[f(\Sigma_r)+\cdots+f(\Sigma_{r-1}+1)][f(\Sigma_{r-1})+\cdots+f(1)]\cdots[f(\Sigma_1)+\cdots+f(1)]}. \quad \text{(C.16)}$$

All the other terms of (C.12) can be rewritten in a similar manner. Each time S_{r+1} is shifted one more place to the right, we express one more term in the denominator (from the left) as a sum of f's beginning with $f(u)$ and then affect the permutation of the f's which makes the numerator identical to the one written out explicitly in (C.11). The result of these manipulations is that

$$\begin{aligned} &\sum_{P(f)} [\{S_{r+1}, S_r, \cdots, S_1\} + \cdots + \{S_r, \cdots, S_1, S_{r+1}\}] \\ &= \sum_{P(f)} v[S_{r+1}; f(u), \cdots, f(\Sigma_{r+1})]v[S_r; f(\Sigma_r), \cdots, f(\Sigma_{r-1}+1)]\cdots v[S_1; f(\Sigma_1), \cdots f(1)] \\ &\times \Bigg\{ \frac{1}{[f(\Sigma_r)+\cdots+f(1)][f(\Sigma_{r-1})+\cdots+f(1)][f(\Sigma_1)+\cdots+f(1)]} \\ &- \frac{1}{[f(\Sigma_r)+\cdots+f(\Sigma_{r-1}+1)][f(\Sigma_{r-1})+\cdots+f(1)]\cdots[f(\Sigma_1)+\cdots+f(1)]} \\ &+ \frac{1}{[f(\Sigma_r)+\cdots+f(\Sigma_{r-1}+1)][f(\Sigma_r)+\cdots+f(\Sigma_{r-2}+1)][f(\Sigma_{r-2})+\cdots+f(1)][f(\Sigma_1)+\cdots+f(1)]} \\ &-\cdots+(-1)^{r-1}\frac{1}{[f(\Sigma_r)+\cdots+f(\Sigma_{r-1}+1)]\cdots[f(\Sigma_r)+\cdots+f(\Sigma_1+1)][f(\Sigma_1)+\cdots+f(1)]} \\ &+(-1)^r \frac{1}{[f(\Sigma_r)+\cdots+f(\Sigma_{r-1}+1)]\cdots[f(\Sigma_r)+\cdots+f(1)]} \Bigg\}. \end{aligned} \quad \text{(C.17)}$$

It is not difficult to verify that the expression in the square bracket of (C.17) is zero.

The extension of the proof to include multiple commutations expressions can be accomplished without difficulty. First we define $G_{rst\ldots}(z)$ to indicate the contribution to $G(z)$ arising from terms in which there are r single commutator vertices, s double commutator vertices, t triple commutator vertices, etc. Now if we attempt to apply the above proof to $G_{rst\ldots}(z)$, it is immediately observed that the structure of the product of $S_i - 1$ matrix elements which correspond to S_i is in general different for every permutation of S_1 to S_{r+1} which changes the position of S_i. However, if the J operators are rearranged appropriately each time S_1 to S_{r+1} is permuted, this structure is not changed. If we now redefine $\sum_{P(S)}$ to mean the sum over this double rearrangement, the proof that $G_{rst\ldots}(z)$ has no pole at the origin follows by complete analogy to the one presented for $G_1(z)$.