Spacings of Nuclear Energy Levels

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(Received September 30, 1957)

The distribution of spacings of nuclear energy levels in many heavy nuclei at an excitation energy of 5 to 9 Mev is obtained by careful correction of the observed distributions for the effect of failure to observe all levels. Results of transmission measurements on U^{234} and U^{236}, as measured with the Brookhaven fast chopper, are presented. The experimental spacings of the zero-spin nuclides are considered first since all the levels from slow neutron capture have the same spin. The results show a deficiency of small spacings relative to the exponential distribution, which corresponds to a random occurrence of levels. In the analysis it is shown that there is no local correlation of neutron widths and level spacings. The "level repulsion" effect is also found for the nuclides of nonzero spin, for which the data are more abundant but the analysis is complicated by the presence of two spin systems. The distribution obtained is in agreement with one suggested by Wigner based on a probability of level occurrence proportional to the spacing S. The corrections here developed are also applied to the reduced neutron width distribution and this corrected distribution is in good agreement with the Porter-Thomas distribution.

INTRODUCTION

THE techniques of slow-neutron spectroscopy have advanced rapidly during the last few years, both in the resolving power of the instruments and in the methods of analysis of the data. As a result of these improvements, parameters of many neutron resonances are now available, particularly for heavy nuclei. The heavy nuclei have given much data because there are many resonances in the low-energy region for which the new instruments and analysis methods are well adapted. In addition, the fact that only $l=0$ interactions take place at low energy simplifies the interpretation greatly. The parameters of each resonance represent properties of the energy level in the compound nucleus to which the resonance corresponds. These energy levels are at an excitation energy of from 5 to 9 Mev, equal to the neutron binding energy.

For elements above atomic weight 100 there are values of neutron widths now available for several hundred resonances and of radiation widths for about fifty resonances.[1] The measured parameters have revealed the general distribution laws of these parameters, although not in detail as yet. The radiation widths are rather constant from level to level in individual nuclides, and in fact do not exhibit large variations even from nuclide to nuclide.[2–5] In contrast, the neutron widths range over values differing by factors as large as several hundred in individual nuclides, and the average values for various nuclides also show wide variations.[6,7] These findings are in good agreement with the expectations based on current nuclear theory[8]; the radiation widths being very constant because they are averages over several hundred final states available for gamma emission, whereas the neutron widths vary markedly from level to level because each neutron width corresponds to a single exit channel, rather than an average. Concerning the spacings of levels, the average spacing, D, is now reasonably well known for a number of nuclides. The relationship of D to other nuclear properties such as the even-odd characteristics, the excitation energy, the proximity to closed shells, and spin has been investigated,[9–12] but not in detail. At the present time more values of the level spacings are needed in order to specify accurately the dependence of D on all the the individual important parameters.

There has been little detailed information available on the spacings, S, between adjacent levels in individual nuclides. This lack may seem a bit surprising for apparently a very easy property to measure concerning the levels is their separation in energy. However, because of the great difficulty in ascertaining whether every level has been detected, it is difficult to say with certainty whether a legitimate set of spacings has been measured for a given nuclide. Another difficulty in gaining information on the distribution of spacings arises because in all but zero-spin target nuclei the observed levels correspond to two different spin states, whose individual distributions are independent. For capture of slow neutrons by a target nucleus of spin I, levels of spin $I+\frac{1}{2}$ or $I-\frac{1}{2}$ can be formed. For practically all measurements of neutron resonances no determination of the spin is possible, hence the observed levels

[1] *Neutron Cross Sections*, compiled by D. J. Hughes and J. A. Harvey, Brookhaven National Laboratory Report BNL-325 (Superintendent of Documents, U. S. Government Printing Office, Washington, D. C., 1955) and D. J. Hughes and R. B. Schwartz, Supplement I to BNL-325.
[2] D. J. Hughes and J. A. Harvey, Nature **173**, 942 (1954).
[3] H. H. Landon, Phys. Rev. **100**, 1414 (1955).
[4] J. S. Levin and D. J. Hughes, Phys. Rev. **101**, 1328 (1955).
[5] A. Stolovy and J. A. Harvey, Phys. Rev. **108**, 353 (1957).
[6] Harvey, Hughes, Carter, and Pilcher, Phys. Rev. **99**, 10 (1955).
[7] D. J. Hughes and J. A. Harvey, Phys. Rev. **99**, 1032 (1955).
[8] C. E. Porter and R. G. Thomas, Phys. Rev. **104**, 483 (1956).
[9] H. W. Newson and R. H. Rohrer, Phys. Rev. **94**, 654 (1954).
[10] Sailor, Landon and Foote, Jr., Phys. Rev. **96**, 1014 (1954).
[11] John A. Harvey, Phys. Rev. **98**, 1162 (1955).
[12] T. D. Newton, Can. J. Phys. **34**, 804 (1956).

correspond to the two possible spin states combined at random. If the spacing distribution is exponential in each spin state then these two distributions combined will also be exponential. In that case no complication will be introduced by the presence of two spin systems. However, if the levels are to some extent regular in their occurrence, the distribution law of the two spin states combined will appear to be more random than the single spin state distribution. This distorting effect of the combination of the two separate spin systems makes the experimental determination of the distribution of spacings of each spin state much more difficult.

Because the spacings of levels are determined by the structure of the nucleus it is obviously of interest to measure the actual distribution law of the individual spacings for comparison with theory. In addition, there is a practical use of the level spacing distribution. This use has to do with the prediction of cross sections in the region of several kev, above the energy at which detailed resonances can be measured, but in a region where cross sections are needed for design of reactors. In order to predict the cross sections in the kev region it is necessary to know the distribution laws of neutron and radiation widths and, in addition, the distribution of the level spacings.

There has been practically no theoretical work done on the distribution of spacings between nuclear energy levels. It seems that the assumption has usually been made that the levels are distributed at random. This assumption amounts to saying that the probability of a level occurring in the interval dS at a distance S from a particular level does not depend on S and is given by dS/D. This gives a probability distribution of level spacings S proportional to $e^{-S/D}$, an exponential distribution of spacings. However, it was pointed out in 1929 by von Neumann and Wigner[13] that in many-body problems in quantum mechanics there will always be a repulsion of energy levels. Recently Wigner[14] has considered the question of the spacings of nuclear energy levels of zero-spin target nuclei and has concluded that at least for small values of S the probability of finding a level in an interval dS is proportional to (SdS/D). If this proportionality to S should hold for large values of S the distribution law of spacings would then be proportional to $S\exp[-(\pi/4D^2)S^2]$, a distribution that goes to zero for small S. If the probability of finding a level is proportion to S only for small values of S, the distribution would probably be exponential for large spacing values but there would be fewer small spacing values than from an exponential distribution.

The target nucleus U^{238} was particularly useful in getting the first information on the distribution law of spacings. As it is even-even, only one compound nucleus spin, $\frac{1}{2}$, results from capture of slow neutrons, hence the difficulty of the two spin systems is avoided. Furthermore, because of its importance to nuclear reactors the energy levels of U^{238} have been carefully investigated. The first results for U^{238} showed that the levels occurred in a surprisingly regular manner, and not at all like the exponential distribution of spacings expected for a random occurrence of levels. The number of levels investigated in U^{238}, however, was only about 10 or 15 in number, hence, the statistical accuracy was not sufficient to draw any definite conclusion about the distribution law of spacings. The surprising regularity of spacings in U^{238}, however, stimulated further work. On the basis of data on U^{238} and preliminary results on Th^{232}, Harvey[11] reported that the spacings appeared more uniform than would have been expected if the levels occurred at random. However, Fujimoto, Fukuzawa, and Okai[15] considered the distribution of 268 level spacings computed from resonance data which had been published, and concluded that the distribution was consistent with an exponential distribution. Gurevich and Pevsner[16] concluded that there was a repulsion of levels for zero-spin nuclei and also for levels of the same spin state for the nonzero-spin target nuclei. However, in all these treatments, no corrections were applied for the small resonances which were not observed. Measurements were undertaken with the Brookhaven fast chopper with the nuclides Th^{232}, U^{234}, and U^{236}, in order to obtain more data similar to that for U^{238}, and the odd nuclides results were carefully studied as well.

After presenting the experimental data on two of these even-even nuclides in the next section, we shall consider the various corrections that must be made in order to obtain the actual spacing distribution from the observed location of levels. Because of the finite resolving power of the instruments, distortions of the actual spacing distribution inevitably occur. It is a matter of some difficulty to correct the observed spacing distribution for these experimental effects, and the corrections were considered in detail. In the process the neutron width distribution was investigated, as well as the possible correlation of individual neutrons widths to local spacing. After the spacing law for the zero-spin nuclei is obtained, we shall then consider the nonzero-spin nuclei, for which many more data are available, but for which the added complication of the combination of two spin systems is present.

II. EXPERIMENTAL DATA ON U^{234} AND U^{236}

In order to increase the available data on the distribu- of spacings of even-even target nuclei, measurements were made on the nuclides U^{234} and U^{236}. Previous to the present work very little was known about the resonance structure of these two nuclides. In 1949 a resonance was assigned to U^{234} at 5 ev based on trans-

[13] J. von Neumann and E. Wigner, Physik. Z. **30**, 467 (1929).
[14] E. P. Wigner, Oak Ridge National Laboratory Report ORNL-2309, 1957 (unpublished), p. 59.
[15] Fujimoto, Fukuzawa, and Okai (private communication).
[16] I. I. Gurevich and M. I. Pevsner, J. Exptl. Theoret. Phys. U.S.S.R. **31**, 162 (1956) [translation: Soviet Phys. JETP **4**, 278 (1957)]; also Nuclear Phys. **2**, 575 (1957).

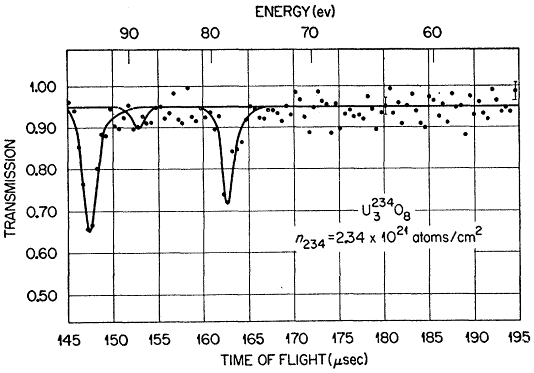

FIG. 1. The transmission as a function of time-of-flight at 20 m for a sample of $(U^{234})_3O_8$ ($n_{234}=2.34\times10^{21}$ atoms/cm^2) in the energy range from 55 to 100 ev. The straight line is the base line from which areas are measured to determine the neutron widths of the resonances.

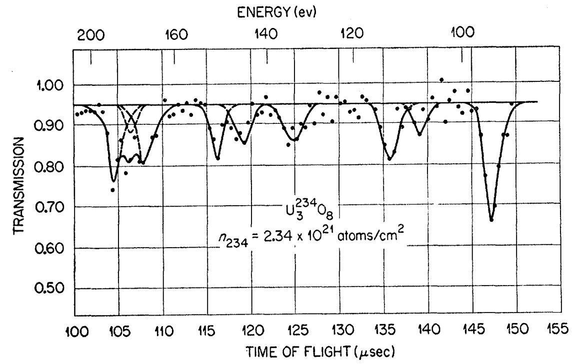

FIG. 2. The transmission as a function of time-of-flight at 20 m for a sample of $(U^{234})_3O_8$ ($n_{234}=2.34\times10^{21}$ atoms/cm^2) in the energy range from 90 to 200 ev.

mission measurements at Columbia[17] using two samples of enriched U^{235} containing different percentages of U^{234}. In measurements made with the Oak Ridge chopper in 1952[18] using a sample enriched to 96.65% in U^{236}, a resonance was found at 5.3 ev in U^{236}. It was expected that the resonance structure of U^{234} and U^{236} would be similar to that of U^{238}, and that many resonances would be observed below 200 ev.

The samples obtained from Oak Ridge National Laboratory in the form of oxides were as follows: a 186-mg U^{234} sample with 95.55% U^{234}, 3.02% U^{235} and 0.98% U^{238}, and a 304-mg U^{236} sample with 95.36% U^{236}, 4.14% U^{235}, 0.43% U^{238} and 0.07% U^{234}. The Brookhaven fast chopper[6,19] with the bank of 128 BF_3 counters and the 100-channel analyzer was used for all the measurements. In order to get somewhat thicker samples, only one slit was used and the other was blocked off. The advantage of the fast chopper with regard to its ability to use small samples is well illustrated by this work on U^{234} and U^{236}.

Transmission measurements were made in the energy range from a few electron volts to a few kev. At the maximum operating rotor speed of 10 000 rpm the chopper produces a one-μsec burst. Since the collection time in the BF_3 counters is about 0.5 μsec, this results in a resolution of 0.06 μsec/m for high-energy neutrons when one uses a 20-meter flight path and 0.5-μsec channels. At somewhat lower energies the flight time in the detector, 0.7 μsec at 100 ev and 1.4 μsec at 25 ev, produces resolutions of 0.07 and 0.095 μsec/m at these two energies, corresponding to energy spreads of 1.9 ev and 0.33 ev. Since the correction for the small resonances that may be missed depends on the energy resolution it is necessary to know the resolution quite accurately. The natural widths of the resonances in U^{234} and U^{236} vary from 0.03 ev at low energies to about 0.1 ev at the higher energies, and the true shapes of the resonances are not seen with the present resolution. Also, the Doppler widths of the resonances are several times the natural widths, Γ.

Figures 1 and 2 show the transmission curves of the U^{234} sample in the energy range from about 60 to 200 ev with a sample thickness of 2.34×10^{21} atoms/cm^2 of U^{234}. Each curve represents about 40 hours of operating time with the sample, in which about 2500 counts were accumulated per μsec channel. The areas of the transmission "dips" are estimated to be accurate to about 10% for a large dip such as the one at 95 ev and about 50% for a small dip such as the one at 89 ev. Between the prominent resonances it is estimated that a small resonance producing at least a 2% dip would have been observed. At lower energies several small transmission dips occurred at energies corresponding to the large resonances in U^{235} and U^{238} and as their strengths agreed with those expected from the contributions from these from these other isotopes, they were assigned to these isotopes and discarded.

Since the energy resolution was greater than the natural width of the resonances, only an area analysis was possible. For thin samples, the area of a transmission dip gives $\sigma_0\Gamma$ directly and hence Γ_n. The method of analysis, using a set of standard curves, has been discussed by Hughes.[20] For each resonance the area was measured over an energy interval such that the wing correction was in general less than 10%. The data for U^{234} are summarized in Table I. Only for the 5.2-ev resonance was more than one sample thickness run. When one uses the thick-thin method of analysis, this level results in a value for Γ_γ of $(22\pm9)\times10^{-3}$ ev. The computations of Γ_n in Table I are based on an assumed Γ_γ of 25×10^{-3} ev; however, since the samples are thin the results are very insensitive to the assumed Γ_γ. The errors on the neutron widths are determined entirely by the errors of the measured areas.

[17] Havens, Rainwater, and E. Melkonian (unpublished work, 1949).
[18] Pawlicki, Smith, and Thurlow (unpublished work, 1952).
[19] Seidl, Hughes, Palevsky, Levin, Kato, and Sjöstrand, Phys. Rev. **95**, 476 (1954).

[20] D. J. Hughes, J. Nuclear Energy **1**, 237 (1955).

TABLE I. Neutron widths and energies of the resonances in U^{234} up to 400 ev. The computations of Γ_n are based on a Γ_γ of 25×10^{-3} ev. Δ is the Doppler width and the reduced neutron width, Γ_n^0, equals $\Gamma_n/[E_0(\text{in ev})]^{\frac{1}{2}}$.

E_0 (ev)	Δ (ev)	n (atoms/cm²)	Area (% error) (ev)	Γ_n (10^{-3} ev)	Reduced width Γ_n^0 (10^{-3} ev)
5.20	0.048	2.02×10^{19}	0.061 (15)		
		6.2×10^{20}	0.37 (10)	4.4 ± 0.3	1.93 ± 0.15
		2.34×10^{21}	0.71 (5)		
31.4	0.12	2.34×10^{21}	0.54 (10)	7.7 ± 2.0	1.4 ± 0.4
46.4	0.14	2.34×10^{21}	0.014 (50)	0.07 ± 0.04	0.010 ± 0.005
49.4	0.15	2.34×10^{21}	0.57 (12)	11 ± 4	1.6 ± 0.5
78.3	0.19	2.34×10^{21}	0.43 (12)	6.4 ± 1.5	0.72 ± 0.17
88.7	0.20	2.34×10^{21}	0.09 (50)	0.9 ± 0.5	0.10 ± 0.05
95.3	0.21	2.34×10^{21}	0.82 (10)	28 ± 7	2.9 ± 0.8
106.9	0.22	2.34×10^{21}	0.24 (25)	3.1 ± 1.1	0.30 ± 0.10
112.1	0.22	2.34×10^{21}	0.56 (10)	13 ± 3	1.2 ± 0.3
132.9	0.24	2.34×10^{21}	0.55 (20)	14 ± 5	1.2 ± 0.5
145.9	0.25	2.34×10^{21}	0.59 (20)	17 ± 7	1.4 ± 0.5
154.0	0.26	2.34×10^{21}	0.63 (15)	19 ± 6	1.5 ± 0.5
179.0	0.28	2.34×10^{21}	1.18 (15)	70 ± 30	5.2 ± 1.9
184.0	0.29	2.34×10^{21}	0.61 (35)	20 ± 12	1.5 ± 0.9
191	0.29	2.34×10^{21}	1.47 (15)	110 ± 40	8 ± 3
274	0.35	2.34×10^{21}	0.62 (40)	26 ± 17	1.6 ± 1.0
295	0.36	2.34×10^{21}	1.18 (30)	80 ± 50	5 ± 3
319	0.38	2.34×10^{21}	1.37 (25)	110 ± 60	6 ± 3
357	0.40	2.34×10^{21}	0.60 (60)	30 ± 20	1.6 ± 1.1
369	0.40	2.34×10^{21}	2.0 (40)	220 ± 150	12 ± 8

In the energy range up to 155 ev, 12 resonances are listed. Above this energy it is evident from a plot of the number of levels against neutron energy that some resonances are missed. When one applies a 15% correction for all the small resonances missed in the energy range up to 155 ev (discussed later), the average level spacing, D, is 12 ± 3 ev. The average reduced neutron width, $\bar{\Gamma}_n^0$, becomes $(1.1\pm0.3)\times10^{-3}$ ev. The strength function $(\bar{\Gamma}_n^0/D)$, determined from a plot of $\sum\Gamma_n^0$ *vs* E and including the higher energy resonances, is $(1.2\pm0.3)\times10^{-4}$. The reduced neutron width distribution is consistent with either a Porter-Thomas or an exponential distribution.

The transmission measurements of U^{236} were very similar to those of U^{234}. Since about 50% more material was available, the sample measured was somewhat thicker. The three low-energy resonances in U^{238} were observed, as well as several resonances in U^{235}.

The data on the resonances in U^{236} are given in Table II. No area is listed for the 133-ev level since this small transmission dip was much broader than that expected for a single resonance and it is possible that it is not a single resonance. In the energy range up to 140 ev, 9 resonances are listed. Above this energy, resonances are definitely missed. Upon applying a 14% correction for small resonances missed up to 140 ev, D becomes 14 ± 4 ev and $\bar{\Gamma}_n^0(1.9\pm0.6)\times10^{-3}$ ev. The $\bar{\Gamma}_n^0/D$ ratio, including higher energy resonances is $(1.3\pm0.4)\times10^{-4}$. The reduced neutron widths, as for U^{234}, are in agreement with either a Porter-Thomas or an exponential distribution.

TABLE II. Neutron widths and energies of resonances in U^{236} up to 400 ev. The computations of Γ_n are based on a Γ_γ of 25×10^{-3} ev.

E_0 (ev)	Δ (ev)	n (atoms/cm²)	Area (% error) (ev)	Γ_n (10^{-3} ev)	Reduced width Γ_n^0 (10^{-3} ev)
5.49	0.049	4.04×10^{21}	0.56 (5)	1.76 ± 0.21	0.75 ± 0.12
30.2	0.12	4.04×10^{21}	0.219 (10)	0.61 ± 0.11	0.111 ± 0.019
34.6	0.12	4.04×10^{21}	0.43 (20)	2.6 ± 1.2	0.44 ± 0.20
44.5	0.14	4.04×10^{21}	0.92 (10)	19 ± 5	2.8 ± 0.8
72.3	0.18	4.04×10^{21}	1.24 (10)	40 ± 10	4.7 ± 1.1
87.4	0.20	4.04×10^{21}	1.27 (10)	44 ± 11	4.7 ± 1.1
121.0	0.23	4.04×10^{21}	1.31 (15)	53 ± 19	4.8 ± 1.7
126.0	0.24	4.04×10^{21}	0.56 (20)	8 ± 4	0.7 ± 0.3
133	0.24	4.04×10^{21}			
198	0.30	4.04×10^{21}	1.67 (10)	94 ± 25	6.7 ± 1.7
216	0.31	4.04×10^{21}	1.45 (15)	80 ± 30	5.4 ± 2.0
280	0.35	4.04×10^{21}	1.6 (25)	100 ± 50	6 ± 3
308	0.37	4.04×10^{21}	1.9 (25)	130 ± 70	7 ± 4
384	0.41	4.04×10^{21}	2.3 (25)	190 ± 100	10 ± 5

The level spacings of U^{234} and U^{236} are somewhat smaller than that of U^{238}, 16 ± 3 ev. The order is consistent with the fact that the binding energy of U^{234} is about 200 kev higher than that of U^{236}, which in turn is about 200 kev higher than that of U^{238}.

III. LEVEL SPACING DISTRIBUTION FOR ZERO-SPIN TARGET NUCLEI

When one considers the distribution of level spacings of U^{234} and U^{236} based on the data in Tables I and II, it is apparent that there is a definite deficiency of small spacings. This is true even for the energy region below 160 ev in U^{234}, in which one feels that most of the resonances are being observed. This "cutoff" energy is determined by means of a plot of the number of resonances *vs* energy (see Fig. 3 of reference 6, for example). The same deficiency of small spacings is true for the distributions of spacings of U^{238} and Th^{232} determined from parameters listed in BNL-325.[1]

However, even in an energy range where the resolution width is much less than the average level spacing, there is still a sizable probability for missing small resonances, and careful corrections must be made to obtain the correct level spacing distribution. The loss of levels is made more serious because the sizes of the resonances (determined by their neutron widths) have a broad distribution, with the very small ones the most probable. Since the probability of missing a small resonance near another resonance is high, the correction for the small spacings in the distribution is very important.

In making the correction for the failure to observe levels, it is important to determine if there is any correlation between an individual value of Γ_n and the spacing to the adjacent levels. Thus, if levels of small neutron widths were correlated with small spacings, the loss of small levels near large ones would be quite different than if there were no correlation between Γ_n and S. A strict interpretation of the treatment given by Blatt and Weisskopf[21] (particularly Fig. 8.3) would lead one to believe that small spacings might be correlated with small Γ_n^0's of adjacent levels. In order to investigate the possible correlation between the size of a level, given essentially by Γ_n^0, and the local spacings, a comparison of the individual values of Γ_n^0 and the spacings to the nearest levels was carried out. A definite ratio of the *average* reduced neutron width

[21] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).

to the *average* spacing, D, is predicted by the theory of the cloudy crystal ball but this theory deals only with average properties of levels, not the individual values.

The most direct way to investigate the correlation between the neutron widths Γ_n and the spacings to the nearest levels is illustrated in Fig. 3 for U^{234} for the resonances up to 154 ev. In this figure the reduced neutron widths, $\Gamma_n{}^0$ (the value of Γ_n at 1 ev) for individual levels are plotted against the average distance to the levels on either side of the given level. The solid line in Fig. 3 has a slope determined from the average reduced neutron widths and the average of the local spacings. If the neutron width were strictly proportional to the spacing to the nearest levels, each point would be expected to fall on the solid line. However, the scatter of the observed points show that there is no strong correlation between neutron width and the local spacing.

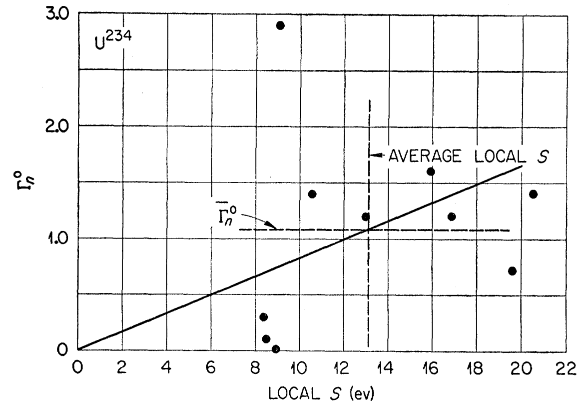

FIG. 3. A plot for U^{234} of the reduced neutron width of a level *vs* the average of the distance to the levels on either side of the given level. The straight line has a slope given by the average reduced neutron width divided by the average local spacing.

In order to study the possibility of a correlation in a somewhat different manner, the results for the four even-even nuclei investigated are shown in Fig. 4, in which the number of levels is plotted as a function of the ratio of ($\Gamma_n{}^0$/local S) to (average $\Gamma_n{}^0$/average local S). Here all points would fall at unity if there were a strict correlation between each $\Gamma_n{}^0$ and the local spacing. However, a very broad distribution is observed, which is approximately that expected for the case of no correlation between $\Gamma_n{}^0$ and the local spacing. For the case of no correlation the expected distribution is, of course, given by the individual distribution laws for the neutron widths and for the spacings, these two distributions being combined as independent. The curve in Fig. 4 was computed by assuming a Thomas-Porter distribution of neutron widths and the distribution of spacings discussed later, namely, $(4S/D^2)e^{-2S/D}$. The agreement between the experimental results and the calculation shows that there is very little correlation between neutron widths and spacings to the adjacent levels and, hence, that these distributions can be treated as independent in computing the corrections necessary to determine the distribution law of level spacings.

Let us consider the transmission curves of the sample of U^{234} shown in Figs. 1 and 2. In the regions between resonances we shall assume that any resonance that gives less than a 2% transmission dip would not have been detected and any resonance larger than this would have been observed, this limit corresponding to about twice the statistical standard deviation. In the energy region around 85 ev, this limit corresponds to a minimum observable area of 0.03 ev. From the U^{234} sample thickness we can readily compute that this corresponds to a reduced neutron width of 0.03×10^{-3} ev. Since the average reduced neutron width of U^{234} is 1.1×10^{-3} ev, the resonances that have been missed in this energy region must have reduced neutron widths that are less than 3% of the average reduced neutron width. If thicker samples had been measured, the loss of small resonances in the regions between observed resonances would have been decreased. For example, if a sample three times thicker had been measured, resonances with reduced neutron widths $>1\%$ of the average reduced neutron width would have been observed. At higher energies where the energy resolution gets poorer, the loss of small resonances increases. For example, at 160 ev the resolution is 3.6 ev and, hence, any area less than 0.07 ev would not have been observed. This area corresponds to a reduced neutron width of 0.11×10^{-3} ev or 10% of the average reduced neutron width. At low energies very small resonances can be detected between the larger observed resonances. For example, at 8 ev resonances with reduced neutron widths of only 0.1% of the average value would have been observed with the particular U^{234} sample used.

Near the large observed resonances, of course, it is possible to miss much larger resonances than for the energy regions between resonances. If a resonance occurs within a half a resolution width of a large

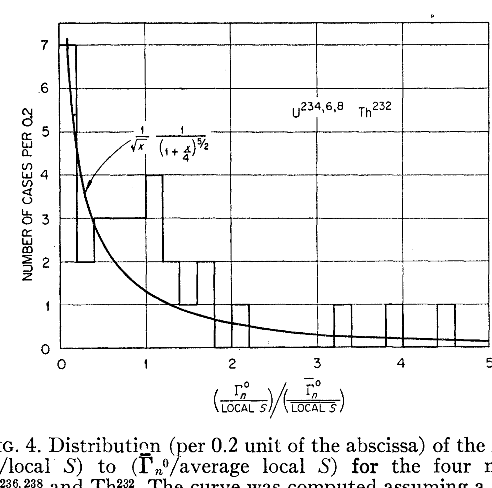

FIG. 4. Distribution (per 0.2 unit of the abscissa) of the ratio of ($\Gamma_n{}^0$/local S) to ($\bar{\Gamma}_n{}^0$/average local S) for the four nuclides $U^{234,236,238}$ and Th^{232}. The curve was computed assuming a Porter-Thomas distribution of neutron widths and a level spacing distribution of the form $(4/D^2)Se^{-2S/D}$. A normalizing factor of 2.33 has been omitted from the equation on the curve.

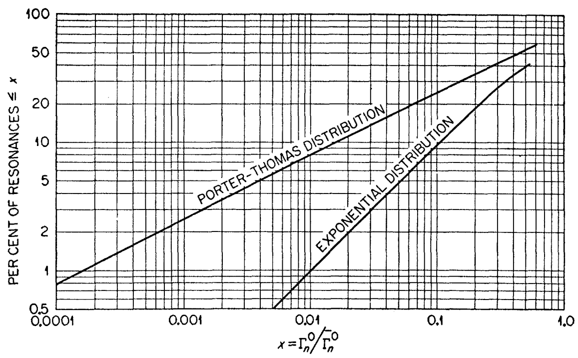

FIG. 5. The percentage of resonances smaller than a particular size for the Porter-Thomas and the exponential reduced neutron width distributions as a function of the size of the resonance.

observed resonance it will be missed if its dip area is less than half the area of the observed transmission dip. For example, in the region of the large resonance at 95 ev, any resonance from 94 to 96 ev which is missed must have had a reduced neutron width less than 45% of the average value. On the wings of a resonance from one half to one resolution width away we have assumed that any resonance with 4 times the area of the limit that we computed between resonances would have been observed. Thus, any resonance which might be in the energy range from 96 to 97 ev would be missed unless its reduced neutron width were greater than 12% of the average. If we are close to a big transmission dip, such as the one at 95 ev, little advantage is obtained in using a thicker sample, even though the dip due to the small resonance increases with sample thickness since the large resonance dip increases in width.

In order to determine the fraction of all the resonances that are missed corresponding to the size limits just mentioned, it is necessary to know the distribution of the reduced neutron widths, which determine the size of levels. For example, if the distribution is exponential,

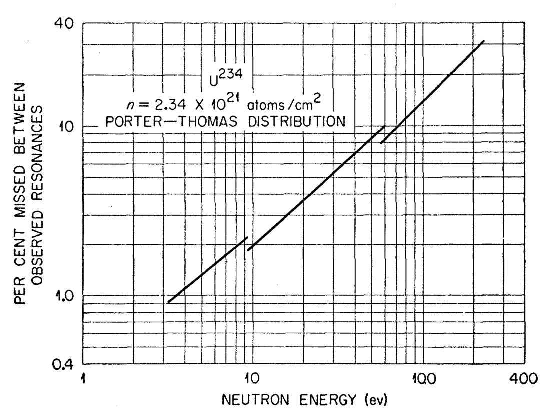

FIG. 6. The percentage of resonances which are missed between observed resonances for U^{234} as a function of neutron energy assuming a Porter-Thomas distribution of reduced neutron widths. In the vicinity of large resonances the loss is much greater.

only 1% of the resonances have neutron widths less than 1% of the average and only 9.5% have $\Gamma_n^0 < 0.1\bar{\Gamma}_n^0$. However, for a Porter-Thomas distribution[8] 8% of the resonances have widths less than $0.01\bar{\Gamma}_n^0$ and 24.5% have widths less than $0.1\bar{\Gamma}_n^0$. The percentages of resonances smaller than a particular value are plotted in Fig. 5 for both the exponential and Porter-Thomas distributions. In computing the correction for missed levels, we shall use the Porter-Thomas distribution since it fits the experimental neutron width distribution better. From the estimate already made of the upper limit of levels missed as a function of energy for U^{234}, together with Fig. 5, we easily obtain (Fig. 6) the fractional loss of resonances *between* the observed resonances *vs* energy in the case of U^{234}. The data below 9 ev were taken at a rotor speed of 3000 rpm, and those from 9 to 60 ev at 6000 rpm, which account for the discontinuities at 9 and 60 ev. In the vicinity of large resonances the loss of small resonances is much greater, as has been explained. From the percentage of resonances which are missed both between and near observed resonances, it is possible to correct the observed level spacing distribution for the loss of the small resonances. It is obvious that greatest loss of resonances occurs near the large observed resonances, hence, giving a large correction to the number of small spacings and not affecting the large spacings very much. Thus, the correction to the small spacings is very important.

The correction for the loss of small resonances has been made for three spacing distributions (1) an exponential distribution (2) a distribution of the form $4(S/D^2)e^{-2S/D}$, and (3) a form $(\pi/2D^2)S \times\exp[-(\pi/4D^2)S^2]$. The distribution (1) means that the levels occur at random, (2) is an empirical form that corresponds to a deficiency of small spacings, and (3) is the Wigner distribution already mentioned. Since we have already computed the fraction of resonances that are lost as a function of energy, both between and near resonances, the number missed at any particular energy follows directly from the fraction missed and the assumed distribution of level occurrence [constant for (1), and proportional to S for (3)]. The effect of adding missed levels in this way is that the number of large spacings derived from the data in Table I must be decreased and the number of small spacings increased.

The entire energy range up to 154 ev for the case of U^{234} was analyzed in this manner and the percentage loss of levels *vs* energy was computed. Since the thickness of the U^{236} sample was just a little greater than that for U^{234}, no detailed analysis of its transmission curve was made. It was estimated that the loss of resonances between observed resonances was about 30% less than that of U^{234} as a result of the increased sample thickness, but that the correction for small levels near large ones was the same as for U^{234}. The Th^{232} and U^{238} data were obtained with samples much thicker (a factor of 15) than the U^{234} samples, hence, a detailed analysis was carried out for these nuclides, the same correction being used for both.

No data were used at energies higher than the point at which the loss of resonances had reached 15% (based on the exponential spacing distribution). In U^{234}, 11 spacings were used up to 154 ev; in U^{236}, 8 spacings up to 133 ev; in U^{238}, 10 spacings up to 192 ev; and in Th^{232}, 6 spacings up to 131 ev. The data for U^{238} were taken from BNL-325 Supplement I and that for Th^{232} were BNL fast chopper data listed in a review article by Harvey and Schwartz.[22] The small resonance at 10.2 ev in U^{238} was not included in the analysis since it is possible that this is a *p*-wave resonance.[23] Recent high-resolution data from Columbia[24] indicate many more resonances in U^{238} above 200 ev but do not show any new resonances below this energy. In order to improve the statistical accuracy of the data, the distributions of the 4 nuclides were combined by plotting them relative to the average level spacing of each nuclide. Since there is some error in the measured *D* for each nuclide, the procedure distorts the combined distribution a slight amount. The results are shown in Figs. 7, 8, 9 with corrections based on the three distributions considered. The correction increases the number of spacings from the observed number of 35 to 40, 38 and 37 for the distributions (1), (2), and (3), respectively.

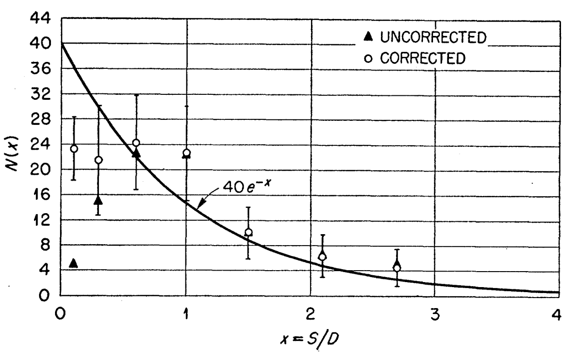

FIG. 7. The level spacing distribution for the four zero-spin target nuclides $U^{234,236,238}$ and Th^{232} as a function of the spacing between adjacent resonances, *S*, divided by the average level facing, *D*. The corrections were determined assuming that the resonances occur at random, which corresponds to an exponential distribution. The corrections increase the number of spacings from the observed 35 to 40 (a 14% correction).

Because of the large statistical uncertainty it is impossible to establish a definite distribution law on the basis of the four zero-spin nuclides alone. However, the deficiency of small spacings seems definite and of the three distributions considered, the third (Wigner) is the best fit. We shall now consider the results for nonzero nuclides for which more data are available.

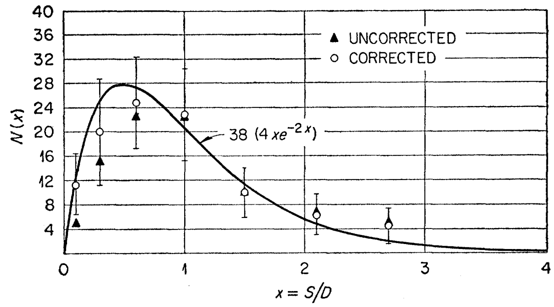

FIG. 8. The level spacing distribution for the four zero-spin target nuclides as a function of S/D. The corrections were computed assuming a distribution of spacings of the form $Se^{-2S/D}$. The corrections increase the number of spacings from 35 to 38.

IV. SPACING DISTRIBUTION FOR NUCLIDES OF NONZERO SPIN

Much more extensive measurements have been made for the nuclides other than those of zero spin discussed in the last section. However, for the nuclides of nonzero spin the complication exists of the presence of two spin systems that tends to obscure the distribution law of spacings within a single spin system. Before discussing the available experimental data on the nuclides of nonzero spin, we shall consider the way in which spacing distributions are altered when two spin systems are combined.

If the levels in each individual spin system occur at random, the combined distribution would also be random and the nuclides of nonzero spin could be treated in exactly the same way as the zero-spin nuclides. However, as we have seen in the previous section, it seems definite that the distribution law is not exponential and, hence, we must compute the combined distribution. It is not a difficult matter to compute the combined distribution when the distributions for a single spin state have simple analytical forms, such as $(4/D^2)Se^{-2S/D}$ and $(\pi/2D^2)S\exp[-(\pi/4D^2)S^2]$ discussed in the previous section.

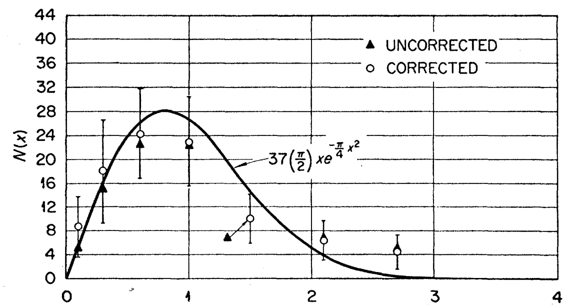

FIG. 9. The level spacing distribution for the four zero-spin target nuclides as a function of S/D. The corrections were computed assuming that the probability of level occurrence is proportional to *S*, which corresponds to the Wigner distribution of the form $S\exp[-(\pi/4D^2)S^2]$. The curve is drawn normalized to 37 spacings.

[22] John A. Harvey and R. B. Schwartz, *Progress of Nuclear Energy. Ser. I. Physics and Mathematics* (Pergamon Press, Ltd., London, 1957), Vol. 2.

[23] Bollinger, Cote, Dahlberg, and Thomas, Phys. Rev. **105**, 661 (1957).

[24] Rosen, Desjardins, Havens, and Rainwater, Bull. Am. Phys. Soc. Ser. II, **2**, 41 (1957).

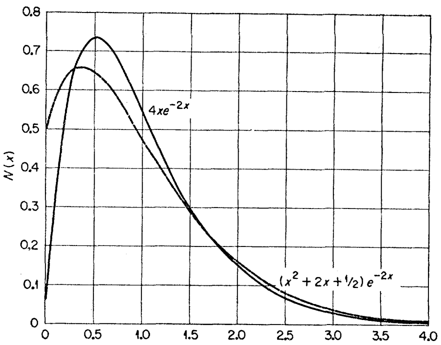

FIG. 10. Combination at random of two distributions of the form $4xe^{-2x}$ to get the combined distribution $(x^2+2x+\frac{1}{2})e^{-2x}$. The combined distribution gives the probability of obtaining any spacing regardless of its spin of the level.

The complete formula that can be used to combine two distributions of arbitrary forms, $P_1(S)$ and $P_2(S)$, with average spacings D_1 and D_2, has the following form[25]:

$$P(S)=\frac{1}{D_1D_2}\left\{P_1(S)\int_0^\infty xP_2(x+S)dx+P_2(S)\int_0^\infty xP_1(x+S)dx+2\int_0^\infty P_1(x+S)dx\int_0^\infty P_2(x+S)dx\right\}. \quad (1)$$

By means of Eq. (1) it is possible to transform the distributions used in the previous section so that they may be compared with nonzero spin nuclides, for which many measured spacings are available. For two distributions of the form $(4/D^2)Se^{-2S/D}$, each with the same average level spacing D per spin state, the combined distribution is

$$e^{-2y}(y^2+2y+\tfrac{1}{2}),$$

where y is the distance between adjacent levels divided by $D/2$, and $D/2$ is the average spacing of resonances of both spin states regardless of the spin state. The single and the combined distributions are shown in Fig. 10 for this case, from which it is seen that the combined distribution is quite similar to the individual distribution for large spacings but is closer to an exponential distribution than the individual distribution for small spacings. For two distributions of the form $(\pi/2D^2)S\exp[-(\pi/4D^2)S^2]$, each with the same average level spacing D per spin state, the combined distribution is

$$\tfrac{1}{2}\exp(-\pi y^2/8)+\frac{\pi y}{8}\exp(-\pi y^2/16)\times\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{4}y\sqrt{\pi}}\exp(-x^2)dx\right),$$

where again y is the distance between adjacent levels.

[25] A. M. Lane, Oak Ridge National Laboratory Report ORNL-2309, 1957 (unpublished), p. 113.

If the individual distributions do not have the same average spacing, the change in the distribution resulting from the combination will be somewhat less, as would be expected. The limited accuracy of the experimental data does not justify computations assuming the level spacing per spin state to be different or selecting distributions more complicated than the two outlined here.

The experimental data considered for the nonzero spin nuclei are tabulated in BNL-325 and its Supplement I. The nuclides included, with the number of spacings per nuclide in parentheses, are as follows: In^{113}(6), In^{115}(7), Sn^{117}(4), Cs^{133}(11), ${}_{56}Ba^{135}$(10), Pr^{141}(5), Nd^{143}(6), Nd^{145}(4), Eu^{151}(13), Eu^{153}(7), Tb^{159}(15), Ho^{165}(14), Tm^{169}(9), Lu^{175}(11), Hf^{179}(14), Ta^{181}(9). The total number of intervals used was 145. All of these nuclides were measured with the Brookhaven chopper under conditions and with sample thickness equivalent to the U^{234} data. Most of the nuclides have average spacings similar to that of U^{234}, although some (Eu^{151} and Eu^{153}) are an order of magnitude less and others (Sn^{117} and Pr^{141}) have spacings much larger. It was decided that the correction derived for the U^{234} would be satisfactory to apply to these nonzero-spin nuclides and no elaborate examination of all the transmission curves was made. The fissionable isotopes like U^{235} were not included in the analysis since the spacings of U^{235} are complicated by the presence of interference between resonances.

As for the zero-spin nuclides, the corrections for missed levels were made for the three distributions in the combined form for two spin states, assumed equal in number. For the exponential distribution the correction is just the same as for the even nuclides and the results are shown in Fig. 11. As with the zero-spin nuclei, the corrected points are in reasonable agreement with an exponential distribution except for the smallest spacing, and those for which y is about unity. In Fig. 12 are shown the experimental spacing distributions and the corrections based on the other two distributions already considered. Here the agreement is much better, with either distribution, than with the random distribu-

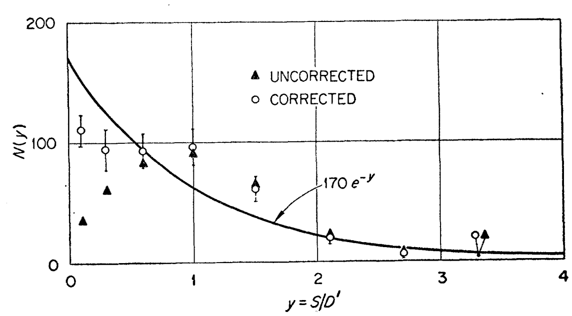

FIG. 11. The level spacing distribution for the nonzero-spin nuclides as a function of S/D'. D' is the average spacing of levels of both spin states regardless of their spin and is taken to be equal to $D/2$, where D is the level spacing per spin state. The corrections were computed by assuming an exponential distribution of spacings, which increase the number of spacings from 145 to 170.

tion of Fig. 11. Distribution 3 (Wigner) seems to agree better than 2 but the difference is not great. However, the conclusion of the zero-spin results, that repulsion of levels exists, is supported by the odd-spin nuclides, and with greater statistical accuracy. Actually, the results indicate somewhat more repulsion than the assumed distributions, as would be expected if the two spin systems are not equally populated.

V. NEUTRON WIDTH DISTRIBUTION

The considerations given in Sec. III with regard to the size of a resonance which would have been missed as a function of neutron energy can be used to correct the observed neutron width distributions for the 4 nuclides $U^{234,236,238}$ and Th^{232}. From the size of reduced neutron width which would not have been observed *vs* energy for each nuclide we can determine the efficiency for detecting a particular size resonance averaged over the energy range up to the "cutoff" energy. The efficiency for detecting very small resonances is obviously low while that for detecting large resonances is almost unity. The results when the 4 even-even nuclides are combined is as follows. The efficiency is ~ 0 for detecting a resonance whose Γ_n^0 is only $0.0001\bar{\Gamma}_n^0$, $\sim 25\%$ for $\Gamma_n^0 = 0.001\bar{\Gamma}_n^0$, $\sim 53\%$ for $\Gamma_n^0 = 0.01\bar{\Gamma}_n^0$, $\sim 80\%$ for $\Gamma_n^0 = 0.1\bar{\Gamma}_n^0$ and $\sim 100\%$ for $\Gamma_n^0 = \bar{\Gamma}_n^0$. When this efficiency factor is applied to the experimental data it makes an important correction to the number of small neutron widths. For example, of the 39 neutron widths considered for these 4 nuclides, three were observed with $\Gamma_n^0/\bar{\Gamma}_n^0$ between 0 and 0.04. Because of this efficiency factor and the neutron width distribution, a correction of a factor of 2 must be applied in this interval. However, rather than plot the neutron width distribution we have chosen to plot the distribu-

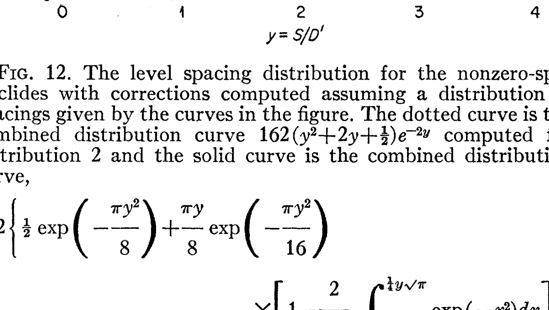

FIG. 12. The level spacing distribution for the nonzero-spin nuclides with corrections computed assuming a distribution of spacings given by the curves in the figure. The dotted curve is the combined distribution curve $162(y^2+2y+\frac{1}{2})e^{-2y}$ computed for distribution 2 and the solid curve is the combined distribution curve,

$$162\left\{\frac{1}{2}\exp\left(-\frac{\pi y^2}{8}\right)+\frac{\pi y}{8}\exp\left(-\frac{\pi y^2}{16}\right)\times\left[1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{4}y\sqrt{\pi}}\exp(-x^2)dx\right]\right\},$$

computed for the Wigner distribution.

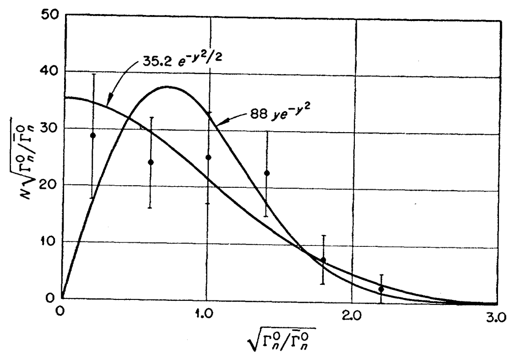

FIG. 13. The distribution of the square roots of the reduced neutron widths for $U^{234,236,238}$ and Th^{232} as a function of $(\Gamma_n^0/\bar{\Gamma}_n^0)^{\frac{1}{2}}$. The points have been corrected for the loss of small resonances. The Gaussian shaped curve is the Porter-Thomas distribution and the curve $88y\exp(-y^2)$ corresponds to an exponential distribution of reduced neutron widths.

tion of the square roots of the neutron widths,[7] i.e., $N((\Gamma_n^0/\bar{\Gamma}_n^0)^{\frac{1}{2}})$, which is a property more closely related to nuclear theory.[8] The results for 39 reduced neutron widths are shown in Fig. 13. No correction has been applied for the distortion produced because of the fact that the average neutron widths are not accurately known for the individual nuclides since it is estimated that this correction is much smaller than the statistical errors on the points. The experimental points are in better agreement with a Porter-Thomas distribution, the Gaussian curve on Fig. 13, than an exponential distribution of reduced widths, corresponding to the curve $88y\exp(-y^2)$.

In an analysis of 145 neutron widths of resonances from both zero and nonzero spin target nuclei, Hughes and Harvey[7] estimated that the corrections for small resonance which were missed would be less than the statistical accuracy on the experimental points and concluded that the data were in agreement with either a Porter-Thomas or an exponential distribution of neutron widths. However, as a result of the careful examination of the transmission data outlined in this paper we see that the corrections for small size resonances are very important. For example, the first point in Fig. 2 of reference 7 should be raised a factor of 2. After this correction has been applied, the data are in better agreement with the Porter-Thomas distribution than an exponential distribution of reduced neutron widths. This is in agreement with the results of a detailed treatment by Porter and Thomas[8] who concluded that their distribution is consistent with the data, while an exponential distribution is not.

ACKNOWLEDGMENTS

The authors would like to express their thanks to E. P. Wigner and L. Dresner for suggestions and assistance on several theoretical points, to V. E. Pilcher for assistance in obtaining and analyzing the data to determine resonance parameters, and to Miss J. S. Murphy for detailed computations in making the corrections for missed resonances.