where $d\Omega'$ is the differential of solid angle in the center-of-mass system. The integral is carried out over the aperture of the telescope in k_0, Ω' space. N is the number of target nucleons/cm^2 with the particular value of initial momentum. $N(d\sigma/d\Omega')$ is the probability per unit solid angle of a photon with energy k_0 producing a meson at an angle $\pi-\delta$ in the center-of-mass system. The product $(dn/dk_0)dk_0$ is the number of photons in the energy range $k_0\pm\frac{1}{2}dk_0$. If the aperture of the telescope is sufficiently small, as it was designed to be, then we may replace the integrand by its value at the center of the aperture and take it outside of the integral. Making successive transformations of the integral from k_0,Ω' space to k,Ω' space, to k,Ω space and finally to T_p,Ω space, where the integral becomes $\Delta T_p\Delta\Omega$, we obtain

$$\text{Counts/beam integrator pulse} = (d\sigma/d\Omega')\Delta T_p\Delta\Omega NG/E_0 \times\{(\partial \cos\alpha'/\partial \cos\alpha)(1-\beta_n \cos\theta_n)(\partial k/\partial T_p)f(k)/k\},$$

where ΔT_p is the width of the proton energy interval accepted by the telescope, $\Delta\Omega$ is the solid angle subtended by the telescope, G is the total energy in the photon beam per beam integrator pulse, E_0 is the maximum bremsstrahlung energy, $\partial \cos\alpha'/\partial \cos\alpha$ is the solid angle transformation function (photon energy held constant), β_n is the initial velocity of the target nucleon, (θ_n,ϕ_n) is the initial direction of the target nucleon (the photon direction is $\theta=0$; the plane containing photon and recoil proton defines $\phi=0$), $\partial k/\partial T_p$ is the partial derivative of photon energy with respect to recoil proton energy with the laboratory proton angle held constant, and $f(k)$ is the bremsstrahlung function defined so that the number of photons/beam integrator pulse in the Δk is $Gf(k)\Delta k/kE_0$.

In deriving the above equation, account was taken of the fact that at arget nucleon moving toward the synchrotron sees a larger photon flux than if it were moving away from the synchrotron. It is now seen that the cross section at each of the dots in Fig. 4 should be weighted by the expression inside the braces in the foregoing equation.

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Scattering of μ^- Mesons by Nuclei*

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The effect of finite nuclear size on the scattering of μ^- mesons is calculated for a nucleus $Z=80$, $R=1.2A^{\frac{1}{3}}\times10^{-13}$ cm. The nucleus is considered as a uniformly charged sphere. The μ^- mesons have $v/c=0.2$ corresponding to an energy $E=2.1$ Mev. It is found that the left-right asymmetry in the scattering of polarized μ^- mesons is decreased from that for a point nucleus. However, considerable asymmetry remains at large angles.

INTRODUCTION

THE scattering of negative μ mesons by a point Coulomb field is similar to that for electrons if the μ meson is taken to be a Dirac particle. For the same $\beta=v/c$, the only difference is that the μ cross section is smaller in the ratio of the squares of the masses of the two particles. The angular distributions and polarizations are exactly the same in the two cases.

Recent experiments[1] on the decay of π mesons, $\pi\rightarrow\mu+\nu$, have indicated that the μ mesons are polarized in the direction of their momentum in the center-of-mass system. It is possible, by using this effect, to obtain a beam of transversely polarized μ mesons. The scattering of such a beam by a point Coulomb field will have a left-right asymmetry,[2] S, which has been calculated by Sherman.[3] Measurement of this asymmetry would indicate the original spin direction (parallel or antiparallel to the momentum) of the μ meson and this could be used as a check on the hypothesis of conservation of leptons.

The purpose of this paper is to examine the effect of the finite size of a charge distribution such as is found in heavy nuclei on the scattering cross section and on S for μ^- mesons. For electrons the finite size of the nucleus does not become important until highly relativistic velocities are reached, at which point there are no polarization effects. μ mesons, because of the larger mass, have smaller wavelengths than electrons for a corresponding β and in fact the finite size affects the scattering strongly already at the lowest energies.

We derive in the next section a general method for correcting the Dirac particle point Coulomb scattering results for finite nuclear size for a uniformly charged nucleus of radius R. We then apply the results to the

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† Part of this work was performed while one of the authors (J.F.) was at Brookhaven National Laboratory, Upton, New York.
[1] Garwin, Lederman, and Weinrich, Phys. Rev. **105**, 1415 (1957).
[2] H. A. Tolhoek, Revs. Modern Phys. **28**, 277 (1956).
[3] N. Sherman, Phys. Rev. **103**, 1601 (1956).

calculation of the cross section for scattering of a μ^- meson beam and the left-right asymmetry for transverse polarization for the case $\beta=0.2$ ($E=2.1$ Mev), the nucleus having charge number $Z=80$, $kR=0.77$. This corresponds to using a nuclear radius $R=1.2A^{\frac{1}{3}}\times10^{-13}$ cm. The small value of kR enables the calculation to be done with a desk computer. Machine calculations are advisable if a series of such calculations are to be performed, especially for higher energies.

THEORY

The scattering of relativistic Dirac particles of mass, m, and total relativistic energy, E, in the potential, V, is governed by the Dirac radial equations[4,5]

$$\begin{aligned}\frac{1}{\hbar c}(E-V+mc^2)F_\kappa+\frac{dG_\kappa}{dr}-\frac{(\kappa-1)}{r}G_\kappa=0,\\ -\frac{1}{\hbar c}(E-V-mc^2)G_\kappa+\frac{dF_\kappa}{dr}+\frac{(\kappa+1)}{r}F_\kappa=0,\end{aligned}\tag{1}$$

where $\kappa=\pm(j+\frac{1}{2})$ for $j=l\pm\frac{1}{2}$.

The asymptotic form of G_κ,

$$G_\kappa\sim\frac{1}{r}\sin\left(kr+\frac{Ze^2}{\hbar v}\ln2kr-\tfrac{1}{2}l\pi+\eta_\kappa\right),\tag{2}$$

determines the phase shift, η_κ. The logarithmic term appears since the potential falls off as $1/r$.

The scattering cross section averaged over φ is then given by

$$\sigma(\theta)=|f(\theta)|^2+|g(\theta)|^2,\tag{3}$$

with

$$\begin{aligned}f(\theta)&=\frac{1}{2ik}\sum_{\kappa=1}^{\infty}\kappa\{[\exp(2i\eta_\kappa)-1]P_{\kappa-1}(\cos\theta)\\&\qquad+[\exp(2i\eta_{-\kappa})-1]P_\kappa(\cos\theta)\},\\ g(\theta)&=\frac{1}{2ik}\sum_{\kappa=1}^{\infty}\{-\exp(2i\eta_\kappa)P_{\kappa-1}^1(\cos\theta)\\&\qquad+\exp(2i\eta_{-\kappa})P_\kappa^1(\cos\theta)\}.\end{aligned}\tag{4}$$

For a given initial spin direction $\boldsymbol{\sigma}$, the cross section as a function of θ and φ is given by[2]

$$\frac{d\sigma(\theta,\varphi)}{d\Omega}=\{|f(\theta)|^2+|g(\theta)|^2\}\left\{1+S(\theta)\frac{\boldsymbol{\sigma}\cdot(\hat{p}_1\times\hat{p}_2)}{\sin\theta}\right\},\tag{5}$$

where $\hat{p}_1$ and $\hat{p}_2$ are unit vectors in the directions of initial and final meson momenta, respectively, and

$$S(\theta)=\frac{i(fg^*-f^*g)}{|f|^2+|g|^2}.\tag{6}$$

The left-right asymmetry in the scattering, that is the relative number of scatterings at $\varphi=0$ minus the number at $\varphi=\pi$ divided by the sum of these two scatterings, for a given θ is

$$\frac{L-R}{L+R}=\left(\frac{\boldsymbol{\sigma}\cdot(\hat{p}_1\times\hat{p}_2)}{\sin\theta}\bigg|_{\varphi=0}\right)S(\theta).\tag{7}$$

For a transversely polarized beam (see Fig. 1),

$$(L-R)/(L+R)=S(\theta).\tag{8}$$

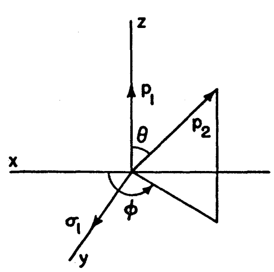

FIG. 1. The relative directions of spin and momenta for a transversely polarized beam.

Polarizations after double scattering of an unpolarized beam can also be calculated by using S. Formulas for such polarizations are given by Mott and Massey[4] and by Tolhoek.[2]

For a point Coulomb potential, $V=-\alpha\hbar c/r$ where $\alpha=Ze^2/\hbar c$, the solutions of (1) which are regular at the origin are:

$$G_\kappa=-M_\kappa\,\mathrm{Im}\,\xi_\kappa/r,\quad F_\kappa=N_\kappa\,\mathrm{Re}\,\xi_\kappa/r,\tag{9}$$

$$\xi_\kappa=\left(\frac{s-iq}{-\kappa-iq'}\right)^{\frac{1}{2}}e^{-ikr}(2kr)^sF(s+iq,\,2s+1,\,2ikr),\tag{10}$$

where $s=(\kappa^2-\alpha^2)^{\frac{1}{2}}$, $q=\alpha/\beta$, $q'=q(1-\beta^2)^{\frac{1}{2}}$ with $\beta=v/c$, and $F(s+iq,\,2s+1,\,2ikr)$ is the confluent hypergeometric function. M_κ and N_κ are constant normalization factors with $N_\kappa=-[(E/mc^2-1)/(E/mc^2+1)]^{\frac{1}{2}}M_\kappa$.

The asymptotic expression for the confluent hypergeometric function leads to the asymptotic form (2) with

$$\eta_\kappa=-\arg\left(\frac{s-iq}{-\kappa-iq'}\right)^{\frac{1}{2}}-\arg\Gamma(s+1+iq)+\tfrac{1}{2}(l-s)\pi,\tag{11}$$

and the normalization constant

$$M_\kappa=\frac{|\Gamma(s+1+iq)|}{\Gamma(2s+1)}e^{\pi q/2}.\tag{12}$$

TABLE I. Changes in Coulomb phase shifts due to finite nuclear size.

κ	1	−1	2	−2	3
$-\delta_\kappa$(degrees)	76.5	26.3	12.6	1.3	0.8

[4] N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Oxford University Press, Oxford, 1949), second edition, p. 74.
[5] L. K. Acheson, Phys. Rev. **82**, 488 (1951).

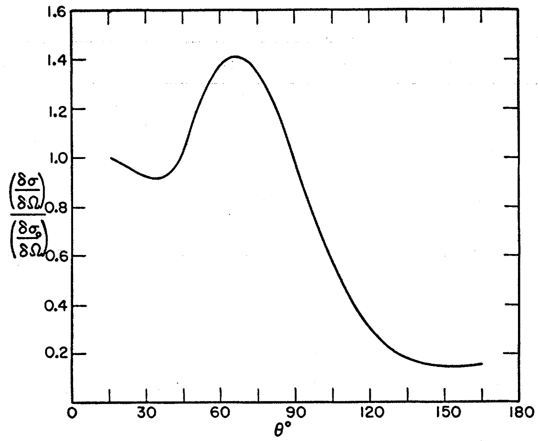

FIG. 2. The relative cross section for a uniform charge distribution. $d\sigma_0/d\Omega$ is Sherman's result for a point nucleus.

The use of the η_κ given by (11) in Eqs. (3), (4), and (6) would then lead to the scattering cross section and asymmetry for a point Coulomb scatterer.

For a finite size nucleus, the phase shift is determined by matching the logarithmic derivative of the inside wave function to a linear combination of the regular and irregular Coulomb solutions at the boundary of the nucleus.

Eliminating F_κ in Eqs. (1) and making the substitution $g_\kappa = r[d(\ln rG_\kappa)/dr]$ leads to a Ricatti-type differential equation for the inside logarithmic derivative:

$$r\frac{dg_\kappa}{dr}+g_\kappa^2+g_\kappa\left(-1+\frac{r(dV/dr)}{(E-V+mc^2)}\right)+\frac{[(E-V)^2-m^2c^4]}{\hbar^2c^2}r^2-\kappa(\kappa-1)-\frac{r(dV/dr)\kappa}{(E-V+mc^2)}=0. \qquad (13)$$

The solution of Eq. (13) for a uniformly charged nucleus is given in the appendix. The matching condition at the radius, R, of the nucleus then is

$$g_\kappa(R)=R\left[\frac{d}{dr}\{\ln[ArG_\kappa^+(r)+BrG_\kappa^-(r)]\}\right]_{r=R}, \qquad (14)$$

where G_κ^+ is the regular Coulomb solution given by (6) and G_κ^- is the irregular solution obtained from G_κ^+ by replacing s by $-s$.

The phase shift for finite size, η_κ', is then determined from the asymptotic relation

$$\sin(kr+q\ln kr-\tfrac{1}{2}l\pi+\eta_\kappa')\sim A\sin(kr+q\ln kr-\tfrac{1}{2}l\pi+\eta_\kappa^+)+B\sin(kr+q\ln kr-\tfrac{1}{2}l\pi+\eta_\kappa^-), \qquad (15)$$

where η_κ^+ and η_κ^- are the phase shifts for G_κ^+ and G_κ^-, respectively, as determined by Eq. (11).

If one lets $\delta_\kappa=\eta_\kappa'-\eta_\kappa^+$ and $\epsilon_\kappa=\eta_\kappa^- -\eta_\kappa^+$, Eq. (15) leads to

$$\tan\delta_\kappa=\frac{\sin\epsilon_\kappa}{(A/B)_\kappa+\cos\epsilon_\kappa}. \qquad (16)$$

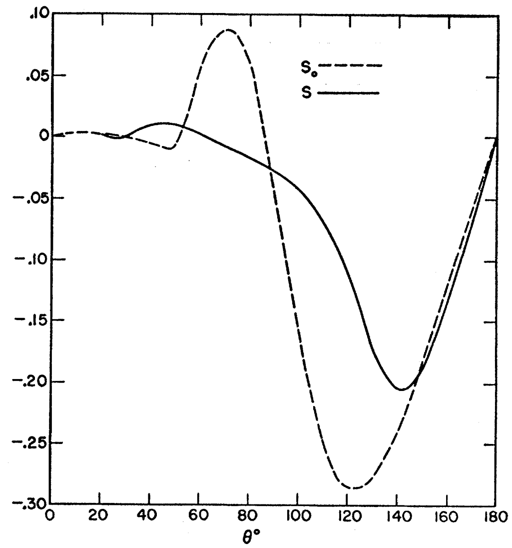

FIG. 3. The asymmetry parameter $S(\theta)$ for a uniform charge distribution. S_0 is Sherman's result for a point nucleus.

The ratio $(A/B)_\kappa$ from Eq. (14) is

$$\left(\frac{A}{B}\right)_\kappa=-(2kR)^{-2s}\frac{|\Gamma(1-s-iq)|}{|\Gamma(1+s-iq)|}\frac{\Gamma(1+2s)}{\Gamma(1-2s)}\frac{\mathrm{Im}\,\{e^{-ikR}\exp(i\varphi^-)}{\mathrm{Im}\,\{e^{-ikR}\exp(i\varphi^+)}\frac{[(g_\kappa+s+ikR)F^- -R(dF^-/dr)]_{r=R}\}}{[(g_\kappa-s+ikR)F^+ -R(dF^+/dr)]_{r=R}\}}, \qquad (17)$$

$$\exp(i\varphi^\pm)=\left(\frac{\pm s-iq}{-\kappa-iq'}\right)^{\frac{1}{2}},$$

and $F^\pm=F(\pm s+iq,\ \pm 2s+1,\ 2ikR)$.

The expression for $f(\theta)$ and $g(\theta)$ can be written directly in terms of δ_κ and the regular Coulomb phase shifts, η_κ^+:

$$f(\theta)=f_C(\theta)+\frac{1}{2ik}\sum_{\kappa=1}^{\infty}\kappa\{\exp(2i\eta_\kappa^+[\exp(2i\delta_\kappa)-1]\times P_{\kappa-1}(\cos\theta)+\exp(2i\eta_{-\kappa}^+)[\exp(2i\delta_{-\kappa})-1]P_\kappa\},$$

$$g(\theta)=g_C(\theta)+\frac{1}{2ik}\sum_{\kappa=1}^{\infty}\{-\exp(2i\eta_\kappa^+)[\exp(2i\delta_\kappa)-1]\times P_{\kappa-1}^1+\exp(2i\eta_{-\kappa}^+)[\exp(2i\delta_{-\kappa})-1]P_\kappa^1\}, \qquad (18)$$

where $f_C(\theta)$ and $g_C(\theta)$ are the values of $f(\theta)$ and $g(\theta)$ for point Coulomb scattering. The functions $f_C(\theta)$ and $g_C(\theta)$ can be calculated from the tables of Sherman

TABLE II. Differential cross section and asymmetry parameter.

θ	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°
$d\sigma/d\Omega$(b)	6260	374	87.8	38.5	16.6	6.86	2.84	1.27	0.738	0.615	0.630
$\frac{d\sigma/d\Omega}{d\sigma_0/d\Omega}$ [a]	1.01	0.921	1.03	1.37	1.35	0.990	0.572	0.300	0.183	0.155	0.158
$100\,S(\theta)$	0.3	−0.1	1.1	0.2	−1.3	−2.7	−5.3	−11	−19	−19	−10
$100\,S_0(\theta)$ [a]	0.2	−0.2	−1.0	5.6	8.2	−3.6	−20	−28	−26	−19	−10

[a] $d\sigma_0/d\Omega$ and $S_0(\theta)$ refer to the point-nucleus results of Sherman.

through

$$\begin{aligned} kf_C(\theta)&=G(\theta)-iq'F(\theta),\\ kG_C(\theta)&=\tan(\tfrac{1}{2}\theta)G(\theta)+iq'\cot(\tfrac{1}{2}\theta)F(\theta), \end{aligned} \tag{19}$$

where $F(\theta)$ and $G(\theta)$ are given by Sherman.

RESULTS AND CONCLUSIONS

The change in the Coulomb phase shifts due to the finite nuclear size is tabulated in Table I. The finite size of the nucleus has the effect of decreasing the scattering considerably at large angles (Table II and Fig. 2).

The left-right asymmetry, S, in the scattering of transversely polarized μ^- mesons as a function of scattering angle is plotted in Fig. 3 both for a finite size charge and for a point Coulomb field. At large angles a considerable left-right asymmetry remains after correcting for finite nuclear size. One might have expected this asymmetry to disappear owing to the finite nuclear size with its consequent weakening of the spin-orbit interaction. However, the Rutherford or charge scattering is also weakened and hence a considerable asymmetry remains. It is not unreasonable to expect that this left-right asymmetry will persist at higher energies.

It is to be noted that for μ^+ mesons of 2-Mev energy, one should not expect to see asymmetries since the Coulomb repulsion of the nuclear scattering center prevents the μ^+ meson from reaching the region where the spin-orbit interaction is important.

Finally, the results of this calculation show that μ mesons when scattered from nuclei can provide information on the charge distribution of nuclei. This information would supplement that obtained from the work of Fitch and Rainwater[6] on μ-mesonic atoms and that of Hofstadter[7] on high-energy electron scattering. It is to be noted that measurements of left-right asymmetries in the scattering of transversely polarized μ mesons provides information not obtainable from electron scattering since for electrons the finite nuclear size is not important at energies where $S(\theta)$ is large.

[6] V. L. Fitch and J. Rainwater, Phys. Rev. **92**, 789 (1953).
[7] E.g., R. Hofstadter, Revs. Modern Phys. **28**, 214 (1956).

ACKNOWLEDGMENTS

We wish to express thanks to Miss H. Oberthal, who carried out much of the numerical work involved in this paper.

APPENDIX

The potential energy of a negatively charged μ meson inside a uniformly charged nucleus is

$$V=-(Ze^2/2R)[3-(r/R)^2]. \tag{A1}$$

Equation (13) can be written for potential (A1) in the form

$$(\epsilon-bx)g_\kappa{}^2+2x(\epsilon-bx)\frac{dg_\kappa}{dx}-(\epsilon-3bx)g_\kappa+J(x)=0 \tag{A2}$$

where

$$\begin{aligned} J(x)=-\kappa(\kappa-1)&+[\rho^2\epsilon^2(\epsilon-2)+\kappa(\kappa-3)b]x\\ &-\rho^2b\epsilon(3\epsilon-4)x^2+\rho^2b^2(3\epsilon-2)x^3-\rho^2b^3x^4\\ x=(r/R)^2,&\quad b=(Ze^2/R)/2mc^2,\\ \epsilon=(E+mc^2)/mc^2+3b,&\quad \rho=R/(\hbar/mc). \end{aligned}$$

The solution of (A2) can be written as

$$g_\kappa(x)=\sum_{n=0}^{\infty}a_nx^n,$$

where

$$\begin{aligned} a_0&=\kappa \quad \text{for } \kappa>0,\\ &=-(\kappa-1) \quad \text{for } \kappa<0, \end{aligned}$$

and

$$a_n=\left[\frac{b}{\epsilon}(2n-5+a_0)a_{n-1}-\frac{\theta_n}{\epsilon}+\sum_{m=1}^{n-1}a_m\left(\frac{b}{\epsilon}a_{n-m-1}-a_{n-m}\right)\right](2n-1+2a_0)^{-1}. \tag{A3}$$

θ_n is given by

$$J(x)=\sum_{n=0}^{4}\theta_nx^n,\ \text{and}\ \theta_n=0\ \text{for}\ n>4.$$

The value of $g_\kappa(r)$ for $r=R$ is then

$$g_\kappa(R)=\sum_{n=0}^{\infty}a_n. \tag{A4}$$