Some Properties of the Shift and Penetration Factors in Nuclear Reactions*

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The shift and penetration factors are shown to have certain monotonic properties in the variables ρ, η, and l for both positive and negative energies. The behavior of the shift factor in the neighborhood of zero energy (threshold) is discussed.

I. INTRODUCTION AND SUMMARY

THE Wigner theory of nuclear reactions[1] has the central feature that it separates the various nuclear reaction parameters into two types: internal quantities, such as the reduced widths which are independent of energy; and energy-dependent external quantities, which are functions of the (spherically symmetric) interaction of separated pairs of particles. The Wigner theory is necessarily complicated, and in the region of widely separated levels a drastic simplification, the "one-level approximation,"[2] is employed. In this approximation, the external parameters required for the width and energy shift are the penetration factor, P, and the shift factor, S, defined in Eqs. (8) and (4). These factors also occur in the general case, but enter in a more complicated manner.

It is the purpose of the present note to detail a few general properties of the external quantities S and P, for the particular case that only the Coulomb field (for definiteness taken to be repulsive), and the centrifugal barrier, are operative in the external region. Many of these results are obvious on physical grounds, such as, for example, the fact that P is a positive, monotonically increasing function of the energy; others relating to the shift function are more involved. Besides their usefulness for facilitating a numerical treatment of the functions P and S, such results are of value in furnishing an intuitive feeling for the over-all behavior of these functions.

The point of departure for the present work is an unpublished result of R. G. Thomas [Eq. (9)] giving a very convenient form for an investigation such as follows. For the general case, we show that P monotonically decreases with η and l; monotonically increases with ρ, and correspondingly monotonically increases with energy. For the shift function, S, we demonstrate that the shift monotonically increases with η. For increasing ρ, we show that S/ρ decreases monotonically; S itself, however, apparently does not behave so simply. (See also the remarks in Sec. IV.) In consequence, we can show, at best, only that S/ρ is a monotonically decreasing function of energy. The particular case of $E=0$ is given in detail.

For the case where only the centrifugal barrier occurs, the general formula is shown to reduce to a remarkably simple form for both P and S. It is possible to show for this case that both P and S are monotonic with changing energy.

The one-level approximation requires, however, that the external functions also be considered for *negative* energy, corresponding to closed channels. The required generalization of P, S to this region has been given by Thomas.[3] For negative energy, S is shown to be monotonic in η, ρ, and l; but not in energy.

A particularly interesting situation occurs at threshold, that is, the behavior of S and P near zero energy. A typical example physically, which led in part to this investigation, is the case of the reaction ${}_1H^3+{}_1H^2\to He^{5*}$ and the inverse reaction ${}_0n^1+{}_2He^4\to He^{5*}$, currently under investigation at this laboratory.[4] For these reactions the shift factors are required in the neighborhood of the (d,t) threshold; the linear approximation[3] will clearly not suffice. Application of our results to this, and other reactions, will be reported elsewhere.

It is well known[5] that the behavior of P near zero energy suffices to illustrate the general rules of Wigner[6] for those cases where the threshold occurs in an entrance or exit channel. Our investigation below yields a similar conclusion for S. The shift factor near threshold is shown to be continuous in energy for $\eta>0$, and continuous through l energy derivatives for $\eta=0$. The case $\eta=0$, $l=0$ has therefore a discontinuous slope; specifically $S_{0,0}\propto|E|^{\frac{1}{2}}$, $E<0$; $S_{0,0}=0$, $E\geq 0$. The one-level approximation thus shows for $\eta=0$, $l=0$ a singular behavior which is just midway between the two extreme general behaviors given by Wigner[6] ("cusp") and recently by Breit[7] ("tilted S").

II. SUMMARY OF FORMULAS

The one-level approximation leads to a scattering matrix of the Breit-Wigner form:

$$S_{\alpha\alpha'}=\exp[i(\xi_\alpha+\xi_{\alpha'})][\delta_{\alpha\alpha'}+ig_\alpha g_{\alpha'}/(E_\lambda+\Delta_\lambda-E-\tfrac{1}{2}i\Gamma_\lambda)]. \quad (1)$$

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[1] E. P. Wigner, Phys. Rev. **70**, 15 (1946); Proc. Am. Phil. Soc. **90**, 27 (1946); Phys. Rev. **70**, 606 (1946); E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).
[2] We take this to mean that not only is the R matrix approximated by a single pole, but also that $R^{(\infty)}$ is set to zero.
[3] R. G. Thomas, Phys. Rev. **88**, 1109 (1952).
[4] Bonner, Prosser, and Slattery, Bull. Am. Phys. Soc. Ser. II, **2**, 180 (1957).
[5] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), Chap. VIII.
[6] E. P. Wigner, Phys. Rev. **73**, 1002 (1948).
[7] G. Breit, Phys. Rev. **107**, 1612 (1957).

The notation and application of this result is discussed in Blatt and Biedenharn[8] and in many other places too numerous to detail. The relevant features for the present discussion are the total energy shift, Δ_λ, and the total width, Γ_λ.

The total energy shift, Δ_λ, is composed of the partial shifts,

$$\Delta_\lambda=\sum_i \Delta_{\lambda i}, \tag{2}$$

where the sum i extends over all (open or closed) channels. The partial shifts $\Delta_{\lambda i}$ are determined[9] by:

$$\Delta_{\lambda i}=\gamma_{\lambda i}^2(S_{\eta,l}+b_i), \tag{3}$$

where $\gamma_{\lambda i}^2$ is the reduced width, b_i the boundary condition matrix, and

$$S_{\eta,l}\equiv-\rho[F_l(\eta,\rho)F_l'(\eta,\rho)+G_l(\eta,\rho)G_l'(\eta,\rho)]\times[F_l^2(\eta,\rho)+G_l^2(\eta,\rho)]^{-1}, \tag{4}$$

with $F_l(\eta,\rho)$ and $G_l(\eta,\rho)$ denoting the regular and irregular Coulomb wave functions[10] (the prime denotes $d/d\rho$). Equation (4) is taken as the general definition of the shift factor for positive energies (open channels).

For negative energies (closed channels), we have

$$S_{\eta,l}^{(-)}\equiv-\rho W'/W, \tag{5}$$

with $W=W_{-\eta,l+\frac{1}{2}}(2\rho)$ being the Whittaker function.

The total width Γ_λ is similarly composed of the partial widths,

$$\Gamma_\lambda=\sum_i \Gamma_{\lambda i}, \tag{6}$$

where the partial widths $\Gamma_{\lambda i}$ are given by

$$\Gamma_{\lambda i}=2\gamma_{\lambda i}^2 P_{\eta,l}, \tag{7}$$

$$P_{\eta,l}\equiv\rho[F_l^2(\eta,\rho)+G_l^2(\eta,\rho)]^{-1}. \tag{8}$$

This last equation is the general definition of the penetration factor for positive energies. For negative energies, $P_{\eta,l}=0$.

These formulas are often given alternative forms depending upon the choice of reference energy and boundary condition applied to the internal region.[9] The form given above requires, however, that $b_i=-S_i(E_{\mathrm{res}})$. That this is necessary can be seen from the fact, obtained from the results of Sec. IV, that otherwise difficulty would arise from the contribution of distant closed channels.

III. MONOTONIC PROPERTIES FOR POSITIVE ENERGIES

(a) Penetration Factor

Thomas has shown that the quantity $F_l^2(\eta,\rho)+G_l^2(\eta,\rho)\equiv A_{\eta,l}^2$ may be expressed in the form of an integral:

$$A_{\eta,l}^2=2\rho\int_0^\infty dz e^{-2\rho z}Q(z),$$

$$Q(z)=\exp(2\eta\tan^{-1}z)(1+z^2)^l \times {}_2F_1\left(-l-i\eta,\,-l+i\eta,\,1;\frac{z^2}{1+z^2}\right). \tag{9}$$

(a proof of this is included in an appendix for convenience).

It will be seen immediately that the integrand is everywhere positive and real.[11] It follows that the penetration factor is positive; and is an increasing function of ρ, since $\partial A/\partial\rho$ is everywhere negative. In a similar fashion one sees that $\partial A/\partial l$ and $\partial A/\partial\eta$ are both positive, so that P decreases monotonically with both l and η. Since $\partial\eta/\partial E$ is negative, the result that dP/dE is always positive also follows at once.

(b) Shift Function

To draw conclusions for the shift function is not quite so trivial as in the above case for the penetration factor. Consider first the variation with ρ. An alternative form for A is useful here:

$$A_{\eta,l}^2=1+\int_0^\infty dz e^{-2\rho z}Q'(z),\quad Q'(z)=dQ/dz. \tag{10}$$

The function Q' is again everywhere positive and real.[12] Using this, we find:

$$(d/d\rho)(S/\rho)=\tfrac{1}{2}A^{-4}[(A^2)'(A^2)'-A^2(A^2)'']. \tag{11}$$

Using (10), the bracket may be written:

$$[\cdots]=4\left\{\left(\int ze^{-2\rho z}Q'dz\right)^2-\left(\int e^{-2\rho z}Q'dz\right)\times\left(\int e^{-2\rho z}z^2Q'dz\right)\right\}-4(A^2)''. \tag{12}$$

The term in curly brackets is everywhere negative. To see this, we write this term formally as a double integral:

$$\{\cdots\}=\int\int dxdye^{-2\rho(x+y)}Q'(x)Q'(y)y(x-y). \tag{13}$$

This double integral, however, is symmetrical in x and

[8] J. M. Blatt and L. C. Biedenharn, Revs. Modern Phys. **24**, 267 (1952).

[9] R. G. Thomas, Phys. Rev. **81**, 148 (1951); **88**, 1109 (1952); **97**, 224 (1955).

[10] *Tables of Coulomb Wave Functions*, National Bureau of Standards Applied Mathematics Series 17 (U. S. Government Printing Office, Washington, 1952), Vol. 1.

[11] The series for the hypergeometric function converges absolutely over the range of integration, so that it is permissible to draw such conclusions from a termwise examination of the series.

[12] It is an immediate result that $(A^2)'=(d/d\rho)(A^2)$ is negative everywhere and behaves oppositely to A^2.

y. Upon making this explicit, one finds

$$\{\cdots\}=-\tfrac{1}{2}\int\int dxdye^{-2\rho(x+y)}Q'(x)Q'(y)(x-y)^2, \quad (14)$$

which is now obviously everywhere negative.

As $(A^2)''\geqslant 0$, the result now follows that

$$(d/d\rho)(S/\rho)\leqslant 0. \quad (15)$$

The behavior of S with respect to changes in η is more involved to examine. Use the form in Eq. (9) and symmetrize the results. One finds

$$\begin{aligned}\frac{\partial S}{\partial\eta}&=A^{-4}\int_0^\infty\int_0 dxdye^{-2\rho(x+y)}Q(x)Q(y)M(x,y),\\ M(x,y)&=(x-y)[(d/d\eta)\ln(Q(x)/Q(y))].\end{aligned} \quad (16)$$

The burden of the proof is that the sign of the term in brackets varies as $\mathrm{sgn}(x-y)$.

To see this, we note that the η derivative of $\ln Q$ contains two terms,

$$\frac{\partial(\ln Q)}{\partial\eta}=2\tan^{-1}x+\frac{\partial}{\partial\eta}(\ln {}_2F_1), \quad (17)$$

and clearly $\mathrm{sgn}(\tan^{-1}x-\tan^{-1}y)=\mathrm{sgn}(x-y)$, so that only the ${}_2F_1$ is of concern. The relevant quantity [abbreviating ${}_2F_1(\cdots)$ by F], is:

$$\frac{\partial}{\partial\eta}\ln\left(\frac{F(x)}{F(y)}\right)=[F(x)F(y)]^{-1}\left[F(y)\frac{\partial F(x)}{\partial\eta}-F(x)\frac{\partial F(y)}{\partial\eta}\right]. \quad (18)$$

Introduce now the series definition:

$$\begin{aligned}F(x)&=\sum_{n=0}c_n\left(\frac{x^2}{1+x^2}\right)^n,\\ c_n&=(n!)^{-2}(-l+i\eta)_n(-l-i\eta)_n\geqslant 0,\\ F(y)\frac{\partial F(x)}{\partial\eta}-F(x)\frac{\partial F(y)}{\partial\eta}&=\sum_{m,n}\left(\frac{x^2}{1+x^2}\right)^n\left(\frac{y^2}{1+y^2}\right)^m c_nc_m\frac{\partial}{\partial\eta}\ln\left(\frac{c_n}{c_m}\right).\end{aligned} \quad (19)$$

Noting now that

$$\begin{aligned}&\mathrm{sgn}\frac{\partial}{\partial\eta}\ln(c_n/c_m)=\mathrm{sgn}(n-m),\\ &\mathrm{sgn}\left[\left(\frac{x^2}{1+x^2}\right)^n\left(\frac{y^2}{1+y^2}\right)^m-\left(\frac{x^2}{1+x^2}\right)^m\left(\frac{y^2}{1+y^2}\right)^n\right]\\ &\qquad=\mathrm{sgn}(x-y)\,\mathrm{sgn}(n-m),\end{aligned} \quad (20)$$

we find that

$$\mathrm{sgn}\frac{\partial}{\partial\eta}\ln\left(\frac{F(x)}{F(y)}\right)=\mathrm{sgn}(x-y), \quad (21)$$

and therefore $\partial S/\partial\eta\geqslant 0$.

Combining the η and ρ dependence yields the result that

$$(d/dE)(S/\rho)\leqslant 0. \quad (22)$$

The variation of S with l is apparently more complicated, for similar results have not been obtained for $\eta\neq 0$.

(c) Limit $E=0$

The general form Eq. (9) is quite unsuited to investigate the limit $E\rightarrow 0_+$. Since, however, $A_{\eta,l}{}^2\rightarrow G_l{}^2$ and $(A_{\eta,l}{}^2)'\rightarrow(G_l{}^2)'$, the limiting form for S is readily obtained from the results[13]:

$$\begin{aligned}G_l(\eta,\rho)&\rightarrow 2(2\lambda)^{l+\frac{1}{2}}\rho^{-l}[(2l+1)!C_l(\eta)]^{-1}\\ &\qquad\times K_{2l+1}([8\lambda]^{\frac{1}{2}}),\\ G_l'(\eta,\rho)&\rightarrow -2(2\lambda)^{l+\frac{1}{2}}\rho^{-l-1}[(2l+1)!C_l(\eta)]^{-1}\\ &\qquad\times\{lK_{2l+1}([8\lambda]^{\frac{1}{2}})+(2\lambda)^{\frac{1}{2}}K_{2l}([8\lambda]^{\frac{1}{2}})\},\end{aligned} \quad (23a)$$

where

$$\begin{aligned}(2l+1)!C_l(\eta)&=2^le^{-\frac{1}{2}\pi\eta}|\Gamma(l+1+i\eta)|,\\ \lambda&=\rho\eta=\hbar^{-2}Z_1Z_2e^2Ma,\end{aligned} \quad (23b)$$

and K denotes the modified Hankel function. As a result, one finds:

$$S_{\eta,l}\rightarrow S_{\eta,l}{}^{(0)}=l+(2\lambda)^{\frac{1}{2}}K_{2l}([8\lambda]^{\frac{1}{2}})/K_{2l+1}([8\lambda]^{\frac{1}{2}}). \quad (24)$$

The behavior of this limiting form of the shift function is indicated by:

(a) $$S_{\eta,l}{}^{(0)}\rightarrow\begin{cases}l+\lambda^2/l, & l\neq 0\\ l-2\lambda^2\ln 2\lambda, & l=0,\end{cases} \quad (25)$$
as $\lambda\rightarrow 0$

(b) $$S_{\eta,l}{}^{(0)}\sim(2\lambda)^{\frac{1}{2}} \quad (26)$$
as $\lambda\rightarrow\infty$.

It is easily shown that $S_{\eta,l}{}^{(0)}$ is a monotonically increasing function of λ.

From the general results above that S/ρ decreases for increasing energy, it is clear that zero energy should yield a maximum for this quantity. It is not surprising then for S to be also large at zero energy. One notes that, in general, S *cannot* vary monotonically with energy; for, if this were true in general it would imply (using previous results) that behavior with increasing a and Z should be everywhere *opposite*—while the above zero-energy limit shows that here they behave *similarly*.

(d) Special Case of $Z=0$

The general formula of Eq. (9) allows an interesting result to be derived for the neutron case. Using Kummer's transformation,

$$Q(z)={}_2F_1(-l,l+1,1;-z^2),\quad \eta=0. \quad (27)$$

This function is now a polynomial in z^2, with $l+1$ positive terms. Integrating termwise, we find

$$A_{0,l}{}^2={}_3F_0(-l,l+1,\tfrac{1}{2};-\rho^{-2}), \quad (28)$$

where the ${}_3F_0$ is defined by the terminating series,

$$\begin{aligned}{}_3F_0(\cdots)&=\sum_{m=0}^{l}(-l)_m(l+1)_m(\tfrac{1}{2})_m(m!)^{-1}(-)^m\rho^{-2m},\\ (a)_m&\equiv\Gamma(a+m)/\Gamma(a).\end{aligned} \quad (29)$$

[13] Erdelyi, Magnus, Oberhettinger, and Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, Chap. VI. The result is originally due to Breit and co-workers: Yost, Wheeler, and Breit, Phys. Rev. **49**, 174 (1936). See also J. G. Beckerley, Phys. Rev. **67**, 11 (1945).

This series for $A_{0,l}^2$ gives a simple prescription for the penetration factor, and is ideally suited for a machine calculation.

The shift function now takes the form:

$$S_{0,l}=\tfrac{1}{2}l(l+1)\rho^{-2}\,{}_3F_0(1-l,\,l+2,\,\tfrac{3}{2};\,-\rho^{-2})/\,{}_3F_0(-l,\,l+1,\,\tfrac{1}{2};\,-\rho^{-2}). \quad (30)$$

This shift function (in a manner similar to the proof for the variation with η, above) can be shown to be monotonically decreasing for ρ, and therefore E. In addition, one may also show that $S_{0,l+1}\geqslant S_{0,l}$.

IV. MONOTONIC PROPERTIES FOR NEGATIVE ENERGIES

The shift factor is given by Eq. (5); it is convenient to write this as

$$S_{\eta,l}^{(-)}=-l+\left(\int_1^\infty e^{-\rho z}\bar{Q}(z)dz\right)^{-1}\times\left[\delta_\eta{}^0\delta_l{}^0+\int_1^\infty e^{-\rho z}z\bar{Q}'(z)dz\right], \quad (31)$$

$$\bar{Q}(z)=(z^2-1)^l\left(\frac{z-1}{z+1}\right)^\eta.$$

Using methods exactly as in Sec. III, and omitting for the moment the case $l=0$, $\eta=0$, we find that

$$\frac{dS^{(-)}}{d\rho}=-\tfrac{1}{2}\left(\int e^{-\rho z}\bar{Q}dz\right)^{-2}\int_1^\infty\int dxdy e^{-\rho(x+y)}\times\bar{Q}(x)\bar{Q}(y)\bar{M}(x,y), \quad (32)$$

$$\bar{M}(x,y)=(x-y)[x\bar{Q}'(x)/\bar{Q}(x)-y\bar{Q}'(y)/\bar{Q}(y)].$$

It is easily shown that for $x,\ y\geqslant 1$, $\operatorname{sgn}\bar{M}(x,y)=\operatorname{sgn}(y-x)$. It follows that $dS^{(-)}/d\rho\geqslant 0$, upon noting explicitly that for $l=0$, $\eta=0$ the result is also true.

In a similar fashion, the relevant functions $\bar{M}(x,y)$ being elementary, one sees that $\partial S^{(-)}/\partial\eta\geqslant 0$ and $\partial S^{(-)}/\partial l\geqslant 0$.

Since now an increase in either ρ or η increases $S^{(-)}$, it no longer follows that $S^{(-)}$ is monotonic in E.

For $\eta=0$, the shift factor $S^{(-)}$ has the simple form of a ratio of polynomials.[14]

$$S_{0,l}^{(-)}=\rho+\tfrac{1}{2}l(l+1)\rho^{-1}\,{}_2F_0(1-l,\,l+2;\,-\tfrac{1}{2}\rho^{-1})/\,{}_2F_0(-l,\,l+1;\,-\tfrac{1}{2}\rho^{-1}). \quad (33)$$

V. THRESHOLD BEHAVIOR

The threshold behavior of the penetration factor has been extensively discussed in reference 5, and need not be repeated here. The results of the one-level approximation are in complete accord with the general behavior shown by Wigner.[6]

Let us consider now the shift factor, for a situation where the threshold channel, t, is neither an entrance nor an exit channel. According to (2), the total shift Δ_λ will involve the shift, $\Delta_{\lambda t}$, of this channel as the energy varies across threshold.

Now for positive energy, in the vicinity of $E=0$, *if* $\eta>0$, the amplitude $A_{\eta,l}^2$ is essentially determined by G_l^2, since $(F/G)\sim e^{-2\pi\eta}\ll 1$. Very near zero energy, therefore, S is completely determined by the irregular function G_l. We may therefore use for $S^{(+)}$ the form:

$$\bar{S}=-\rho(F+iG)'/(F+iG)=-\rho Y'/Y, \quad (34)$$

which for very near zero energy is equivalent to the correct S. But $\bar{S}$ is just the form which continues into $S^{(-)}$; hence $S^{(+)}$ and $S^{(-)}$ continue correctly into one another as functions of $E^{\frac{1}{2}}$. To see, however, that the variable is really E, instead of $E^{\frac{1}{2}}$, we note that for positive energy $S\rightarrow-\rho G_l'/G_l$, and this is an *even* function of k ($k\eta=$constant). Thus, across zero energy, $S^{(+)}$ goes smoothly into $S^{(-)}$ as a function of E, for $\eta>0$.

For the case where $\eta=0$, we can apply a variant of this argument directly, upon noting that the irregular function now dominates the regular function by the factor k^{2l+1}. The irregular function is, as before, even in k; thus the previous argument shows that the functions $S^{(+)}$ and $S^{(-)}$ are the same functions of k^2 up to order k^{2l+1}. Hence for $E=0$, S is continuous to l derivatives in the energy.

For $\eta=0$, the shift functions are ratios of polynomials. Consider, for example, the case $l=0$.

$$S_{0,0}^{(+)}=0,\quad S_{0,0}^{(-)}=\rho=ka=(2Ma^2/\hbar^2)^{\frac{1}{2}}|E|^{\frac{1}{2}}. \quad (35)$$

Thus as the energy decreases across this threshold, the total shift, Δ_λ, shows a discontinuous derivative. For the general case, Wigner has shown[6] a cusp at threshold is possible. Recently, Breit[7] has given a general proof that the cusp behavior may be modified, for the general case, so that the slope on both sides of the singularity may have the same sign. (To use Breit's descriptive phrase, the curve is like "the central portion of the letter S turned on its side"). We see that the one-level approximation yields a not unreasonable compromise between these two extreme possible behaviors, and appears to be satisfactory as an approximation.

ACKNOWLEDGMENTS

This investigation of the properties of the S and P factors was initiated by the late Dr. R. G. Thomas in a discussion with one of the authors in 1956. In particular, the results of Sec. III(a) and parts of III(b) essentially reproduce the work of Thomas, (which will be published as part of an extensive survey of nuclear reaction theory by Dr. A. M. Lane and Dr. R. G. Thomas).

We are indebted to Professor Gregory Breit for the favor of a prepublication form of his paper.

[14] Erdelyi, Magnus, Oberhettinger, and Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 264, Eq. (5).

APPENDIX

The function $Y=F+iG$ is given by:

$$Y_l(\eta,\rho)=[\Gamma(l+1-i\eta)/\Gamma(l+1+i\eta)]^{\frac{1}{2}}\times e^{\frac{1}{2}\pi i(l+1-i\eta)}W_{i\eta,\,l+\frac{1}{2}}(2i\rho), \quad \text{(A-1)}$$

where W is a Whittaker function. Thus,

$$A_{\eta,\,l}^2=YY^*=e^{\pi\eta}W_{i\eta,\,l+\frac{1}{2}}(2i\rho)W_{-i\eta,\,l+\frac{1}{2}}(-2i\rho). \quad \text{(A-2)}$$

However, we have the relation[15]

$$W_{\kappa,\,\mu}(z)W_{\lambda,\,\mu}(\zeta)=[\Gamma(1-\kappa-\lambda)]^{-1}(z\zeta)^{\mu+\frac{1}{2}}e^{-\frac{1}{2}(z+\zeta)} \times\int_0^\infty e^{-t}t^{-\kappa-\lambda}(z+t)^{-\frac{1}{2}+\kappa-\mu}(\zeta+t)^{-\frac{1}{2}+\lambda-\mu} \times{}_2F_1(\tfrac{1}{2}-\kappa+\mu,\,\tfrac{1}{2}-\lambda+\mu;\,1-\kappa-\lambda;\,\Theta)dt, \quad \text{(A-3)}$$

where

$$\Theta=\frac{t(z+\zeta+t)}{(z+t)(\zeta+t)}.$$

Introducing values for the various parameters, κ, λ, $\cdots$, and employing Kummer's relations for the hypergeometric function yield the result given in Eq. (9).

[15] W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1949), p. 91.

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Resonance Scattering of Slow Neutrons on Indium*

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The scattering of slow neutrons by indium was observed by using a crystal spectrometer as a monoenergetic neutron source. Thin scattering samples were placed in the neutron beam with the plane of the samples at a small angle to the incident beam. The samples therefore appeared thick for the transmitted neutrons but thin for neutrons scattered at right angles to the direction of the incident beam. A measurement of the scattered and the transmitted neutrons gives σ_s/σ_t. The total cross section σ_t was also measured in the energy range from 0.3 to 11 ev and the value of σ_s then computed. Both the total cross section and σ_s/σ_t results could be well matched to the Breit-Wigner formulas from 0.3 to 3 ev with the same set of parameters, assuming that the spin of the compound state in In^{116} for the 1.456-ev level was 5. No determination of the spin state of the compound nucleus could be made for higher levels.

I. INTRODUCTION

The Breit-Wigner single-level equation gives the variation of scattering, capture, and total cross sections with the energy of the incident neutron. These formulas are given in Eqs. (1), (2), and (3).[1] In Eqs. (1′), (2′), and (3′) are presented alternative formulations which abbreviate certain groupings of the primary parameters by expressions which are convenient for analysis.

$$\sigma_s=4\pi gR_R^2+\frac{4\pi g\lambda\!\!\!^{-}{}_0^2\Gamma_n^2}{4(E-E_0)^2+\Gamma^2}+\frac{16\pi g\lambda\!\!\!^{-}{}_0\Gamma_nR_R(E-E_0)}{4(E-E_0)^2+\Gamma^2}+4\pi(1-g)R_{NR}^2, \quad (1)$$

$$\sigma_c=\left(\frac{E_0}{E}\right)^{\frac{1}{2}}\frac{4\pi g\lambda\!\!\!^{-}{}_0^2\Gamma_n\Gamma_\gamma}{4(E-E_0)^2+\Gamma^2}, \quad (2)$$

$$\sigma_t=\sigma_s+\sigma_c, \quad (3)$$

where the symbols represent the following: σ_s = scattering cross section, σ_c = capture cross section, σ_t = total cross section, E = energy of incident neutron, E_0 = resonance energy, $2\pi\lambda\!\!\!^{-}{}_0$ = neutron wavelength at resonance, Γ_n = partial width for neutron decay, Γ_γ = partial width for electromagnetic radiation, $\Gamma=\Gamma_n+\Gamma_\gamma$ = total width of resonance, R_R = radius of nucleus for resonance interaction, and R_{NR} radius of nucleus for nonresonance interaction.

$$\sigma_s=\sigma_p+\frac{\sigma_{s0}\Gamma^2}{4(E-E_0)^2+\Gamma^2}+\frac{I(E-E_0)}{4(E-E_0)^2+\Gamma^2}, \quad (1')$$

$$\sigma_c=\left(\frac{E_0}{E}\right)^{\frac{1}{2}}\frac{\sigma_{c0}\Gamma^2}{4(E-E_0)^2+\Gamma^2}, \quad (2')$$

$$\sigma_t=\sigma_s+\sigma_c, \quad (3')$$

where

$$\sigma_p=4\pi gR_R^2+4\pi(1-g)R_{nR}^2,$$

$$\sigma_{s0}\Gamma^2=4\pi g\lambda\!\!\!^{-}{}_0^2(\Gamma_n/\Gamma)^2\Gamma^2,$$

$$\sigma_{c0}\Gamma^2=4\pi g\lambda\!\!\!^{-}{}_0^2(\Gamma_n/\Gamma)(\Gamma_\gamma/\Gamma)\Gamma^2,$$

$$I=16\pi g\lambda\!\!\!^{-}{}_0R_R\Gamma_n.$$

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[1] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1951).