Polarization and Angular Correlation in the Production and Decay of Particles of Spin $\frac{1}{2}$ and Spin $\frac{3}{2}$

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Some relationships describing the angular correlation and polarization effects in the production and subsequent decay of particles of arbitrary spins are given. They are specialized to the cases of production and decay of particles of spin $\frac{1}{2}$ and $\frac{3}{2}$. Expressions for the angular distribution and polarization of the decay products are reduced to tractable forms involving the physical vectors of the problem and a minimal number of parameters describing the production and decay interactions. The information that may be obtained from the analysis of the angular correlations of the decay products is discussed.

1. INTRODUCTION

THE angular distribution of the π meson produced in hyperon decay provides information regarding the hyperon spin. If this spin is one-half then there is a probability of one-half that the decay direction will lie in one of the two polar quadrants defined relative to the line of flight of the hyperon. The measured value of the fraction of decays into the two polar quadrants is 0.55 ± 0.021. The statistical probability that the measured value would differ as much as this from a true value of one-half is 1.7%.[1] In view of this indication that the spin of the hyperon may not be one-half, it becomes of interest to determine the detailed consequences of the hyperon's spin begin greater than one-half. The purpose of this paper is to examine, for the case of spin $\frac{1}{2}$ and spin $\frac{3}{2}$ hyperons, the correlation between the direction and polarization of the decay nucleon and the directions defined by the production process. The case in which the hyperon production process is the capture of a K particle from orbital states has been treated by Treiman and Gatto.[2,3] We consider here the case in which the hyperons are produced in high-energy π-p collisions. This is a generalization of the work of Adair,[4] who considered the special case of production at angles near zero or 180 degrees.

It is assumed throughout our paper that the π meson and K meson are spin-zero particles. The strange particles are assumed to be single particles and not parity doublets.[5] Conservation of parity is not assumed in the decay process. For the decay of spin-$\frac{1}{2}$ hyperons the consequences of a violation of conservation of parity have been discussed by Lee, Steinberg, Feinberg, Kabir, and Yang.[6] The generalization of these results to the case of spin-$\frac{3}{2}$ particles is discussed here. Consequences of noninvariance with respect to time reversal in the decay of both spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ particles are also discussed.

In the analysis of polarization phenomena statistical mixtures of states must be considered, and a density matrix formulation is convenient.[7] The spin-space density matrix $\mathcal{U}(\theta\phi)$ is defined by the relation

$$\langle A\rangle_{\theta\phi}=\mathrm{Tr}\,A\mathcal{U}(\theta\phi), \tag{1.1}$$

where $\langle A\rangle_{\theta\phi}$ is the expectation value of the spin operator A if the measurement is made on particles moving in the direction $\theta\phi$. The matrices A and $\mathcal{U}(\theta\phi)$ are square matrices of dimension $(2S+1)$, where S is the spin quantum number. It is convenient to introduce a complete orthonormal set of matrices in this space. We use the matrices $T_\kappa{}^Q$ defined as follows[8]:

$$\begin{aligned}\langle S'\mu'|T_\kappa{}^Q|S''\mu''\rangle&=\left(\frac{2Q+1}{2S'+1}\right)^{\frac{1}{2}}C_{\mu''\kappa\mu'}{}^{S''QS'}\\&=\left(\frac{2Q+1}{2S'+1}\right)^{\frac{1}{2}}C_{S''Q}(S'\mu',\mu''\kappa),\end{aligned} \tag{1.2}$$

where the six-index symbols on the right are the usual Clebsch-Gordan coefficients.[9] The matrices $T_\kappa{}^Q$ are real and their Hermitian conjugates $\bar{T}_\kappa{}^Q$ are their respective transposes. By use of the completeness property of the $T_\kappa{}^Q$ the $\mathcal{U}(\theta\phi)$ may be expanded in the form

$$\mathcal{U}(\theta\phi)=\bar{\alpha}_\kappa{}^Q(\theta\phi)T_\kappa{}^Q=\alpha_\kappa{}^Q(\theta\phi)\bar{T}_\kappa{}^Q. \tag{1.3}$$

The coefficients $\alpha_\kappa{}^Q(\theta\phi)$ and $\bar{\alpha}_\kappa{}^Q(\theta\phi)$ defined by the above equations are complex conjugates owing to the Hermiticity of the density matrix. They are scalars that determine the density matrix and give the state of polarization of the system. In virtue of the orthonor-

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[1] Alvarez, Bradner, Falk-Vairant, Gow, Rosenfeld, Solmitz, and Tripp, University of California Radiation Laboratory Report UCRL-3775 (unpublished). These authors consider the results quoted to be only rather weak evidence for a hyperon spin $>\frac{1}{2}$.

[2] S. B. Treiman, Phys. Rev. **101**, 1216 (1956); R. Gatto, Nuovo cimento 3, 1142 (1956).

[3] R. Spitzer and H. P. Stapp, University of California Radiation Laboratory Report UCRL-3796 Rev (unpublished).

[4] R. K. Adair, Phys. Rev. **100**, 1540 (1955).

[5] T. D. Lee and C. N. Yang, Phys. Rev. **104**, 822 (1956).

[6] Lee, Steinberger, Feinberg, Kabir, and Yang, Phys. Rev. **106**, 1367 (1957).

[7] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), third edition, p. 130; L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952); U. Fano, Revs. Modern Phys. **29**, 74 (1957).

[8] Our normalization is different from that of other authors; see C. Eckart, Revs. Modern Phys. **2**, 305 (1930).

[9] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).

mality condition,

$$\mathrm{Tr}\, T_\kappa{}^Q \bar{T}_{\kappa'}{}^{Q'} = \delta_{QQ'}\delta_{\kappa\kappa'}, \tag{1.4}$$

the $\alpha_\kappa{}^Q(\theta\phi)$ and $\bar{\alpha}_\kappa{}^Q(\theta\phi)$ may be expressed as

$$\alpha_\kappa{}^Q(\theta\phi) = \mathrm{Tr}\, \mathfrak{U}(\theta\phi) T_\kappa{}^Q, \quad \bar{\alpha}_\kappa{}^Q(\theta\phi) = \mathrm{Tr}\, \mathfrak{U}(\theta\phi) \bar{T}_\kappa{}^Q. \tag{1.5}$$

We shall be interested in processes in which the initial states are described by the spin-orbit variables (S',μ',θ',ϕ') and the final states by the spin-orbit variables (S,μ,θ,ϕ). If the initial system is a plane wave moving in the direction $\theta'\phi'$ with a spin quantum number S', then the parameters $\alpha_\kappa{}^Q(S,\theta\phi)$, which describe the spin-space characteristics of the reaction products that emerge in the direction $\theta\phi$ and in the state with spin S, are given in terms of the parameters $\alpha_{\kappa'}{}^{Q'}(S',\theta'\phi')$, which describe the spin-space characteristics of the initial plane wave, by the fundamental equation[10]

$$I(\theta\phi)\alpha_\kappa{}^Q(S,\theta\phi) = \frac{N}{4\pi} \sum_{LL'L''L'''} \sum_{JJ'} R_{SL;\,S'L'}{}^J R_{SL'';\,S'L'''}{}^{J'*} (2J+1)(2J'+1)[(2L+1)(2L'+1)(2L''+1)(2L'''+1)]^{\frac{1}{2}}$$
$$\times \sum_{\Lambda\Lambda'\lambda\lambda'} Y_\Lambda{}^\lambda(\theta\phi) Y_{\Lambda'}{}^{\lambda'}(\theta'\phi') C_{000}{}^{LL''\Lambda} C_{000}{}^{L'''L'\Lambda'} (-1)^{L+L'+\Lambda} \sum_{Q'\kappa'} \alpha_{\kappa'}{}^{Q'}(S',\theta'\phi')(2Q'+1)^{\frac{1}{2}}$$
$$\times \sum_{Z\zeta} (2Z+1)^{\frac{1}{2}} C_{\lambda\zeta\kappa}{}^{\Lambda ZQ} C_{\kappa'\lambda'\zeta}{}^{Q'\Lambda'Z} X(L''L\Lambda,J'JZ,SSQ) X(L'''L'\Lambda',J'JZ,S'S'Q'). \tag{1.6}$$

The X coefficient is the one defined by Fano,[11] the $Y_L{}^m(\theta\phi)$ are the usual spherical harmonics,[9] the $R_{SL;\,S'L'}{}^J$ are reaction matrix elements determined by the specific nature of the reaction, and the coefficient N is a normalization factor. If the initial system is a plane-wave state with momentum k', and N is taken as $(2\pi/k')^2$, then $I(\theta\phi)$ is the differential cross section. The value of $I(\theta\phi)$ may be determined by the condition [implied by Eqs. (1.1), (1.4), and (1.5), together with the requirement that the expectation value of a pure number is equal to that number]

$$\alpha_0{}^0(S,\theta\phi) = (2S+1)^{-\frac{1}{2}}. \tag{1.7}$$

The above formula relates directly the expectation values of operators in the initial and final states. It is sometimes convenient to consider the reaction matrix itself. According to our definition,[12] the matrix element $\langle S\mu|\mathcal{R}(\theta\phi;\theta'\phi')|S'\mu'\rangle$, when multiplied by $(2\pi/k')(v/v')^{\frac{1}{2}}$, where v' and v are the initial and final relative velocities, gives the reaction (or scattering) amplitude $f_\mu{}^S(\theta\phi)$ when the initial state is a plane wave of unit particle density in the spin state $\chi_{\mu'}{}^{S'}$. If the z axis is chosen to lie along the outgoing direction the matrix $\mathcal{R}(\theta\phi;\theta'\phi')$ may be expressed in the form,

$$\mathcal{R}(00;\theta'\phi') = \sum_{Q\kappa} a_\kappa{}^Q(S,00)\bar{T}_\kappa{}^Q,$$

where

$$a_\kappa{}^Q(S,00) = \frac{(-1)^{S'-S}}{(4\pi)^{\frac{1}{2}}} \sum_{LL'J} R_{SL;\,S'L'}{}^J Y_{L'}{}^\kappa(\theta'\phi')$$
$$\times (2L+1)^{\frac{1}{2}}(2J+1)(-1)^L C_{\kappa 0\kappa}{}^{L'LQ}$$
$$\times W(LJQS';SL'), \tag{1.8}$$

where W is the Racah coefficient.[13] If the initial and final spin quantum numbers, S' and S respectively, are equal then the matrices $\bar{T}_\kappa{}^Q$ are square matrices. Otherwise they are nonsquare, with $(2S'+1)$ columns and $(2S+1)$ rows. The indices S and S' on the $R_{SL;\,S'L'}{}^J$ will be suppressed in the remainder of the paper.

2. REACTION FORMULAS FOR SPIN-$\frac{1}{2}$ AND SPIN-$\frac{3}{2}$ PARTICLES

The form of the angular distribution and polarization of the reaction products of the decay of a spin-$\frac{1}{2}$ hyperon into one spin-zero particle and one spin-$\frac{1}{2}$ particle may be obtained from Eqs. (1.5) and (1.6) by dropping the contributions from all initial states with $L'\neq 0$. If the unit vector along the momentum of the fermion in the decay products is denoted by $\mathbf{V}$ and the polarization vector of the initial system is denoted by $\mathbf{P}_i$, the angular distribution of the decay products is given by

$$I(\mathbf{V}) = (N/4\pi)[|R_0|^2 + |R_1|^2 + 2\,\mathrm{Re}(R_0R_1^*)\mathbf{P}_i\cdot\mathbf{V}], \tag{2.1}$$

and the final polarization is

$$\begin{aligned}\mathbf{P}(\mathbf{V}) &= I(\mathbf{V})^{-1}(N/4\pi)\{2\,\mathrm{Re}(R_0R_1^*)\mathbf{V} \\ &\quad - 2\,\mathrm{Im}(R_0R_1^*)(\mathbf{P}_i\times\mathbf{V}) + |R_0|^2\mathbf{P}_i \\ &\quad + |R_1|^2[2(\mathbf{P}_i\cdot\mathbf{V})\mathbf{V} - \mathbf{P}_i]\} \\ &= I(\mathbf{V})^{-1}(N/4\pi)[2\,\mathrm{Re}(R_0R_1^*)\mathbf{V} \\ &\quad - 2\,\mathrm{Im}(R_0R_1^*)(\mathbf{P}_i\times\mathbf{V}) \\ &\quad + (|R_0|^2 + |R_1|^2)(\mathbf{P}_i\cdot\mathbf{V})\mathbf{V} \\ &\quad - (|R_0|^2 - |R_1|^2)(\mathbf{P}_i\times\mathbf{V})\times\mathbf{V}].\end{aligned} \tag{2.2}$$

[10] For a detailed derivation of this equation see reference 3, where several special cases of interest are also given. We have assumed that $S+S'$ is integral. A formula essentially equivalent to Eq. (1.6) has been derived by A. Simon, Phys. Rev. **92**, 1050 (1953).

[11] U. Fano, National Bureau of Standards Report NBS-1214 (unpublished), p. 48. Algebraic formulas and tables of the X coefficients are given by H. Matsunobu and H. Takebe, Progr. Theoret. Phys. Japan **14**, 589 (1955). The phase and normalization of our X coefficient are the same as those of the U coefficient of Matsunobu and Takebe.

[12] For a complete description of the formalism used to obtain the results quoted in this paper see reference 3.

[13] G. Racah, Phys. Rev. **62**, 438 (1942). The values of the W coefficient have been tabulated by L. C. Biedenharn, Oak Ridge National Laboratory Report ORNL-1098, 1952 (unpublished).

The R_L are an abbreviation of $R_{L0}^{\frac{1}{2}}$. They are the two fundamental parameters that completely describe the decay process. If we take $N=1$ and normalize the R_L so that $|R_0|^2+|R_1|^2=1$ then $I(\mathbf{V})d\Omega$ is the probability that the final nucleon will have its velocity in the solid angle $d\Omega$ about the direction $\mathbf{V}$.

The polarization vector of the hyperon, $\mathbf{P}_i$, is determined by the production mechanism. If parity is conserved in the production the $\mathbf{P}_i$ must be normal to the plane of production. Its magnitude may be expressed in terms of the fundamental parameters of the production process. For the case in which the intrinsic parities of the initial and final states of the production process are equal these expressions are well-known.[14] The extension of these results to the case in which the intrinsic parities of the initial and final states differ is given in reference 3. The contribution from D states and the case in which parity is not conserved in the production process are also examined there.

The case in which the initial fermion of the production process is a spin-$\frac{1}{2}$ particle and the final fermion is a spin-$\frac{3}{2}$ particle may be described in a form similar to the above. For this purpose we introduce the symbols

$$\begin{aligned} T(\mathbf{u}_1)&=(4\pi/3)^{\frac{1}{2}}Y_1{}^{\kappa}(\mathbf{u}_1)\bar{T}_{\kappa}{}^1,\\ T(\mathbf{u}_1,\mathbf{u}_2)&=(4\pi/5)^{\frac{1}{2}}Y_2{}^{\kappa}(\mathbf{u}_1,\mathbf{u}_2)\bar{T}_{\kappa}{}^2,\\ T(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)&=(4\pi/7)^{\frac{1}{2}}Y_3{}^{\kappa}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)\bar{T}_{\kappa}{}^3. \end{aligned} \tag{2.3}$$

Here the $\mathbf{u}_i$ are arbitrary vectors and the symbol $Y_N{}^{\kappa}(\mathbf{u}_1,\cdots,\mathbf{u}_N)$ represents the function of the vectors $\mathbf{u}_i$ that is linear in each argument, is symmetric in all its arguments, and which becomes $Y_N{}^{\kappa}(\theta\phi)$ when all its arguments are set equal to the unit vector in the direction $\theta\phi$. If unit vectors along the initial and final momenta are denoted by $\mathbf{K}'$ and $\mathbf{K}$ respectively and $\mathbf{N}$ is the unit vector in the direction $\mathbf{K}'\times\mathbf{K}$, than the $\mathcal{R}$ matrix describing the production may be expressed as the following superposition of these T matrices:

$$\begin{aligned} \mathcal{R}(\mathbf{K},\mathbf{K}')=\sqrt{2}(2\pi/k')^{-1}[&g_1(\theta)T(\mathbf{N})+g_2(\theta)T(\mathbf{K},\mathbf{K})\\ &+g_3(\theta)T(\mathbf{K},\mathbf{K}')+g_4(\theta)T(\mathbf{K}',\mathbf{K}')\\ &+h_1(\theta)T(\mathbf{K})+h_2(\theta)T(\mathbf{K}')\\ &+h_3(\theta)T(\mathbf{N},\mathbf{K})+h_4(\theta)T(\mathbf{N},\mathbf{K}')]. \end{aligned} \tag{2.4}$$

The explicit form of the g_i and h_i when only S- and P-wave final states contribute is given in Table I. The normalization factors in Eq. (2.4) have been chosen so that the differential reaction cross section for the case of an unpolarized initial fermion is

$$\begin{aligned} I_0(\theta)=\textstyle\sum_i(|g_i|^2+|h_i|^2)&+2\,\mathrm{Re}(h_1h_2{}^*)\cos\theta\\ &+\tfrac{3}{2}\,\mathrm{Re}(h_3h_4{}^*)\cos\theta+2\,\mathrm{Re}(g_2g_3{}^*)\cos\theta\\ &+2\,\mathrm{Re}(g_3g_4{}^*)\cos\theta+\mathrm{Re}(g_2g_4{}^*)(3\cos^2\theta-1)\\ &-\tfrac{1}{4}(|h_3|^2+|h_4|^2+\sin^2\theta|g_3|^2). \end{aligned} \tag{2.5}$$

[14] See L. Wolfenstein and J. Ashkin, reference 7; H. A. Bethe and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston, 1955), p. 79.

If parity is conserved in the reaction then the $h_i(\theta)$ will be zero for the case in which the relative intrinsic parity of the initial particles is the same as that of the final particles; the $g_i(\theta)$ will be zero if these relative intrinsic parities are opposite.

If parity is conserved in the interaction and the initial fermion is unpolarized, the density matrix describing the spin of the final particle must be of the form

$$\begin{aligned} \mathcal{U}(\mathbf{K},\mathbf{K}')=\tfrac{1}{4}&+b(\theta)T(\mathbf{N})+c(\theta)T(\mathbf{K},\mathbf{K})+c'(\theta)T(\mathbf{K},\mathbf{K}')\\ &+c''(\theta)T(\mathbf{K}',\mathbf{K}')+d(\theta)T(\mathbf{N},\mathbf{K},\mathbf{K})\\ &+d'(\theta)T(\mathbf{N},\mathbf{K},\mathbf{K}')+d''(\theta)T(\mathbf{N},\mathbf{K}',\mathbf{K}'). \end{aligned} \tag{2.6}$$

The coefficients in this expression as functions of the $g_i(\theta)$ and $h_i(\theta)$ are given in reference 3. When only S and P final states contribute, the differential reaction cross section reduces to the form

$$I_0(\mathbf{K},\mathbf{K}')=\tfrac{1}{4}\lambda\!\!\!^{-2}[A'+B'\cos\theta+C'\cos^2\theta], \tag{2.7}$$

where, for the case in which the relative intrinsic parities of the initial and final states are the same,

$$\begin{aligned} A'=|R_{11}{}^{\frac12}|^2&+2|R_{02}{}^{\frac32}|^2+(14/5)|R_{11}{}^{\frac32}|^2+(9/5)|R_{13}{}^{\frac52}|^2\\ &+(\tfrac{2}{5})^{\frac12}\,\mathrm{Re}(R_{11}{}^{\frac12}R_{11}{}^{\frac32 *})+3(\tfrac{3}{5})^{\frac12}\,\mathrm{Re}(R_{11}{}^{\frac12}R_{13}{}^{\frac52 *})\\ &\qquad-\tfrac{3}{5}(6^{\frac12})\,\mathrm{Re}(R_{11}{}^{\frac32}R_{13}{}^{\frac52 *}),\\ B'=-2\sqrt{2}\,&\mathrm{Re}(R_{11}{}^{\frac12}R_{02}{}^{\frac32 *})+(4/5^{\frac12})\,\mathrm{Re}(R_{11}{}^{\frac32}R_{02}{}^{\frac32 *})\\ &\qquad+6(6/5)^{\frac12}\,\mathrm{Re}(R_{02}{}^{\frac32}R_{13}{}^{\frac52 *}),\\ C'=-(12/5)&|R_{11}{}^{\frac32}|^2+(18/5)|R_{13}{}^{\frac52}|^2\\ &-3(\tfrac{2}{5})^{\frac12}\,\mathrm{Re}(R_{11}{}^{\frac12}R_{11}{}^{\frac32 *})-9(\tfrac{3}{5})^{\frac12}\,\mathrm{Re}(R_{11}{}^{\frac12}R_{13}{}^{\frac52 *})\\ &\qquad+(9/5)6^{\frac12}\,\mathrm{Re}(R_{11}{}^{\frac32}R_{13}{}^{\frac52 *}). \end{aligned} \tag{2.8}$$

When the contribution of the P final state is much smaller than that of the S final state and the quadratic final P-state terms can be neglected, the parameters in Eq. (2.6) are given in terms of these same reaction matrix elements by

$$\begin{aligned} I_0(\theta)b(\theta)&=\tfrac{1}{4}\lambda\!\!\!^{-2}\sin\theta\alpha_1,\\ I_0(\theta)c(\theta)&=0,\\ I_0(\theta)c'(\theta)&=\tfrac{1}{4}\lambda\!\!\!^{-2}\{2\alpha_3+\alpha_4\},\\ I_0(\theta)c''(\theta)&=\tfrac{1}{4}\lambda\!\!\!^{-2}\{\alpha_2-5\cos\theta\alpha_3\},\\ I_0(\theta)d(\theta)&=0,\\ I_0(\theta)d'(\theta)&=0,\\ I_0(\theta)d''(\theta)&=\tfrac{1}{4}\lambda\!\!\!^{-2}\{\sin\theta\alpha_5\}, \end{aligned} \tag{2.9}$$

TABLE I. Coefficients of T matrices in Eq. (2.4) when only S- and P-wave final states contribute.

$$\begin{aligned} g_1(\theta)&=(\lambda/4\pi)[-\tfrac{1}{2}iR_{11}{}^{\frac12}-i(\tfrac{5}{2})^{\frac12}R_{11}{}^{\frac32}]\sin\theta,\\ g_2(\theta)&=0,\\ g_3(\theta)&=(\lambda/4\pi)[R_{11}{}^{\frac12}-(\tfrac{2}{5})^{\frac12}R_{11}{}^{\frac32}+2(\tfrac{3}{5})^{\frac12}R_{13}{}^{5/2}],\\ g_4(\theta)&=(\lambda/4\pi)[-\sqrt{2}R_{02}{}^{\frac32}-5(\tfrac{3}{5})^{\frac12}R_{13}{}^{5/2}\cos\theta],\\ h_1(\theta)&=(\lambda/4\pi)[R_{10}{}^{\frac12}+\tfrac{1}{2}(\tfrac{2}{5})^{\frac12}R_{12}{}^{\frac32}+\tfrac{3}{2}(\tfrac{3}{5})^{\frac12}R_{12}{}^{5/2}],\\ h_2(\theta)&=(\lambda/4\pi)[-\sqrt{2}R_{01}{}^{\frac32}-\tfrac{3}{2}(\tfrac{2}{5})^{\frac12}R_{12}{}^{\frac32}\cos\theta-(9/2)(\tfrac{3}{5})^{\frac12}R_{12}{}^{5/2}\cos\theta],\\ h_3(\theta)&=0,\\ h_4(\theta)&=(\lambda/4\pi)[-i3(\tfrac{2}{5})^{\frac12}R_{12}{}^{\frac32}+i(\tfrac{3}{5})^{\frac12}R_{12}{}^{5/2}]\sin\theta. \end{aligned}$$

where the α_i are

$$
\begin{aligned}
\alpha_1 &= \mathrm{Im}[-(\tfrac{5}{2})^{\frac{1}{2}} R_{11}{}^{\frac{1}{2}} R_{02}{}^{\frac{3}{2}*} + \tfrac{2}{5} R_{11}{}^{\frac{3}{2}} R_{02}{}^{\frac{3}{2}*} \\
&\qquad + (9/5)(\tfrac{3}{2})^{\frac{1}{2}} R_{02}{}^{\frac{3}{2}} R_{13}{}^{\frac{5}{2}*}], \\
\alpha_2 &= -|R_{02}{}^{\frac{3}{2}}|^2, \\
\alpha_3 &= \mathrm{Re}[\tfrac{4}{5}(6/5)^{\frac{1}{2}} R_{02}{}^{\frac{3}{2}} R_{13}{}^{\frac{5}{2}*} + (6/5(5)^{\frac{1}{2}}) R_{11}{}^{\frac{3}{2}} R_{02}{}^{\frac{3}{2}*}], \qquad (2.10) \\
\alpha_4 &= \mathrm{Re}[\sqrt{2} R_{11}{}^{\frac{1}{2}} R_{02}{}^{\frac{3}{2}*} + (8/5(5)^{\frac{1}{2}}) R_{11}{}^{\frac{3}{2}} R_{02}{}^{\frac{3}{2}*} \\
&\qquad - \tfrac{3}{5}(6/5)^{\frac{1}{2}} R_{02}{}^{\frac{3}{2}} R_{13}{}^{\frac{5}{2}*}], \\
\alpha_5 &= \mathrm{Im}[(18/5) R_{02}{}^{\frac{3}{2}} R_{11}{}^{\frac{3}{2}*} - \tfrac{3}{5}(6^{\frac{1}{2}}) R_{02}{}^{\frac{3}{2}} R_{13}{}^{\frac{5}{2}*}].
\end{aligned}
$$

The case in which the initials and final intrinsic parities are different is described by the same equations modified by the replacement of $R_{L,L'}{}^J$ by $R_{L,L'\pm1}{}^J$, where the choice of sign is fixed by the vector addition law; this implies $|J-L'|=\frac{1}{2}$.

The foregoing expressions give the form of the angular distribution and spin-space characteristics of the hyperons produced in association with K particles in pion-nucleon collisions if the K particle and hyperons are spin 0 and spin $\frac{3}{2}$, respectively. In the subsequent decay of this hyperon into a pion plus nucleon, each term in the hyperon density matrix $\mathfrak{U}(\mathbf{K},\mathbf{K}') \equiv \mathfrak{U}_H$ gives a characteristic angular distribution and also a characteristic angular dependence for the polarization of the final nucleon. In order to exhibit these angular dependences in a convenient way we write Eq. (2.6) in the form

$$
\mathfrak{U}_H = \tfrac{1}{4} + \textstyle\sum_i \tau_1{}^i T(\mathbf{u}^i) + \sum_j \tau_2{}^j T(\mathbf{u}_1{}^j, \mathbf{u}_2{}^j) + \sum_k \tau_3{}^k T(\mathbf{u}_1{}^k, \mathbf{u}_2{}^k, \mathbf{u}_3{}^k). \qquad (2.11)
$$

In this formula the $\mathbf{u}_N{}^n$ are vectors that are to be selected in a way that gives the desired form of $\mathfrak{U}_H$. For example we obtain the form of $\mathfrak{U}_H$ given in Eq. (2.6) by the choice $\mathbf{u}^1=\mathbf{N}$, $\mathbf{u}_1{}^1=\mathbf{K}$, $\mathbf{u}_2{}^1=\mathbf{K}$, $\mathbf{u}_1{}^2=\mathbf{K}$, $\mathbf{u}_2{}^2=\mathbf{K}'$, etc. The various $\tau_j{}^i$ are to be identified with the coefficients $b(\theta)$, $c(\theta)$, etc., appearing in Eq. (2.6). The angular distribution of the decay products is given in terms of the $\tau_j{}^i$ by

$$
\begin{aligned}
I(\mathbf{V}) = (4\pi)^{-1}\{ &(|R_1|^2+|R_2|^2) + \textstyle\sum_i 5^{-\frac{1}{2}} \tau_1{}^i (\mathbf{u}^i\cdot\mathbf{V}) 4\,\mathrm{Re}(R_1R_2{}^*) - \sum_j \tau_2{}^j [3(\mathbf{u}_1{}^j\cdot\mathbf{V})(\mathbf{u}_2{}^j\cdot\mathbf{V}) - (\mathbf{u}_1{}^j\cdot\mathbf{u}_2{}^j)](|R_1|^2+|R_2|^2) \\
&- \textstyle\sum_k 5^{-\frac{1}{2}} \tau_3{}^k [5(\mathbf{u}_1{}^k\cdot\mathbf{V})(\mathbf{u}_2{}^k\cdot\mathbf{V})(\mathbf{u}_3{}^k\cdot\mathbf{V}) - (\mathbf{u}_1{}^k\cdot\mathbf{V})(\mathbf{u}_2{}^k\cdot\mathbf{u}_3{}^k) - (\mathbf{u}_2{}^k\cdot\mathbf{V})(\mathbf{u}_3{}^k\cdot\mathbf{u}_1{}^k) - (\mathbf{u}_3{}^k\cdot\mathbf{V})(\mathbf{u}_1{}^k\cdot\mathbf{u}_2{}^k)] 6\,\mathrm{Re}(R_1R_2{}^*)\}. \qquad (2.12)
\end{aligned}
$$

The polarization vector of the nucleon in the final state is given by

$$
\begin{aligned}
I(\mathbf{V})\mathbf{P} = (4\pi)^{-1}[&2\,\mathrm{Re}(R_1R_2{}^*)\mathbf{V} + \textstyle\sum_i 5^{-\frac{1}{2}} \tau_1{}^i [2(|R_1|^2+|R_2|^2)(\mathbf{u}^i\cdot\mathbf{V})\mathbf{V} + 4(|R_1|^2-|R_2|^2)\mathbf{V}\times(\mathbf{u}^i\times\mathbf{V}) \\
&- 8\,\mathrm{Im}(R_1R_2{}^*)(\mathbf{u}^i\times\mathbf{V})] - \textstyle\sum_j \tau_2{}^j 2\,\mathrm{Re}(R_1R_2{}^*)[3(\mathbf{u}_1{}^j\cdot\mathbf{V})(\mathbf{u}_2{}^j\cdot\mathbf{V}) - (\mathbf{u}_1{}^j\cdot\mathbf{u}_2{}^j)]\mathbf{V} - \sum_k 5^{-\frac{1}{2}} \tau_3{}^k \{[(|R_1|^2 \\
&+ |R_2|^2)(\mathbf{V}\cdot\mathbf{u}_1{}^k)\mathbf{V} + (|R_1|^2-|R_2|^2)\mathbf{V}\times(\mathbf{u}_1{}^k\times\mathbf{V}) - 2\,\mathrm{Im}(R_1R_2{}^*)(\mathbf{u}_1{}^k\times\mathbf{V})] \\
&\times[5(\mathbf{u}_2\cdot\mathbf{V})(\mathbf{u}_3\cdot\mathbf{V}) - 3(\mathbf{u}_2\cdot\mathbf{u}_3)] + \mathrm{Sym.}\}]. \qquad (2.13)
\end{aligned}
$$

The symbol Sym. in the preceding line represents the sum of the two terms needed to symmetrize the contents of the braces.

3. DISCUSSION

The consequences of nonconservation of parity in the decay of a spin-$\frac{1}{2}$ hyperon may be obtained from an examination of Eqs. (2.1) and (2.2) if it is noted that conservation of parity requires either R_0 or R_1 to vanish. The measurement of the anisotropy in the decay cross section as a test of parity conservation has been suggested by Lee *et al.*[6] Another test would be the measurement of the longitudinal polarization of decay nucleons that move in the plane of production. This latter method is more difficult but has the advantage that it is independent of the magnitude of the hyperon polarization. If the hyperon polarization is large, however, then the measurements the nucleon angular distribution and of various components of the nucleon polarization allow R_0 and R_1 to be determined up to an over-all phase. These two parameters give a complete phenomenological description of the decay process.

The measurement of the final polarization also permits a direct test of the invariance of the decay interaction with respect to time reversal. If the decay Hamiltonian is invariant under time reversal, the quantity $\mathrm{Im}(R_0R_1{}^*)$ must be zero in so far as the decay process may be considered to be first order in the decay Hamiltonian, provided final-state interactions may be neglected. The inclusion of the final-state interactions modifies this condition somewhat.[15] For the case $\Sigma^- \rightarrow n+\pi^-$ the upper limit on $|\mathrm{Im}(R_0R_1{}^*)|$ in the presence of the final-state interactions becomes

$$
|\sin(\delta_0-\delta_1)|\,|R_0|\,|R_1|
$$

where δ_L is the spin-$\frac{1}{2}$ isotopic spin-$\frac{3}{2}$ phase shift for the pion-nucleon system. This places a corresponding limit on the magnitude of the decay nucleon's component of polarization in the direction $\mathbf{P}_i\times\mathbf{V}$.

For the case of spin-$\frac{3}{2}$ hyperons, tests very similar to the above are obtained from an examination of Eqs. (2.12) and (2.13). For this case it is either R_1 or R_2 that must vanish if parity is conserved, and it is $\mathrm{Im}(R_1R_2{}^*)$ that is associated with invariance under time reversal.

If the hyperon is spin $\frac{3}{2}$, the correlation between the directions defined by the production process and those of the decay process are given by Eqs. (2.4) through (2.13). At production threshold, where only the S

[15] For a detailed account of the restrictions imposed upon the reaction matrix elements by invariance conditions and final-state interactions see reference 3.

waves of the final state contribute, the angular distribution for the production is isotropic and the angular distribution of the decay products in the decay center-of-mass frame is of the form $(3\cos^2\Theta'+1)$, where Θ' is the angle, measured in the decay center-of-mass frame, between the direction of the incident nucleon in the production process and the outgoing nucleon of the decay process. This may be compared to the case discussed by Treiman[2] in which it was the initial state of the production process that was an S state. In that case the angular distribution of the decay products was of the form $(3\cos^2\Theta+1)$, where Θ labels the angle between the hyperon velocity and the velocity of the final nucleon. For this limit in which only S waves are produced there will be no asymmetry with respect to the normal to the plane of production. At somewhat higher energies, where the interference between the final S and P waves becomes important, the hyperon-density matrix will contain nonvanishing contributions proportional to $T(\mathbf{N})$, $T(\mathbf{K},\mathbf{K}')$, $T(\mathbf{K}',\mathbf{K}')$, and $T(\mathbf{N},\mathbf{K}',\mathbf{K}')$. The form of the decay angular distribution associated with each of these terms may be obtained from Eq. (2.12). From the $T(\mathbf{N})$ term one obtains a contribution proportional to $\cos\Theta_N$, where Θ_N is the angle between the normal to the production plane and the direction of the nucleon from the decay. This term is analogous to the one that appeared when the hyperon was considered to be spin $\frac{1}{2}$, and it must vanish if parity is conserved in the decay process. The contribution from the $T(\mathbf{N},\mathbf{K}',\mathbf{K}')$ term will also be nonzero only if parity is violated in the decay. The angular distribution associated with this term is obtained from the τ_3 contribution to Eq. (2.12) by setting $\mathbf{u}_1{}^1=\mathbf{N}$, $\mathbf{u}_2{}^1=\mathbf{K}'$, and $\mathbf{u}_3{}^1=\mathbf{K}'$. It is of the form $\cos\Theta_N[5\cos^2\Theta'-1]$. This gives an asymmetry with respect to the normal to the production plane that is greatest for particles that decay in the plane defined by the vectors $\mathbf{N}$ and $\mathbf{K}'$ and which reaches a maximum when $\Theta_N\simeq 58.9°$. The maximum asymmetry from the $T(\mathbf{N})$ term occurs, of course, at $\Theta_N=0$.

In addition to these terms, which reveal parity violations, there is another new term in the angular distribution. This one is a consequence of the $T(\mathbf{K},\mathbf{K}')$ contribution to the hyperon density matrix. According to Eq. (2.12), the angular distribution characteristic of this term is $[3\cos\Theta\cos\Theta'-\cos\theta]$. Each of these terms will also give its characteristic contribution to the polarization of the final nucleon. The form of these contributions is given by Eq. (2.13). At higher energies, where all the terms in the general form of the hyperon density matrix given in Eq. (2.6) contribute, three additional terms may enter in the decay angular distribution. Two are present only if parity is violated, and have the forms

$$\cos\Theta_N[5\cos\Theta\cos\Theta'-\cos\theta]$$

and

$$\cos\Theta_N[5\cos^2\Theta-1].$$

The other has the form $(3\cos^2\Theta-1)$.

We conclude this section with several comments. First, the contributions to the decay angular distribution that are present when parity is not violated give no information about the decay mechanism except its total strength. All of them are proportional to $(|R_0|^2+|R_1|^2)$ for the spin-$\frac{1}{2}$ case and to $(|R_1|^2+|R_2|^2)$ for the spin-$\frac{3}{2}$ case. This form does not allow the contributions from the two final angular-momentum states to be distinguished. For the same reason, however, these terms give information about the production process that is independent of the detailed nature of the decay reaction, and their measurement provides information useful in the study of the strong reactions. Second, if, in the decay angular distribution there should occur a term that is asymmetrical with respect to any direction that lies in the plane of production, then parity must be violated both in the decay and in the production. It is assumed here that the strange particles are single particles—not parity doublets. Third, it is of interest to determine whether the intrinsic parity of the K-hyperon system is the same as the intrinsic parity of the pion-nucleon system. In view of the great dissimilarity in the forms of the $\mathcal{R}$ matrices in these two cases [see Eq. (2.4)], it might be thought that the correlations near threshold between the various angular distributions and polarizations would depend upon the relative intrinsic parities. However, no information about the relative intrinsic parities of the two systems can be obtained from the analysis of the angular distributions and polarizations of hyperons produced in pion-nucleon collisions unless assumptions are made regarding the relative magnitudes of the contributions from various initial angular-momentum states in the production process. This is a consequence of the close similarity, which is discussed below Eq. (2.10), of the formulas that describe the two alternative possibilities.

Finally we note that if the angular distributions in both the production and decay reactions are known, then for incident energies at which only S-wave and S,P-interference terms contribute the parameters α_2, α_3, and α_4 may be determined from (and are in fact overdetermined by) Eqs. (2.7) through (2.12). The production angular distribution determines $A'=-2\alpha_2$ and $B'=6\alpha_3-2\alpha_4$. This leaves a single parameter to be determined by the angular distribution of the decay products (only the part symmetric with respect to reflection in the plane of production is pertinent here). Note that the decay angular distributions at all of the various production angles may be used simultaneously. If $R_{02}{}^{\frac{1}{2}}$ is taken to be real by a suitable choice of the arbitrary phase factor then from the values of these α's the real parts of the other three $R_{LL'}{}^J$ are determined, except for sign, up to a single degree of freedom. The parameters α_1 and α_5 determine to the same extent the imaginary parts of these three $R_{LL'}{}^J$. However, such an analysis of the angular distributions in the production and decay reactions can determine α_1 and α_5 only up to the over-all factor $\mathrm{Re}(R_1R_2^*)/(|R_1|^2+|R_2|^2)$. This factor could be determined by, for example, a measurement of the longitudinal polarization of the final nucleon.

At higher energies, where a phase-shift analysis is not practical, the general parameters $g_i(\theta)$ [or $h_i(\theta)$] must be used in the analysis of the data. Due to their unknown dependence on the production angle θ the analysis at each production angle is a separate problem (unless general power series forms are used). A knowledge of the production cross section and the decay angular distribution for a single production angle $\theta\neq 0$ is sufficient to determine the seven parameters appearing in Eq. (2.6) except for the unknown factor $\mathrm{Re}(R_1R_2^*)/(|R_1|^2+|R_2|^2)$ that occurs with $b(\theta)$ and the various $d(\theta)$. Thus the expressions for $b(\theta)$, etc., in terms of $g_i(\theta)$ [or $h_i(\theta)$] together with Eq. (2.12) provides eight equations for eight unknowns, these being the real and imaginary parts of the four $g_i(\theta)$ [or $h_i(\theta)$] minus one over-all phase factor plus the unknown factor $\mathrm{Re}(R_1R_2^*)/(|R_1|^2+|R_2|^2)$. In principle, therefore, the production process at any angle is completely determined by the angular distribution measurements alone and the polarization measurements are necessary only to distinguish between the various solutions of the system of equations. In practice, however, the complexity of the equations limits their usefulness and polarization measurements would be quite valuable, particularly for the determination of $\mathrm{Re}(R_1R_2^*)$. Furthermore the polarization must be measured if $\mathrm{Im}(R_1R_2^*)$ is to be determined.

4. RELATIVISTIC CORRECTIONS

Although the expressions given above are nonrelativistic in form they may, if properly interpreted, be applied to relativistic problems. The fundamental idea is to apply the formulas to the proper polarization[16] of the fermions. The proper polarization is the polarization as observed in the rest frame of the particle, and it may be described by the nonrelativistic operators. If the covariant reaction matrix is multiplied by appropriate Lorentz transformations it will act directly upon the operators describing the initial covariant proper polarization to give the final covariant proper polarization. Specifically, if the reaction is treated in the center-of-mass frame, the reaction operator $\mathcal{R}_p$ that directly relates the initial and final proper polarizations is given in terms of the usual covariant reaction matrix $\mathcal{R}_r$ by the equation[17]

$$\mathcal{R}_p(k,k')=L(k)\mathcal{R}_r(k,k')L^{-1}(k'),$$

where $L(k)$ is a Lorentz transformation that transforms spinors from their values in a frame in which the center of mass of the reaction is at rest to their values in a rest frame of the final fermion whose four-momentum is k; the transformation $L(k')$ is defined in the same way but relative to the initial particle. The part of the matrix $\mathcal{R}_p$ that describes the transitions between initial and final states having energies of a well-defined magnitude and sign is a reduced matrix of the nonrelativistic form. Moreover, if the Lorentz transformations $L(k)$ and $L(k')$ are chosen to be pure timelike[18] transformations, then the vectors and spin matrices that appear in the reduced $\mathcal{R}$ matrix transform under spatial rotations in the usual nonrelativistic manner. The nonrelativistic reaction matrix and density matrix of the earlier sections may consequently be identified with the reduced part of $\mathcal{R}_p$ and the proper density matrix, respectively.[19]

If the center-of-mass frame of the reaction is not identical with the laboratory frame then there is an ambiguity in the definition of the proper polarization. The correspondence described above between the relativistic and the nonrelativistic formulations is valid specifically in the center-of-mass frame, and the components of proper polarization refer to those rest frames of the initial and final particles that are related to the center-of-mass frame by the transformations $L(k')$ or $L(k)$. In the usual definition of proper polarization the rest frame of the particle is taken to be one generated by the action upon the laboratory frame of a pure timelike Lorentz transformation. In order to obtain the usual proper polarizations from those proper polarizations appearing in our nonrelativistic expressions, the vectors describing the proper polarizations in the latter formalism must be transformed by the sequence of transformations that takes them first to the center-of-mass frame, then to the laboratory frame, and then to the usual rest frame. This sequence of transformations is equivalent to a pure rotation. If the center-of-mass frame is the one generated from the laboratory frame by a pure timelike Lorentz transformation, then the sequence of the three pure timelike transformations produces a rotation of the vectors describing the proper polarization by an amount specified in Eq. (48) of

[16] The term polarization is interpreted to include tensor-type polarizations.

[17] The light-face vector arguments are four-vectors. We have suppressed in this equation the dependence of the operators upon the total energy-momentum vector for the reaction. Since we are considering the center-of-mass frame, this vector is pure timelike. See reference 19 for a detailed discussion.

[18] A pure timelike Lorentz transformation will mean one that leaves unchanged the space components along directions perpendicular to the relative velocity of the two frames.

[19] H. P. Stapp, Phys. Rev. **103**, 425 (1956). This reference gives a detailed justification of the nonrelativistic treatment of spin-$\frac{1}{2}$ particles. The arguments may be extended to particles of higher spin.

reference 19.[20] A detailed treatment of the Dirac-particle case is given in that paper.†

[20] The velocity vectors $\mathbf{V}_a$ and $\mathbf{V}_b$ that occur in Eq. (48) of this reference are the relativistic or covariant velocities $(d\mathbf{x}/d\tau) = \gamma(d\mathbf{x}/dt)$, where t, τ, and γ are time, proper time, and relativistic contraction factor, respectively. This fact is not made sufficiently clear in the reference.

†*Note added in proof.*—If in the production of a spin-$\frac{3}{2}$ hyperon the initial nucleon is unpolarized and only S and P waves contribute in the final state then the angular distribution of the decay products, when averaged azimuthally, takes the form

$$I_D(\theta,\Theta') = \frac{1}{4\pi}\left[\left(\tfrac{3}{2}\cos^2\Theta' + \tfrac{1}{2}\right) - \frac{\chi \sin^2\theta}{I(\theta)}\left(\tfrac{3}{2}\cos^2\Theta' - \tfrac{1}{2}\right)\right].$$

Here θ is the center-of-mass production angle, $I(\theta)$ is the production cross section, χ is a positive constant, and $\cos\Theta' = (V\cdot K')$. Relativistically Θ' is the angle in the hyperon rest frame between the direction of the decay products and the direction lying in the production plane that makes an angle of $(\pi - \theta)$ with the line of flight (see from the hyperon) of the production center of mass. The azimuthal average is about this direction from which Θ' is measured. The constant χ is restricted by the inequality

$$\chi \leq \frac{\lambda^2}{4}\{A' - C' + [(A'+C')^2 - B'^2]^{\frac{1}{2}}\} \leq \frac{\lambda^2}{4}(2A'),$$

where A', B', and C' are the constants given in Eq. (2.7). The form of $I_D(\theta,\Theta')$ given above is an extension to all production angles of a relationship given by Adair,[4] and it will be useful in determining whether the hyperon spin is $\frac{3}{2}$. Its derivation, which is based upon Eqs. (2.5), (2.6), (2.11), and (2.12) and Table I together with the general expressions for the τ_i in terms of the g_i (or h_i) obtained in reference 3, is given by the present authors in the University of California Radiation Laboratory Report UCRL-8005 (unpublished).

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Hyperon Production in Nucleon-Nucleon Collisions*

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The hyperon production process in nucleon-nucleon collisions is analyzed in terms of isotopic spin, and angular momentum near threshold, to deduce certain tests for global symmetry, charge independence, and the parities of Σ, Λ, and K mesons.

Information about the hyperon-nucleon interaction may be obtained from the K-meson spectrum. However, the Σ's and Λ's may interchange owing to this interaction, and Watson's theory of final-state interactions is generalized to allow for this effect. The result is applied to the Σ processes and its significance for the recent calculations of Henley, on the Λ-N final-state interaction is discussed.

1. INTRODUCTION

WE consider the process

$$N + N \rightarrow N + Y + K. \tag{1.1}$$

In Sec. 2 this is analyzed in terms of isotopic spin; certain predictions are made about production ratios, and a relation between cross sections of the type familiar in pion physics is derived.

In Sec. 3 an angular momentum analysis is made near the threshold for Λ production, and near threshold for the process

$$p + p \rightarrow p + \Sigma^+ + K^0, \tag{1.2}$$

which has the advantage that it does not interfere with the Λ production process. We assume that the final baryons are in s states and that the K meson is emitted in an s or p state according to whether it has the same, or opposite, parity as the accompanying hyperon.[1] The angular distribution of the K meson relative to the initial momentum is either isotropic, or of the form $a + b\cos^2\theta$, according to the two possibilities. Thus, the angular distributions should determine the relative parities of the Σ's, Λ's, and K mesons.

In the corresponding pion process,

$$N + N \rightarrow N + N + \pi, \tag{1.3}$$

the spectrum of the pion is strongly affected by the interaction between the final nucleons.[1] In that case, the two-nucleon potential was already phenomenologically well understood. At present little is known of the Y-N interaction and this is a fairly direct way of getting information. In particular, if bound states of $\Lambda^0 N$ or $\Sigma^+ p$ exist, this should be an efficient way of producing them. However, in this problem the effect is complicated by the fact that the final state inter-

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[1] K. M. Watson and K. N. Brueckner, Phys. Rev. 83, 1 (1951); A. H. Rosenfeld, Phys. Rev. 96, 139 (1954).