Brownian Motion of a Mirror in Superfluid Helium*

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This paper treats the Brownian motion of a mirror suspended in a phonon or photon gas. It is shown that the properties of the medium, such as the statistics and the temperature dependence of the excitation concentration, influence the mean square Fourier amplitudes of the motion of the mirror.

I

IN the present theory of superfluid helium we picture the superfluid as a continuum in which small particle-like excitations, photons and rotons, move around. The existence of these excitations is usually inferred through the analysis of thermodynamical and hydrodynamical measurements. Would it be possible to invent a more direct experiment in which the existence of small distinct particle-like excitations could be exhibited directly?

In the nineteenth century the same question was posed to demonstrate the existence of atoms. There the first, so to say, visual demonstration was given through the experiments on Brownian motion. Why not try the same here?

II

If we suspend a small mirror in a gas it will undergo torsional oscillations due to the impact of the gas molecules. By the theory of Brownian motion the mean-square angular displacement is independent of the gas and its pressure and only depends on the temperature. On the other hand, the *details* of the motion depend on the medium. If we Fourier-analyze the angular motion of the mirror, the mean square of the Fourier components depends on the density or pressure.[1] This is quite natural if we observe the behavior of a very dilute gas. The mirror will perform torsional oscillations with its proper frequency, and this motion will be disturbed only seldom by a chance collision with a gas molecule. Thus, essentially all Fourier components will be zero except the few in the neighborhood of the one corresponding to the proper frequency of the mirror. In a dense gas, however, practically all modes will be excited by the constant bombardment of the mirror. If we take the surrounding gas to be a collection of phonons (and rotons) in superfluid He and describe the Brownian motion of a mirror in superfluid He, we could test for the existence and behavior of phonons and rotons. Unfortunately there are formidable difficulties involved in the actual experiment, since the effects are so small. If we take a quartz fiber of a few centimeters in length and one or two microns in diameter and perform the experiment at 2°K, the root-mean-square angular deflection will be of the order of 10^{-4} radian or about 20 seconds of arc. If, suppose, about ten modes are excited, this will give roughly 2 seconds per mode on the average. This is not easy to detect, especially not in view of the external disturbances. Moreover, there is the following additional difficulty. For a quartz fiber of the above size and a mirror of 1 or 2 square-millimeter area, the free period is about one hour. Since we have to take long time averages, in length many times the period, the duration of the experiment will be very large. For this reason we cannot decrease greatly the restoring force in the fiber which would otherwise be useful, since it increases the root-mean-square angular displacement.

III

The motion of the mirror is described by the equation

$$I\ddot{\phi}+r\dot{\phi}+D\phi=\Gamma(t), \qquad (1)$$

where ϕ is the angular displacement of the mirror from its equilibrium position, I its moment of inertia, r the resistance, D the force constant of the filament upon which it is suspended, and Γ is the fluctuating torque.

If $\langle\Gamma^2\rangle_{\mathrm{Av}}$ is the fluctuation in the couple (since $\langle\Gamma\rangle_{\mathrm{Av}}=0$) produced by particles arriving during time Δt, then the condition of equipartition gives the following relation[2]:

$$\tfrac{1}{2}D\langle\phi(t)^2\rangle_{\mathrm{Av}}=\tfrac{1}{4}(\langle\Gamma^2\rangle_{\mathrm{Av}}\Delta t/r)=\tfrac{1}{2}kT. \qquad (2)$$

Hence, if we calculate from a detailed model of the medium $\langle\Gamma^2\rangle_{\mathrm{Av}}$ and r, we must be able to verify relation (2).

If we expand $\Gamma(t)$ and $\phi(t)$ in a Fourier series for the long time interval 0 to τ, $\Gamma(t)=\sum_k(A_k\cos\omega_k t+B_k\sin\omega_k t)$, $\omega_k=2\pi k/\tau$, $\phi=\sum_k\phi_k(t)$, we find[3]

$$\langle\phi_k^2\rangle_{\mathrm{Av}}=\frac{1}{2I^2}\frac{A_k^2+B_k^2}{(\Omega^2-\omega_k^2)^2+(r/I)^2\omega_k^2}, \qquad (3)$$

where $A_k^2+B_k^2=(4/\tau)\langle\Gamma^2\rangle_{\mathrm{Av}}\Delta t$ and $\Omega=(D/I)^{\frac{1}{2}}$ is the eigenfrequency of the suspended mirror.

Our interest is to find $\langle\phi_k^2\rangle_{\mathrm{Av}}$ if the surrounding gas is phonon gas. We omit the discussion of the roton gas,

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[1] G. Uhlenbeck and S. A. Goudsmit, Phys. Rev. 34, 145 (1928). [See also R. H. Fowler, *Statistical Mechanics* (Cambridge University Press, New York, 1936), pp. 775–783.]

[2] R. H. Fowler, reference 1, p. 781.

[3] R. H. Fowler, reference 1, p. 782.

since at sufficiently low temperatures the contribution due to this gas is negligible.

For this we have to evaluate $\langle\Gamma^2\rangle_{\text{Av}}$ and r. [In principle it would be sufficient to calculate $\langle\Gamma^2\rangle_{\text{Av}}$ or r, since by (2) one is a function of the other. As a check, however, it is useful to evaluate both independently.]

Let $\Delta\sigma$ be a small element of area on one side of the mirror at a distance x from the axis of suspension. If G is the momentum transferred to it in time Δt the impulse received will be xG, and the fluctuation in the impulse will be $x^2\langle(G-\bar{G})^2\rangle_{\text{Av}}$. Summing over both sides of the mirror we shall obtain the fluctuation in the total impulse; upon division by $(\Delta t)^2$ we obtain the fluctuation of the total torque. Thus our problem is reduced to evaluating $\langle(G-\bar{G})^2\rangle_{\text{Av}}$.

Take $\Delta\sigma$ to have a normal along the 1 axis. Let i denote a region in phase space which has the following properties: the particle is in the spatial region $v_i=\Delta\sigma\Delta t p_1/m$ for an ordinary gas or $\Delta\sigma\Delta t c p_1/|p|$ for a phonon gas (collision cylinder) *and* its momentum is such that $p_1\leqslant p_1\leqslant p_1+dp_1$; $p_2\leqslant p_2\leqslant p_2+dp_2$; $p_3\leqslant p_3\leqslant p_3+dp_3$. The extension of this region is $\omega_i=v_i dp_1 dp_2\times dp_3/h^3$. Let n_i phase points be in ω_i; then the total momentum transferred to $\Delta\sigma$ in time Δt is $G=\sum_i\times 2p_1(i)n_i$, if we assume specular reflection. If n_i and $n_{i'}$ are statistically independent of each other (as is the case), the fluctuation is given by

$$\langle(G-\bar{G})^2\rangle_{\text{Av}}=4\sum_i p_1(i)\langle(n_i-\bar{n}_i)^2\rangle_{\text{Av}}.$$

What is $\langle(n_i-\bar{n}_i)^2\rangle_{\text{Av}}$ in terms of N_p, the number of particles in unit volume with vector momentum p? By definition $n_i=S_pN_p$, where the summation S_p is performed over all p's in region i. Hence

$$\begin{aligned}\langle(n_i-\bar{n}_i)^2\rangle_{\text{Av}}&=S_pS_{p'}\langle(N_p-\bar{N}_p)(N_{p'}-\bar{N}_{p'})\rangle_{\text{Av}}\\&=S_p\langle(N_p-\bar{N}_p)^2\rangle_{\text{Av}}=S_p[\bar{N}_p+(\bar{N}_p)^2]\qquad(4)\\&=\omega_i\bar{N}_p+\omega_i(\bar{N}_p)^2.\end{aligned}$$

The first equality follows from the definition of n_i, the second from the fact that N_p and $N_{p'}$ are uncorrelated, and the third from the well-known expressions for the fluctuation in the occupation numbers in Bose-Einstein statistics; the last equality follows from the fact that the number of phase points in ω_i is proportional to ω_i. (Since the N_p's are uncorrelated, we see that the n_i's are also uncorrelated, as we asserted above.) For a phonon gas

$$\begin{aligned}\langle(G-\bar{G})^2\rangle_{\text{Av}}&=\frac{4\Delta\sigma\Delta tc}{h^3}\int_0^\infty dp_1\int_{-\infty}^\infty dp_2\int_{-\infty}^\infty dp_3{}^2\frac{p_1{}^3}{|p|}\\&\quad\times\left[\frac{1}{e^{\beta c|p|}-1}+\frac{1}{(e^{\beta c|p|}-1)^2}\right]\\&\equiv(4\Delta\sigma\Delta tc/h^3)K;\quad \beta=(kT)^{-1}.\qquad(5)\end{aligned}$$

The impulse transferred to element $\Delta\sigma$ at a distance x from the axis of suspension is then $(4\Delta tc/h^3)Kx^2\Delta\sigma$; summing over all elements we obtain $(4\Delta tc/h^3)K\kappa^2 2\sigma$, if κ is the radius of gyration and 2σ the total surface area, or σ the area of one side; dividing this by $(\Delta t)^2$, we finally obtain $\langle\Gamma^2\rangle_{\text{Av}}=(8cK/\Delta th^3)\kappa^2\sigma$. What is left is the evaluation of K.

If we transform to polar coordinates, the polar axis being the 1 axis, and expand the factor $(1-e^{-\beta c|p|})^{-1}$ in a binomial series, we can perform the indicated integrations, and sum over the resulting series. We obtain $K=(12\pi^5/90)(kT/c)^5$ or

$$\langle\Gamma^2\rangle_{\text{Av}}=(16\pi^5/15)(kT/h^3t)(kT/c)^4\kappa^2\sigma.$$

In terms of the pressure P (of the excitations) we get $\langle\Gamma^2\rangle_{\text{Av}}=12P(kT/c)\kappa^2\sigma/\Delta t$. If we observe that the mean forward momentum $\bar{p}\sim 3kT/c$, we get $\langle\Gamma^2\rangle_{\text{Av}}\sim 4\bar{p}P\kappa^2\sigma/\Delta t$. In terms of P and $\bar{p}$ it has the same form as for a Boltzmann gas.

The quantity r can be computed as follows. The element $\Delta\sigma$ of the mirror with a normal along the 1 axis should move to the right with the velocity u. The number of collisions from the left with momentum p_1, p_2, p_3 is

$$\Delta\sigma\Delta t[(p_1/|p|)c-u](e^{\beta c|p|}-1)^{-1}dp_1dp_2dp_3/h^3;\qquad (p_1/|p|)c>u.$$

Each impact transfers $2[p_1-(|p|/c)u]$ momentum. The total transfer is

$$(2\Delta\sigma\Delta t/h^3)\int_{(u/c)|p|}^\infty dp_1\int_{-\infty}^\infty dp_2\int_{-\infty}^\infty dp_3(|p|/c)\times[(p_1/|p|)c-u]^2(e^{\beta c|p|}-1)^{-1}.\qquad(6)$$

If $\Delta\sigma$ is located at a distance x from the axis of rotation the total impulse transferred to $\Delta\sigma$ is x times the above expression. Now let us expand the integrand in terms of u, retain only the term linear in u, and put $u=x\dot{\phi}$; the coefficient of $\dot{\phi}$ will give the friction of the element $\Delta\sigma$. Summing over all $\Delta\sigma$, we get the friction of the whole mirror. This way we get $r=(8\pi^5/15h^3)(kT/c)^4\kappa^2\sigma$, where κ is the radius of gyration of the mirror and σ is the surface area of one side of the mirror.

In terms of the pressure, $r\sim 6(P/c)\kappa^2\sigma$ or $2\bar{p}(P/kT)\kappa^2\sigma$. In terms of $\bar{p}$ and P, it has the same form as for a Boltzmann gas.

We easily verify that $\frac{1}{4}(\langle\Gamma^2\rangle_{\text{Av}}\Delta t/r)=\frac{1}{2}kT$, as required.

IV

Let us analyze the details of the motion.

$$\langle\phi_k{}^2\rangle_{\text{Av}}=\frac{2\langle\Gamma^2\rangle_{\text{Av}}\Delta t}{\tau I^2}\frac{1}{(\Omega^2-\omega_k{}^2)^2+(r/I)^2\omega_k{}^2}.\qquad(7)$$

The maximum value will be reached for an $\omega_k\sim\Omega$. For this value of k, we get

$$\langle\phi^2\rangle_{\text{Av}}(\text{resonance})\equiv\langle\phi_R{}^2\rangle_{\text{Av}}=(2/\tau)(\langle\Gamma^2\rangle_{\text{Av}}\Delta t/\Omega^2r^2).\qquad(8)$$

Since $\langle\Gamma^2\rangle_{Av}\Delta t = 2rkT$, we further get

$$\langle\phi_R^2\rangle_{Av} = 4kT/\Omega^2 r\tau; \tag{9}$$

thus the behavior of $\langle\phi_R^2\rangle_{Av}$ as a function of the temperature will be determined by $kT/\tau r$. At 1°K this ratio is about 0.06 (with $\tau\sim 10^3$ sec) and it will increase very sharply as the temperature decreases, since $r\sim T^4$. At first sight it may be surprising that this ratio tends to infinity as $T\to 0$. This is due to the fact that τ cannot be kept constant as $T\to 0$, for the following reason. The long time interval τ for which we performed the Fourier analysis must be much larger than $1/\Omega$, the period of the free oscillations. For Eq. (1) to have a meaning, $1/\Omega$ in turn must be much larger than $\tau_{\text{collision}}$, the average time elapsed between two consecutive phonon-mirror collisions. Consequently $\tau\gg\tau_{\text{collision}}$. If $T\to 0$, $\tau_{\text{collision}}\to\infty$; hence we cannot keep τ (and $1/\Omega$) fixed in the limiting process $T\to 0$. (In practice this is not a very serious point, since at 1°K $\tau_{\text{collision}}$ on 1 cm² is about 10^{-23} sec for a phonon gas, and is still 10 seconds at 10^{-8} °K.) If we keep the ratio $\tau/\tau_{\text{collision}} = N$ fixed in the limiting process, $\langle\phi_R^2\rangle_{Av}$ tends to a finite value. The quantity r can be written as $r = 32(kT/c^2)\kappa^2/\tau_{\text{collision}}$, or $r\tau = 32(kT/c^2)\tau/\tau_{\text{collision}}$, and consequently $kT/r\tau$ is independent of T. As $\tau\to\infty$ the Fourier sum tends to an integral, and consequently we are better off if we deal not with $\langle\phi_k^2\rangle_{Av}$ but with $\langle\phi^2(\omega)\rangle_{Av}d\omega$. In other words, we write

$$\langle\phi^2\rangle_{Av} = \sum_k \langle\phi_k^2\rangle_{Av} = \int_0^\infty \langle\phi^2(\omega)\rangle_{Av}d\omega,$$

where

$$\langle\phi^2(\omega)\rangle_{Av}d\omega = (\langle\Gamma^2\rangle_{Av}\Delta t/\pi I^2)/[(\Omega^2-\omega_k^2)^2+(r/I)^2\omega_k^2].$$

Now the expression is independent of τ. As $T\to 0$ we will have a singularity in the integrand, but the integral is still converging, and converges to the correct value kT/D. Introducing $x=\omega/\Omega$, we can finally write it in dimensionless form as

$$\langle\phi^2\rangle_{Av}(\omega)d\omega = (2\rho\langle\phi^2\rangle_{Av}dx/\pi)/[(1-x^2)^2+\rho^2x^2],$$

where $\langle\phi^2\rangle_{Av} = kT/D$ and $\rho = (r/\Omega I)$ = free period/relaxation time. We see now how the spectrum of the fluctuations behaves. At $x=0$ it starts out with the value $2\rho\langle\phi^2\rangle_{Av}/\pi$, rises to a high and narrow peak at $x=1$ with a height $2\langle\phi^2\rangle_{Av}/\rho\pi$ and width ρ, and drops to zero very fast for $x>1$. Thus we see immediately that the integral over all frequencies will be indeed independent of ρ, being approximately

$$[\langle\phi^2(\omega)\rangle_{Av}]_{\max}\rho = (2\langle\phi^2\rangle_{Av}/\rho\pi)\rho = (2/\pi)\langle\phi^2\rangle_{Av}.$$

A precise evaluation of the integral gives just $\langle\phi^2\rangle_{Av}$ as it should. This is easy to verify for large ρ. Then

$$(2\rho/\pi)\langle\phi^2\rangle_{Av}\int_0^\infty [(1-x^2)^2+\rho^2x^2]^{-1}dx \sim (2\rho/\pi)\langle\phi^2\rangle_{Av}\int_0^\infty [1+\rho^2x^2]^{-1}dx = (2\rho/\pi)\langle\phi^2\rangle_{Av}(\pi/2\rho) = \langle\phi^2\rangle_{Av}.$$

It is interesting to discuss the fraction $\langle\phi^2\rangle_{Av}(x)dx = (2\rho/\pi)/[(1-x^2)^2+\rho^2x^2]$, which gives the fraction of the average potential energy concentrated in the frequency band dx. The temperature dependence is determined by the temperature dependence of ρ. In all cases $\rho\to 0$ as $T\to 0$ and most of the energy is concentrated in a narrow frequency range around 1. For an ideal gas obeying Maxwell-Boltzmann statistics ρ is proportional to $T^{\frac{1}{2}}$, while for a phonon gas ρ is proportional to T^4. Thus, for a phonon gas the narrowness and height of the peak will become more pronounced as we lower the temperature.

One may inquire to what extent this difference in behavior is due to the Einstein-Bose statistics or to the fact that the number of phonons varies with the temperature in the medium. The answer is simple. Let us imagine that we immerse our mirror in an ideal gas whose concentration is the same at each temperature as that of the phonon gas at the same temperature. Then ρ is proportional to $\bar{p}(P/kT)$ or to $T^{\frac{1}{2}}T^3 = T^{7/2}$, since for a phonon gas the density of phonons is proportional to T^3. The peak will be proportional to $T/T^{7/2} = T^{-5/2}$ and it will again tend to infinity as $T\to 0$. As we see, it is the temperature dependence of the excitation density which causes the strong increase in the sharpness of the peak. Of course, the above considerations can be applied without any alteration to the Brownian motion of a mirror in a radiation field. This problem (and the Brownian motion of an electron) was first treated by different methods by Lorentz and by Fokker who have obtained an incorrect result.[4] Pauli's work[5] has shown how this can be remedied. Our method is different from theirs.

V

The experiment would then be as follows. We register the Brownian oscillations of a mirror in superfluid helium, and Fourier-analyze the resulting curve. We compute the mean square of each Fourier component, and plot it as a function of the frequency. We repeat this for several temperatures. The comparison of these curves at different temperatures with the theoretical ones would be the test.

I wish to express my thanks to Professor J. E. Mayer for many interesting discussions.

[4] H. A. Lorentz, Ber. Solvay—Kongress in Brussels, 1911; A. D. Fokker, Arch. néerl. sci. IIIa, 4, 379 (1918).
[5] W. Pauli, Z. Physik 18, 272 (1923).