PHYSICAL REVIEW VOLUME 109, NUMBER 6 MARCH 15, 1958

Proof of Dispersion Relations in Quantized Field Theories*

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(Received October 15, 1957)

The problem of deriving dispersion formulas is reduced to that of the analytic continuation of all functions which are regular in certain domains in the space of several complex variables. The dispersion relations for pion-nucleon scattering are proven for momentum transfers in the center-of-mass system which are smaller than $2\sqrt{2}M_\pi$. This limit can be improved by further analytic continuation. By using causality and spectral conditions the dispersion formulas for forward nucleon-nucleon scattering could be derived only under the unphysical condition $M_\pi > (\sqrt{2}-1)M_N$. We cannot exclude the possibility that this restriction is weakened by taking into account all symmetry properties of the complete four-body Green's function. The situation is similar for the representation of the meson-nucleon vertex function taken on the mass shell of the nucleons. In this case an example of R. Jost shows that the validity of the dispersion formula cannot be guaranteed on the basis of causality, spectrum, and symmetry properties.

1. INTRODUCTION

THE purpose of the present article is to derive some analytic properties of scattering amplitudes on the basis of general assumptions underlying local relativistic quantum field theories. Most of these analytic properties are usually expressed in terms of Hilbert relations, which have come to be called dispersion relations. Such equations have been obtained for various physical processes, and their importance lies in the fact that in several cases they can be directly tested by experiments.[1] In other cases it is at least possible to extract approximate relations between observable quantities. Hence the dispersion formulas make it possible to test experimentally some aspects of the basic "axioms" mentioned earlier.

We shall not give here a detailed discussion of these axioms, since it may be found elsewhere.[2] They consist essentially of the following assumptions,

(a) The existence of linear field operators in Hilbert space,

(b) the transformation laws of these fields under the transformations of the inhomogeneous Lorentz group,

(c) the asymptotic condition, and

(d) the so-called causality condition, which implies that the commutators (or anticommutators) of two field operators shall vanish if these fields are taken at points which have a finite space-like separation.

For the derivation of dispersion relations we need a few additional assumptions about the existence of Fourier transforms and some known properties of the spectrum of intermediate states for the system in question. These will be discussed later.

On the basis of the axioms (a), $\cdots$, (d), one can derive representations for the elements of the scattering matrix, and we shall be especially concerned with the amplitude for the elastic scattering of particles with finite rest mass. Let k and k' (p and p') be the four-momenta of the projectile (target) before and after scattering. In a special Lorentz frame[1] where $\mathbf{p}+\mathbf{p}'=0$, we consider the scattering amplitude T as a function of the projectile energy $\omega=k_0=k_0'$ for fixed finite values of $\Delta=\frac{1}{2}|\mathbf{k}'-\mathbf{k}|$ (2Δ is the amount of the momentum transfer; we disregard here possible charge and spin variables). We are interested in the analytic properties of $T(\omega,\Delta^2)$ as a function of ω, but the representation obtained from the axioms is valid only for *real* ω with $\omega^2 \geq m^2+\Delta^2$, where m is the rest mass of the projectile. (Throughout this paper we set $\hbar=c=1$.) In order to continue the scattering amplitude into the complex ω plane, we must consider it as a function of ω and other variables simultaneously. Thus we are led to use the tools of the theory of functions of several complex variables.

For the case of pion-nucleon scattering a mathematically rigorous proof of dispersion relations for non-forward scattering has been given by Bogoliubov.[3] His proof is valid for values of Δ^2 which are smaller than $\Delta_{\max}{}^2=(m/m+\mu)\mu^2$, where μ is the pion and m is the nucleon mass. Bogoliubov avoids the explicit use of the theory of functions of several complex variables by employing parametrizations and using distribution methods. This makes the proof very involved, and it seems difficult to generalize it to other processes with qualitatively different properties of the spectrum. Because of these difficulties we think that it may be

* A brief report of our results appears in the Proceedings of the "Colloque sur les Problèmes Mathématiques de la Théorie Quantique des Champs," Lille, France, June, 1957.

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[1] For references see, for instance, Goldberger, Nambu, and Oehme, Ann. phys. **2**, 226 (1957).

[2] R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd., **29**, No. 12 (1955); Lehmann, Symanzik, and Zimmermann, Nuovo cimento **1**, 205 (1955); Bogoliubov, Medvedev, and Polivanov, lecture notes, translated at the Institute for Advanced Study, Princeton, 1957 (unpublished); A. S. Wightman, preprint of Lille Conference talk, June, 1957 (this paper contains further references).

[3] N. N. Bogoliubov (private communication, February 1957). We would like to thank Professor Bogoliubov for informing us about the details of his proof.

worthwhile to present a different approach in which the essential part of the proof of dispersion formulas is reduced to the problem of finding the *envelope of holomorphy* of a certain type of domain in the space of several complex variables.[4,5] The *envelope of holomorphy* $E(D)$ of a domain D is the intersection of the *domains of holomorphy* of *all* functions which are regular (holomorphic) in D. The *domain of holomorphy* D_f of a function f, which is analytic in D, is the largest domain into which f may be continued. In the case of one complex variable there exists for every domain D a function $f(z)$ such that $D=D_f$, and hence we have $E(D)=D$. But in the space of two or more complex variables the situation is different and we have the remarkable fact that there exist domains D such that *all* functions which are analytic in D may be continued simultaneously into a larger domain. The largest domain into which *all* these functions can be continued is the envelope of holomorphy of D.

For special domains D, which possess certain symmetry properties, the corresponding $E(D)$ is known.[6] There are also methods for the construction of $E(D)$ for an arbitrary D.[7] The domain D which appears in the proof of dispersion formulas has certain invariance properties and we hope that future mathematical work will make it possible to find the corresponding envelope $E(D)$. In the present paper we consider only a suitable subdomain D_S which is a generalized semitube. For this semitube we can construct the envelope of holomorphy and obtain thus a proof of certain dispersion relations for restricted values of Δ^2. In the case of pion-nucleon scattering we have for instance $\Delta^2<2\mu^2$. The semitube method is not sufficient to prove dispersion relations for nucleon-nucleon scattering; it would only suffice if the mass ratio μ/m where larger than $(\sqrt{2}-1)$. However, this limit is due to such points on the surface of $E(D_S)$ which can be shown to be also surface points of $E(D)$. Hence we cannot hope to continue further on the basis of local commutativity and the support properties derived from the spectral conditions. But there are certain symmetry properties of the four-body Green's function which we have not used, and we cannot exclude the possibility that these permit a continuation beyond $E(D)$ in the relevant region.[8]

The situation is very similar in the case of the meson-nucleon vertex function $F[(k-p)^2, k^2, p^2]=\langle k|j(0)|p\rangle$. We can assure the existence of the representation $[k^2<(m+\mu)^2, p^2<(m+\mu)^2]$

$$F[z,k^2,p^2]=\frac{z^N}{\pi}\int_{(3\mu)^2}^{\infty} d\sigma^2\frac{\rho(\sigma^2,k^2,p^2)}{\sigma^{2N}(\sigma^2-z)}+\sum_{n=0}^{N-1} C_n(k^2,p^2)z^n$$

only for $k^2+p^2<(m+\mu)^2$. In order to reach the mass shell $k^2=p^2=m^2$, we would have to require $\mu>(\sqrt{2}-1)m$.

See Sec. 4 for more detailed discussions of the restrictions in mass ratios and momentum transfer.

2. CONSTRUCTION OF DISPERSION FORMULAS

As an example we consider the elastic scattering of neutral scalar bosons of equal mass. If $S(k',p',k,p)$ describes the relevant S-matrix element for this process, we introduce the usual causal amplitude M_r by the relation

$$S(k',p',k,p)=\langle k'|k\rangle\langle p'|p\rangle+i(2\pi)^4\delta(k'+p'-k-p)\times(16k_0'k_0p_0'p_0)^{-\frac{1}{2}}M_r(k+k';p',p),\quad(2.1)$$

where k, p and k', p' denote the momenta of the particles before and after scattering. Then, by standard methods,[9] we obtain for M_r the representations

$$M_r(k+k';p',p)=2(p_0p_0')^{\frac{1}{2}}i\int d^4x\, e^{\frac{1}{2}i(k+k')\cdot x}\eta(x_0)\times\langle p'|[j(\tfrac{1}{2}x),j(-\tfrac{1}{2}x)]|p\rangle+P(k+k';p',p),\quad(2.2)$$

where $j(x)$ may be defined in terms of the Heisenberg fields $\phi(x)$ by $j(x)=(\Box+m^2)\phi(x)$ and where P is a real polynomial in the components of $k+k'$ with arbitrary coefficients depending on p' and p. In addition to M_r we introduce the corresponding advanced amplitude M_a by

$$M_a(k+k',p',p)=-2(p_0p_0')^{\frac{1}{2}}i\int d^4x\, e^{\frac{1}{2}i(k+k')\cdot x}\eta(-x_0)\times\langle p'|[j(\tfrac{1}{2}x),j(-\tfrac{1}{2}x)]|p\rangle+P(k+k';p',p).\quad(2.3)$$

Then the dispersive and absorptive parts of M_r may be expressed in the form

$$D=\tfrac{1}{2}(M_r+M_a),\quad A=\frac{1}{2i}(M_r-M_a);\quad(2.4)$$

and are real functions of $k+k'$, p' and p. It is convenient for our further discussion to go into the special system in which

$$\tfrac{1}{2}(k+k')=\{\omega,(\omega^2-E_\Delta{}^2)^{\frac{1}{2}}\mathbf{e}\};\quad p'=p_\Delta,\quad p=p_{-\Delta},$$

where

$$p_\Delta=\{E_\Delta,\mathbf{\Delta}\};\quad \mathbf{\Delta}\cdot\mathbf{e}=0;\quad E_\Delta\equiv(m^2+\Delta^2)^{\frac{1}{2}};\quad |\mathbf{e}|=1.$$

In this system the amplitudes M_r and M_a can be

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[4] H. Behnke and P. Thullen, *Theorie der Funktionen mehrerer komplexer Veränderlichen*, Ergebnisse der Mathematische Wissenschaften (Verlag Julius Springer, Berlin and Chelsea Publishing Company, New York, 1934), Vol. 3, No. 3.

[5] S. Bochner and W. T. Martin, *Several Complex Variables* (Princeton University Press, Princeton, 1948).

[6] Hans J. Bremermann, Math. Ann. **127**, 406 (1954).

[7] Hans J. Bremermann, "Construction of the Envelopes of Holomorphy of Arbitrary Domains" (to be published).

[8] We would like to thank Professor R. Jost and Professor H. Lehmann for enlightening discussions concerning this point.

[9] See, for example, Lehmann *et al.* and Bogoliubov *et al.*, reference 2.

written as

$$M_{r,a}(\omega,\Delta^2)=D(\omega,\Delta^2)\pm iA(\omega,\Delta^2)$$

$$=\pm 2E_\Delta i\int d^4x\,\exp[i\omega x_0-i(\omega^2-E_\Delta{}^2)^{\frac{1}{2}}\mathbf{e}\cdot\mathbf{x}]$$

$$\times\eta(\pm x_0)\langle p_\Delta|[j(\tfrac{1}{2}x),\,j(-\tfrac{1}{2}x)]|p_{-\Delta}\rangle$$

$$+\sum_{n=0}^{N-1}C_n(\Delta^2)\omega^{2n}. \quad (2.5)$$

One can easily prove that for fixed Δ^2 the dispersive part is an even function and the absorptive part an odd function of ω.

We assume that the matrix element of the commutator is a tempered distribution in x, but only derivatives of finite order of the δ functions shall appear. As far as the singularities on the light cone are concerned, the latter property is already guaranteed by the causality condition, which implies that the commutator vanishes for $x^2=x_0{}^2-\mathbf{x}^2<0$. The real polynomial in ω^2 has its origin essentially in the possible appearance of $\epsilon(x_0)$ times $\delta^{(n)}(x^2)$ in the integrand of the dispersive part, which is then not defined for $x=0$. This fact introduces a certain arbitrariness in the Fourier transform, which is just expressed by the polynomial with arbitrary coefficients.

From the assumptions we have made about the matrix element in Eq. (2.5) it is clear that these representations of M_r and M_a are defined only for $\omega^2\geq E_\Delta{}^2$. Following Bogoliubov,[10] we consider therefore the amplitudes as functions of an additional variable β such that we have, for $\omega^2>\beta$,

$$M_{r,a}(\omega,\beta,\Delta^2)=\pm 2E_\Delta i\int d^4x\,\exp[i\omega x_0-i(\omega^2-\beta)^{\frac{1}{2}}\mathbf{e}\cdot\mathbf{x}]$$

$$\times\eta(\pm x_0)\langle p_\Delta|[j(\tfrac{1}{2}x),\,j(-\tfrac{1}{2}x)]|p_{-\Delta}\rangle$$

$$+\sum_{n=0}^{N-1}C_n(\beta,\Delta^2)\omega^{2n}. \quad (2.6)$$

We note that the appearance of $\exp[i(\omega^2-\beta)^{\frac{1}{2}}\mathbf{e}\cdot\mathbf{x}]$ in Eq. (2.6) causes no branch points for $M_{r,a}$ as functions of β, since the symmetry properties of the amplitudes allow its replacement by $\cos[(\omega^2-\beta)^{\frac{1}{2}}\mathbf{e}\cdot\mathbf{x}]$.

Before discussing the analytic properties of $M_{r,a}$ let us explore the absorptive part $A(\omega,\beta,\Delta^2)$. For $\beta<0$ the representation

$$A(\omega,\beta,\Delta^2)=2E_\Delta(\tfrac{1}{2})\int d^4x\,\exp[i\omega x_0-i(\omega^2-\beta)^{\frac{1}{2}}\mathbf{e}\cdot\mathbf{x}]$$

$$\times\langle p_\Delta|[j(\tfrac{1}{2}x),\,j(-\tfrac{1}{2}x)]|p_{-\Delta}\rangle \quad (2.7)$$

is defined for all ω. We assume the existence of a complete set of eigenstates of the energy-momentum operator corresponding to non-negative values of the energy. Then we may decompose the matrix element in Eq. (2.7) with respect to these states. Using translation invariance we find a spectral representation of the form

$$A(\omega,\beta,\Delta^2)=\int_0^\infty d\sigma^2\rho(\sigma^2,\beta-\Delta^2,\Delta^2)[\delta(\sigma^2-E_\Delta{}^2-\beta-2E_\Delta\omega)-\delta(\sigma^2-E_\Delta{}^2-\beta+2E_\Delta\omega)], \quad (2.8)$$

where

$$\begin{aligned}\rho(\sigma^2,\beta-\Delta^2,\Delta^2)&=\pi\textstyle\sum_n(2E_\Delta)^{\frac{1}{2}}\langle p_\Delta|j(0)|\mathbf{p}_n\\&=\mathbf{e}[\omega_\beta{}^2(\sigma)-\beta]^{\frac{1}{2}},n\rangle(2p_{n0})^{\frac{1}{2}}(2p_{n0})^{\frac{1}{2}}\langle\mathbf{p}_n\\&=\mathbf{e}[\omega_\beta{}^2(\sigma)-\beta]^{\frac{1}{2}},n|j(0)|p_{-\Delta}\rangle(2E_\Delta)^{\frac{1}{2}}\delta(p_n{}^2-\sigma^2), \quad (2.9)\end{aligned}$$

and

$$\omega_\beta(\sigma)=(1/2E_\Delta)(\sigma^2-E_\Delta{}^2-\beta). \quad (2.10)$$

Here σ denotes the "mass" of the system described by $|p_n,n\rangle$, i.e., the total energy in its own rest frame. The summation extends over all possible intermediate states. We have a continuum of intermediate states with $\sigma\geq 2m$ and a discrete one meson state at $\sigma=m$. For reasons of simplicity let us assume here that there are no other discrete states (e.g., bound states) or continua for $\sigma<2m$. More general cases will be discussed in Sec. 3. Under these assumptions about the spectrum, we find that $\rho(\sigma^2,\beta-\Delta^2,\Delta^2)$ may be written in the form

$$\rho(\sigma^2,\beta-\Delta^2,\Delta^2)=\pi m^2g^2(\beta-\Delta^2)\delta(\sigma^2-m^2)+\Theta(\sigma^2,\beta-\Delta^2,\Delta^2), \quad (2.11)$$

where $\Theta(\sigma^2,\beta-\Delta^2,\Delta^2)\equiv 0$ for $\sigma<2m$. On the basis of Lorentz invariance it is easily seen from Eq. (2.9) that ρ must be a real, non-negative quantity. Furthermore, one finds that the function $g^2(\beta-\Delta^2)$, appearing in the contribution from the one-meson state, can *only* depend on $\beta-\Delta^2$.

We see from Eq. (2.6) that for $\beta<0$ and fixed Δ^2 the amplitudes M_r and M_a are analytic functions of ω for $\mathrm{Im}\omega>0$ and $\mathrm{Im}\omega<0$, respectively. This is a consequence of the causality condition. The amplitudes M_r and M_a for real $\omega=\omega_r$ are obtained by the improper limits

$$\lim_{\epsilon\to 0+}M_{r,a}(\omega_r\pm i\epsilon,\beta,\Delta^2)=M_{r,a}(\omega_r,\beta,\Delta^2).$$

If we take $\beta<-\Delta^2$, we always have a finite gap on the real ω axis where $A(\omega,\beta,\Delta^2)\equiv 0$. Then M_r and M_a are analytic continuations of each other and we have *one* analytic function $M(\omega,\beta,\Delta^2)$, which is regular in the cut ω plane except for a pair of poles at $\omega=\pm\omega_\beta(m)$. The cuts run from $\pm\omega_\beta(2m)$ to $\pm\infty$. From the assumptions we have made about the matrix element of the commutator it follows that for $\omega\to\infty$ ($\mathrm{Im}\omega\neq 0$), $M(\omega,\beta,\Delta^2)$ does not increase stronger than a certain polynomial. Suppose it vanishes in that limit like $1/\omega$. Then we have for $\beta<-\Delta^2$, $\mathrm{Im}\omega\neq 0$, using the Cauchy

[10] Bogoliubov *et al.*, reference 2.

theorem and Eqs. (2.4), (2.8), and (2.11),

$$M(\omega,\beta,\Delta^2)=\frac{1}{\pi}\int_{-\infty}^{+\infty}d\omega'\,\frac{A(\omega',\beta,\Delta^2)}{\omega'-\omega}$$
$$=\frac{1}{\pi}\int_{\omega_\beta(2m)+\beta/2E_\Delta}^{\infty}d\omega'\,\Theta(2E_\Delta\omega'+E_\Delta{}^2,\,\beta-\Delta^2,\,\Delta^2)\times\left[\frac{1}{\omega'-\omega-\beta/2E_\Delta}+\frac{1}{\omega'+\omega-\beta/2E_\Delta}\right]+g^2(\beta-\Delta^2)\frac{h(\beta,\Delta^2)}{\omega_\beta{}^2(m)-\omega^2},\quad(2.12)$$

where $h(\beta,\Delta^2)\equiv-m^2(\beta+\Delta^2)/(4E_\Delta{}^2)$. Note that the lower limit of the integral is independent of β. In the general case we have to supply sufficient powers of ω' in the denominator, which leads to the appearance of an additional polynomial in Eq. (2.12). Since these complications do not cause any principal difficulties in the proof of dispersion formulas, we shall not consider them in detail.

We now wish to continue both sides of Eq. (2.12) from $\beta<-\Delta^2$ to $\beta=E_\Delta{}^2$. To do this we must know the analytic properties of $\Theta(\sigma^2,\beta-\Delta^2,\Delta^2)$ as a function of β. In Sec. 3 we will prove that there exists a $\delta>0$ such that for all $\sigma^2\geq(2m)^2$ and $\Delta^2<\Delta_{\max}{}^2$, $\Theta(\sigma^2,\beta-\Delta^2,\Delta^2)$ is an analytic function of β for $\beta\epsilon S$ (by the notation $a\epsilon A$ we mean that the point a belongs to the set A), where S is the strip

$$S=[\beta:|\mathrm{Im}\beta|<\delta,\ -R<\mathrm{Re}\beta<E_\Delta{}^2+\delta];\quad(2.13)$$

(by "$[a:\cdots]$" we denote "the set of all a which satisfy the condition$\cdots$"), R is any positive number, and δ may go to zero as $R\to\infty$; for all $\beta\epsilon S$ we have $\Theta\equiv0$ for $\sigma^2<(2m)^2$. We know that Θ is a real function of β for real $\beta=\beta_r<0$. Since Θ is analytic in β for $\beta\epsilon S$, it must then also be real for real $\beta=\beta_r\leq E_\Delta{}^2$, $\Delta^2<\Delta_{\max}{}^2$. The limitation on Δ^2 will be discussed in the next section. *In the following we assume only that* $\Delta_{\max}{}^2$ *is finite.*

Using these properties of Θ, we proceed to show that the function $g^2(\beta-\Delta^2)$ must be analytic for $\beta\epsilon S$. Let us denote the integral in Eq. (2.12) by $I(\omega,\beta,\Delta^2)$. If we write Eq. (2.12) in the form

$$[\omega_\beta{}^2(m)-\omega^2][M(\omega,\beta,\Delta^2)-I(\omega,\beta,\Delta^2)]=g^2(\beta-\Delta^2)h(\beta,\Delta^2),\quad(2.14)$$

then the left-hand side is an analytic function of the two complex variables ω and β for $(\omega,\beta)\epsilon D_1$ with

$$D_1=[(\omega,\beta):|\mathrm{Im}\omega|>|\mathrm{Im}(\omega^2-\beta)^{\frac{1}{2}}|,\ 2E_\Delta|\mathrm{Im}\omega|>|\mathrm{Im}\beta|,\ \beta\epsilon S].\quad(2.15)$$

The domain D_1 is the intersection of regions in which $M(\omega,\beta,\Delta^2)$ and $I(\omega,\beta,\Delta^2)$ are analytic functions of ω and β. First we prove that $g^2(\beta-\Delta^2)$ is analytic for $\mathrm{Im}\beta\neq0$, $\beta\epsilon S$, by constructing for each of these points β an ω such that $(\omega,\beta)\epsilon D_1$. We write $\omega=\omega_r+i\omega_i$, $\beta=\beta_r+i\beta_i$, and take $\omega_i\neq0$, $|\omega_i|>|\beta_i|/2E_\Delta$, and $\omega_r>\omega_i\beta_r/\beta_i+\beta_i/4\omega_i$, which is evidently always possible and guarantees that $(\omega,\beta)\epsilon D_1$ for $\beta_i\neq0$, $\beta\epsilon S$. Hence $g^2(\beta-\Delta^2)$ is analytic for $\beta\epsilon S$ except for a possible cut on the real β axis. In order to show that there is actually no such cut, we prove that

$$\lim_{\epsilon\to0+}[g^2(\beta_r+i\epsilon-\Delta^2)-g^2(\beta_r-i\epsilon-\Delta^2)]=0\quad(2.16)$$

for $\beta_r\pm i\epsilon=\beta_\pm\epsilon S$. Let us define $\omega_\pm$ by $\omega_\pm=\omega_r\pm i\epsilon/E_\Delta$. Then $(\omega_\pm,\beta_\pm)\epsilon D_1$ provided we choose $\omega_r>(5/4)E_\Delta+\delta/E_\Delta$ and $\epsilon>0$ and sufficiently small. But for $(\omega,\beta)\epsilon D_1$ we may use the representation (2.6) for $M(\omega,\beta,\Delta^2)$ and find, recalling Eq. (2.4),

$$\lim_{\epsilon\to0+}[M(\omega_+,\beta_+,\Delta^2)-M(\omega_-,\beta_-,\Delta^2)]=2iA(\omega_r,\beta_r,\Delta^2);\quad(2.17)$$

in addition it follows from Eq. (2.12) that

$$\lim_{\epsilon\to0+}[I(\omega_+,\beta_+,\Delta^2)-I(\omega_-,\beta_-,\Delta^2)]=2i[\Theta(2E_\Delta\omega_r+E_\Delta{}^2+\beta_r,\,\beta_r-\Delta^2,\,\Delta^2)-\Theta(-2E_\Delta\omega_r+E_\Delta{}^2+\beta_r,\,\beta_r-\Delta^2,\,\Delta^2)].\quad(2.18)$$

Evidently Eqs. (2.14), (2.17), and (2.18) imply Eq. (2.16). We conclude that $g^2(\beta-\Delta^2)$ may be continued analytically onto the domain S. Since we know from Eq. (2.9) that $g^2(\beta-\Delta^2)$ is real for $\beta=\beta_r<0$, the same must be true for all $\beta\epsilon S$ with $\mathrm{Im}\beta=0$.

If we relax the assumption about the behavior of M for $\omega\to\infty$, we will have on the right-hand side of Eq. (2.14) in addition to the g^2 term a polynomial of the form

$$\sum_{n=0}^{N-1}C_n(\beta,\Delta^2)\omega^{2n}.$$

The proof that the coefficients C_n are analytic functions of β for $\beta\epsilon S$ is completely equivalent to the proof of the corresponding properties for $g^2(\beta-\Delta^2)$.

For the application of dispersion relations it is important to show that $g^2(m)$ is non-negative. In order to prove this we consider the invariant matrix element

$$(2k_0)^{\frac{1}{2}}\langle k|j(0)|p\rangle(2p_0)^{\frac{1}{2}}=m\Gamma[(p-k)^2],\quad(k^2=p^2=m^2),\quad(2.19)$$

in a system where $\mathbf{p}=\mathbf{0}$. From the representation

$$\Gamma[2m(m-k_0)]=-i\left(\frac{2}{m}\right)^{\frac{1}{2}}\int d^4x\,\exp(ik\cdot x)\eta(x_0)\times\langle0|[j(x),j(0)]|0,m\rangle+\sum_{n=0}^{N-1}a_nk_0{}^n,\quad(2.20)$$

and our assumptions about the spectrum one can easily see, using the methods described earlier by one of us (R. O.),[11] that Γ is an analytic function of $\lambda=2m(m-k_0)$ in the cut λ plane, the cut runs from $(2m)^2$ to $+\infty$. Then the function $\Gamma(\lambda)$ is real for $\lambda_i=0$, $\lambda_r<(2m)^2$, since it is for $\lambda_r<0$. Evidently $\Gamma^2(\lambda)$ has the same properties; in addition it is non-negative for $\lambda=\lambda_r<(2m)^2$. We see from Eqs. (2.9), (2.10), and (2.11) that $g^2(\beta-\Delta^2)=\Gamma^2(\lambda)$ with $\lambda=\beta-\Delta^2$. Hence we have shown that $g^2(m)\geq 0$. We could have used the properties of $\Gamma(\lambda)$ mentioned above in order to demonstrate that $g^2(\beta-\Delta^2)$ is analytic for $\beta\epsilon S$. However, the method described earlier is still useful in order to prove the analyticity of the coefficients $C_n(\beta,\Delta^2)$ of the polynomial. Let us now go back to Eq. (2.12). Since Θ and g^2 are analytic functions of β for $\beta\epsilon S$ we realize that the right-hand side of this equation is analytic in ω, β for $(\omega,\beta)\epsilon D_2$, where

$$D_2=[(\omega,\beta):2E_\Delta|\mathrm{Im}\omega|>|\mathrm{Im}\beta|,\beta\epsilon S]. \quad (2.21)$$

Hence we can continue $M(\omega,\beta,\Delta^2)$ into the domain D_2 so as to equal $I(\omega,\beta,\Delta^2)+g^2(\beta-\Delta^2)h(\beta,\Delta^2)[\omega_\beta{}^2(m)-\omega^2]^{-1}$. But any (ω,β) with $\beta=E_\Delta{}^2$, $\mathrm{Im}\omega\neq 0$ is contained in D_2 and hence we obtain from Eq. (2.12) the Hilbert relation

$$M(\omega,E_\Delta{}^2,\Delta^2)=\frac{1}{\pi}\int_{\omega(2m)}^{E_\Delta}d\omega'\Theta(2E_\Delta\omega'+2E_\Delta{}^2,m^2,\Delta^2)\times\left[\frac{1}{\omega'-\omega}+\frac{1}{\omega'+\omega}\right]+\frac{2}{\pi}\int_{E_\Delta}^{\infty}d\omega'\frac{\omega' A(\omega',\Delta^2)}{\omega'^2-\omega^2}+\frac{g^2(m^2)m^2}{2E_\Delta}\left[\frac{1}{\omega(m)-\omega}+\frac{1}{\omega(m)+\omega}\right], \quad (2.22)$$

where $\mathrm{Im}\omega\neq 0$, $\omega(\sigma)=\omega_\beta(\sigma)$ for $\beta=E_\Delta{}^2$ and $A(\omega',\Delta^2)\equiv A(\omega',E_\Delta{}^2,\Delta^2)$. For the M thus continued, it remains to be proved that the improper limits

$$\lim_{\epsilon\to 0+}M(\omega_r\pm i\epsilon,E_\Delta{}^2,\Delta^2)=M_{r,a}(\omega_r,\Delta^2)$$

hold, provided $\omega_r>E_\Delta$. For each $\omega_r{}^2>E_\Delta{}^2+\epsilon^2$ and $\beta=E_\Delta{}^2+2ia\epsilon$, we can always find an a such that $\omega_r-(\omega_r{}^2-E_\Delta{}^2)^{\frac{1}{2}}<a<E_\Delta$. But then we have, for ϵ small enough, $(\omega_r\pm i\epsilon, E_\Delta{}^2\pm 2ia\epsilon)\epsilon D_1\subset D_2$ and consequently

$$\lim_{\epsilon\to 0+}M(\omega_r\pm i\epsilon,E_\Delta{}^2,\Delta^2)=\lim_{\epsilon\to 0+}M(\omega_r\pm i\epsilon,E_\Delta{}^2\pm i2a\epsilon,\Delta^2)=M_{r,a}(\omega_r,\Delta^2), \quad (2.23)$$

because for $(\omega,\beta)\epsilon D_1$ the Fourier representations (2.6) for M_r and M_a are valid. (By $B\supset A$ or $A\subset B$ we denote the fact that all points of A belong also to B.)

Equation (2.22) represents the desired dispersion formula for $M(\omega,\Delta^2)$. The function $\Theta(2E_\Delta\omega'+2E_\Delta{}^2,m^2,\Delta^2)$ appearing on the first integral is the proper extension of the absorptive part on the unphysical region.

[11] Reinhard Oehme, Nuovo cimento **10**, 1316 (1956).

3. ANALYTIC CONTINUATION

In this section we wish to prove that for every fixed $\sigma^2\geq(2m)^2$ and $\Delta^2<\Delta_{\max}{}^2$ the quantity $\Theta(\sigma^2,\beta-\Delta^2,\Delta^2)$ is an analytic function in β for $\beta\epsilon S$, where S is the region defined by Eq. (2.13). Since $\sigma^2\geq(2m)^2$ it is sufficient to show that

$$(\sigma^2-m^2)\Theta(\sigma^2,\beta-\Delta^2,\Delta^2)=(\sigma^2-m^2)\rho(\sigma^2,\beta-\Delta^2,\Delta^2)$$

has the required properties. The quantity ρ is related to $A(\omega,\beta,\Delta^2)$ by

$$A(\omega,\beta,\Delta^2)=\bar{A}(\omega,\beta,\Delta^2)-\bar{A}(-\omega,\beta,\Delta^2), \quad (3.1)$$

with

$$\bar{A}(\omega,\beta,\Delta^2)=\rho(2E_\Delta\omega+E_\Delta{}^2+\beta,\beta-\Delta^2,\Delta^2).$$

Let us define a function $A(p_1p_2p_3p_4)$ by the equation

$$A(p_1p_2p_3p_4)=A(k+k',p',p), \quad (3.2)$$

where $p_1=p'$, $p_2=-p$, $p_3=k'$, $p_4=-k$. For reasons of simplicity we have not introduced a new symbol in Eq. (3.2). By standard methods[10] we have then the representation

$$(2\pi)^4\delta(p_1+p_2+p_3+p_4)A(p_1p_2p_3p_4)=\tfrac{1}{2}\int dx_1dx_2dx_3dx_4\exp[i(p_1x_1+p_2x_2+p_3x_3+p_4x_4)]F(x_1x_2x_3x_4), \quad (3.3)$$

where

$$F(x_1x_2x_3x_4)=\left\langle 0\left|\frac{\delta^2}{\delta\phi(x_1)\delta\phi(x_2)}[j(x_3),j(x_4)]\right|0\right\rangle+\text{degenerate terms}. \quad (3.4)$$

The operator $\delta/\delta\phi(x)$ is that of a functional derivation with respect to the boson field at the point x, where this field is regarded as a classical one while the derivation is being performed. We use it here only as a convenient shorthand and the expectation value F may be easily written out in terms of commutators and step functions. The "degenerate terms" in Eq. (3.4) contain equal time commutators; they do not alter the properties of $A(p_1\cdots p_4)$ in which we are interested, and therefore we do not need to consider them explicitly.

Carrying out the functional derivations in Eq. (3.4), we obtain

$$F(x_1x_2x_3x_4)=\bar{F}(x_1x_2x_3x_4)-\bar{F}(x_1x_2x_4x_3),$$

and correspondingly

$$A(p_1p_2p_3p_4)=\bar{A}(p_1p_2p_3p_4)-\bar{A}(p_1p_2p_4p_3),$$

where

$$F(x_1x_2x_3x_4)=\left\langle 0\left|\left[\frac{\delta j(x_3)}{\delta\phi(x_1)},\frac{\delta j(x_4)}{\delta\phi(x_2)}\right]\right|0\right\rangle+\left\langle 0\left|\left[j(x_3),\frac{\delta^2 j(x_4)}{\delta\phi(x_1)\delta\phi(x_2)}\right]\right|0\right\rangle. \quad (3.5)$$

The condition on the spectrum, which we have discussed in Sec. 2, implies that the Fourier transform of $\langle 0|[j(x_3),\delta^2 j(x_4)/\delta\phi(x_1)\delta\phi(x_2)]|0\rangle$ is zero for $p_3^2<(2m)^2$. Since we are only concerned with the region $p_3^2<(2m)^2$, $p_4^2<(2m)^2$ throughout this paper, we may, without error, take as zero the second term on the right-hand side of Eq. (3.5).

An expression for $j(x)$, equivalent to that used in the last section, is

$$j(x)=i\frac{\delta S}{\delta\phi(x)}S^{+}.$$

The causality condition may now be written as

$$\delta j(x)/\delta\phi(y)=-i\eta(y_0-x_0)[j(y),j(x)]=0 \quad \text{for} \quad (y_0-x_0)<|\mathbf{y}-\mathbf{x}|. \tag{3.6}$$

From the causality condition it follows that $\bar{F}(x_1\cdots x_4)$ is retarded in the variables (x_1-x_3) and (x_2-x_4), and so will be denoted by $\bar{F}_{rr}(x_1\cdots x_4)$. The four functions which are obtained from $\bar{F}_{rr}(x_1\cdots x_4)$ by interchange of the variables x_1 and x_3, x_2 and x_4 any number of times will be denoted ky $\bar{F}_{ij}(x_1x_2x_3x_4)$, with $i, j=r, a$. The subscripts i, j correspond to the advanced or retarded property in the variables (x_1-x_3), (x_2-x_4), respectively. For example, we have

$$\bar{F}_{ar}(x_1x_2x_3x_4)=\left\langle 0\left|\left[\frac{\delta j(x_1)}{\delta\phi(x_3)},\frac{\delta j(x_4)}{\delta\phi(x_2)}\right]\right|0\right\rangle.$$

By the use of the relation

$$\frac{\delta j(x)}{\delta\phi(y)}-\frac{\delta j(y)}{\delta\phi(x)}=i[j(x),j(y)],$$

and the condition on the spectrum, we obtain

$$\begin{aligned}\bar{A}_{rj}(p_1\cdots p_4)-\bar{A}_{aj}(p_1\cdots p_4)&=0\\ &\text{for} \quad p_1^2<(2m)^2, \quad p_3^2<(2m)^2\\ \bar{A}_{ir}(p_1\cdots p_4)-\bar{A}_{ia}(p_1\cdots p_4)&=0\\ &\text{for} \quad p_2^2<(2m)^2, \quad p_4^2<(2m)^2.\end{aligned} \tag{3.7}$$

Also, by the same condition,

$$[(p_1+p_3)^2-m^2]\bar{A}_{ij}(p_1\cdots p_4)=0 \tag{3.8}$$

for $(p_1+p_3)^2<(2m)^2$ or $p_{10}+p_{30}<2m$, and all $i, j=r, a$.

We consider from now on only the function $\bar{B}_{ij}$, $\bar{G}_{ij}$ which are defined by the expressions

$$\bar{B}_{ij}(p_1\cdots p_4)=[(p_1+p_3)^2-m^2]\bar{A}_{ij}(p_1\cdots p_4), \tag{3.9}$$

$$\bar{G}_{ij}(x_1\cdots x_4)=\left[-\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_3}\right)^2-m^2\right]\bar{F}_{ij}(x_1\cdots x_4). \tag{3.10}$$

The analyticity property of $\Theta(\sigma^2, \beta-\Delta^2, \Delta^2)$ which we desire is then proved by Theorem 1[3] (for $\Delta_{\max}^2=m^2$).

Theorem 1.—We are given four generalized functions of the four-vector variables $x_1, \cdots, x_4$,

$$\tilde{f}_{ij}(x_1x_2x_3x_4), \quad i=r, a, \quad j=r, a.$$

The $\tilde{f}_{ij}$ are assumed to be tempered distributions, multiplied by certain step functions, so that their Fourier transforms are defined to within certain arbitrary polynomials. The quantities f_{ij} are assumed invariant under the transformations of the inhomogeneous, orthochronous Lorentz group, and are retarded or advanced in (x_1-x_3), (x_2-x_4) as denoted by the subscripts i, j. The Fourier transforms of the f_{ij}, defined by the expressions

$$(2\pi)^4 f_{ij}(p_1p_2p_3p_4)\delta(p_1+p_2+p_3+p_4)=\int dx_1dx_2dx_3dx_4 \exp[i(p_1x_1+p_2x_2+p_3x_3+p_4x_4)]\tilde{f}_{ij}(x_1x_2x_3x_4),$$

are assumed to have the properties:

$$\begin{aligned}&f_{rj}-f_{aj}\equiv 0 \text{ for } p_1^2<(2m)^2, \text{ and } p_3^2<(2m)^2, \text{ and } j=r \text{ or } a;\\ &f_{ir}-f_{ia}\equiv 0 \text{ for } p_2^2<(2m)^2, \text{ and } p_4^2<(2m)^2, \text{ and } i=r \text{ or } a;\\ &f_{ij}\equiv 0 \text{ for } (p_1+p_3)^2<(2m)^2 \text{ or } (p_{10}+p_{30})<0, \quad i=r \text{ or } a, \quad j=r \text{ or } a.\end{aligned} \tag{3.11}$$

Then we wish to prove that there is a function $\chi(z_1,z_2,z_3,z_4,z_5;z_6)$ of the complex variables $z_1, \cdots, z_5$ and the real variable z_6, which is in general a tempered distribution in z_6 and which has the following properties: (1) For each real z_6, $\chi(z_1,\cdots,z_6)$ is analytic in $z_1, \cdots, z_5$ in the region D,

$$D=[z_1, \cdots, z_5: |z_1-m^2|<\delta, |z_2-m^2|<\delta, |z_3-\gamma|<\delta, |z_4-\gamma|<\delta, |z_5+4\Delta^2|<\delta], \tag{3.12}$$

where $-R\leq\gamma Mm^2$, for any positive number R, and where δ is some small positive number, which may become zero as $R\rightarrow\infty$. We require also $\Delta^2<m^2$. (2) For $p_1, \cdots, p_4$ real, $p_1+p_2+p_3+p_4=0$, $p_{10}+p_{30}>0$, $z_1=p_1^2$, $z_2=p_2^2$, $z_3=p_3^2$, $z_4=p_4^2$, $z_5=(p_1+p_2)^2$, and $z_6=(p_1+p_3)^2$, with $(z_1,\cdots,z_5)\epsilon D$, we have the representation

$$f_{ij}(p_1p_2p_3p_4)=\chi(z_1,z_2,z_3,z_4,z_5;z_6).$$

(3) $\chi\equiv 0$ for $z_6<(2m)^2$.

Proof.—For convenience we introduce the independent four vectors q_1, q_2, and q_3, by

$$p_1=q_1+q_3, \quad p_2=-q_2-q_3, \quad p_3=-q_1+q_3, \quad p_4=q_2-q_3.$$

Then, writing $g_{ij}(q_1q_2q_3)=f_{ij'}(p_1p_2p_3p_4)$, where for $j=r$ then $j'=a$, and for $j=a$ then $j'=r$, we have

$$g_{ij}(q_1q_2q_3)=\int dy_1dy_2dy_3 \exp[i(q_1y_1+q_2y_2+q_3y_3)]\tilde{g}_{ij}(y_1y_2y_3), \tag{3.13}$$

where

$$\tilde{f}_{ij'}(x_1x_2x_3x_4)=\tilde{g}_{ij}(y_1y_2y_3)$$

at $y_1=x_1-x_3$, $y_2=-(x_2-x_4)$, $y_3=(x_1-x_2+x_3-x_4)$.

From Eq. (3.13) and the retarded and advanced character of the g_{ij} we see that the functions g_{ij} are analytic in certain regions. If we take these functions as one function $g(q_1q_2q_3)$, then $g(q_1q_2q_3)$ is analytic in the region $(q_1,q_2)\epsilon W\times W$ for each real q_3, where

$$W=[q: |\mathrm{Im}q_0|>|\mathrm{Im}\mathbf{q}|,\ |\mathrm{Re}q_0|<\infty,\ |\mathrm{Re}\mathbf{q}|<\infty].$$

[Note that $q_3=\frac{1}{2}(p_1+p_3)$ is always real.]

Let S be a set in the four-dimensional real space R_4, which is defined by

$$S=[q: \mathrm{Im}q=0,\ (\mathrm{Re}q+q_3)^2<(2m)^2,\ (\mathrm{Re}q-q_3)^2<(2m)^2]$$

for each q_3. Then the equality properties between the various g_{ij} for real q_1, q_2, and q_1, or q_2 in S, which correspond to Eq. (3.11) for the f_{ij}, may be immediately extended to equalities satisfied for q_1 in S and q_2 in $W\cup R_4$ or vice versa. (By $A\cup B$ we denote the union of the sets A and B.) Explicitly, we have

$$\begin{aligned} &g_{rj}(q_1q_2q_3)-g_{aj}(q_1q_2q_3)\equiv 0 \\ &\qquad\text{for}\quad q_1\epsilon S\quad\text{and}\quad q_2\epsilon W\cup R_4, \\ &g_{ir}(q_1q_2q_3)-g_{ia}(q_1q_2q_3)\equiv 0 \\ &\qquad\text{for}\quad q_2\epsilon S\quad\text{and}\quad q_1\epsilon W\cup R_4, \\ &g_{ij}(q_1q_2q_3)\equiv 0\quad\text{for}\quad q_3{}^2<m^2, \\ &\qquad (q_1,q_2)\epsilon(W\cup R_4)\times(W\cup R_4). \end{aligned}\tag{3.14}$$

We thus see, that, for fixed q_3, the function $g(q_1q_2q_3)$ satisfies the conditions of the edge of the wedge theorem, which we have formulated in the Appendix. Hence by this theorem, we may continue $g(q_1q_2q_3)$ in (q_1,q_2) to be analytic in the region $(W\cup N)\times(W\cup N)$, where N is some neighborhood of the set S.

Since $q_3{}^2\geq m^2$, we may now, without loss of generality, choose a frame of reference in which $\mathbf{q}_3=0$ and write $q_{30}=t$. Then, for a given t, g is a function of q_{10}, q_{20}, and the two 3-vectors $\mathbf{q}_1$, $\mathbf{q}_2$. With respect to $\mathbf{q}_1$ and $\mathbf{q}_2$ it is invariant under the transformations of the orthogonal group and analytic in a region which is also invariant. One can prove that g depends only upon the inner products $\mathbf{q}_1{}^2$, $\mathbf{q}_2{}^2$, $\mathbf{q}_1\cdot\mathbf{q}_2$, and of course upon q_{10} and q_{20}. It will be analytic in these variables in the domain corresponding to $(q_1,q_2)\epsilon(W\cup N)\times(W\cup N)$. The proof in question makes use of the compactness of the orthogonal group and invariance of the Haar measure.[12] Instead of the variables q_{10}, q_{20}, $\mathbf{q}_1{}^2$, $\mathbf{q}_2{}^2$, $\mathbf{q}_1\cdot\mathbf{q}_2$ we choose to consider the variables z_1, z_2, z_3, z_4, z_5, which are defined by the equations

$$z_1=(q_1+q_3)^2,\ z_2=(q_2+q_3)^2,\ z_3=(q_1-q_3)^2,\ z_4=(q_2-q_3)^2,$$
$$z_5=(q_1-q_2)^2,\ z_6=4q_3{}^2\ (q_{30}=t,\ \mathbf{q}_3=0),$$

and are related to the above-mentioned variables by a simple analytic transformation. We write $g=\chi(z_1,\cdots,z_5;z_6)$, where χ is analytic in $z_1,\ \cdots,\ z_5$, for each fixed, real z_6, in the domain corresponding to $(W\cup N)\times(W\cup N)$.

[12] We are indebted to Professor L. Ehrenpreis for discussions concerning this proof.

We are interested in the behavior of χ near the points

$$z_1=m^2,\ z_2=m^2,\ z_3=\gamma,\ z_4=\gamma,\ z_5=-4\Delta^2,\ z_6=4t^2,\tag{3.15}$$

where $-R<\gamma\leq m^2$ and $t\geq m$. At these values of $z_1,\ \cdots,\ z_6$ we have

$$\begin{aligned} q_{10}=q_{20}&=(m^2-\gamma)/4t, \\ \mathbf{q}_1&=\rho(t,\gamma)\mathbf{e}_1+\Delta\mathbf{e}_2, \\ \mathbf{q}_2&=\rho(t,\gamma)\mathbf{e}_1-\Delta\mathbf{e}_2, \end{aligned}\tag{3.16}$$

where $\mathbf{e}_1\cdot\mathbf{e}_2=0$ and $\rho^2(t,\gamma)=[t+(m^2-\gamma)/4t]^2-E_\Delta{}^2$. We wish to show that for $\gamma\leq m^2$, for each $t\geq m$ and $\Delta^2<m^2$, the values of (q_1,q_2) lie in the region of analyticity of $g(q_1q_2q_3)$. This will show that the corresponding value of $z_1,\ \cdots,\ z_5$ lies in the region of analyticity of $\chi(z_1,\cdots,z_5;z_6)$. Let us first take a small $\bar{\delta}>0$ and assume that $\Delta^2<\bar{\delta}^2$. Then we see from Eq. (3.16) that for $\gamma\leq m^2$, $t\geq m$ the inequalities $|\mathrm{Im}\mathbf{q}_i|<\bar{\delta}$, $(\mathrm{Re}q_{i0}\pm t)^2-\mathbf{q}_i{}^2\leq m^2$ hold with $i=1$, 2, and if $\bar{\delta}$ is sufficiently small $(\bar{\delta}<\delta)$ the 4-vectors q_1 and q_2 both lie in N. We see that the edge of the wedge theorem is sufficient to prove the theorem for $0\leq\Delta^2<\bar{\delta}^2$ and hence the dispersion relations for forward and near-forward scattering (derivative amplitudes).[13]

For general Δ^2 it is true that *all* relevant, corresponding values of (q_1,q_2) lie in N for each $t\geq E_\Delta$; but this is no longer the case for $t<E_\Delta$. Hence we need to extend the region of analyticity of g. To consider this in more detail we have drawn in Fig. 1 the curve (III) for

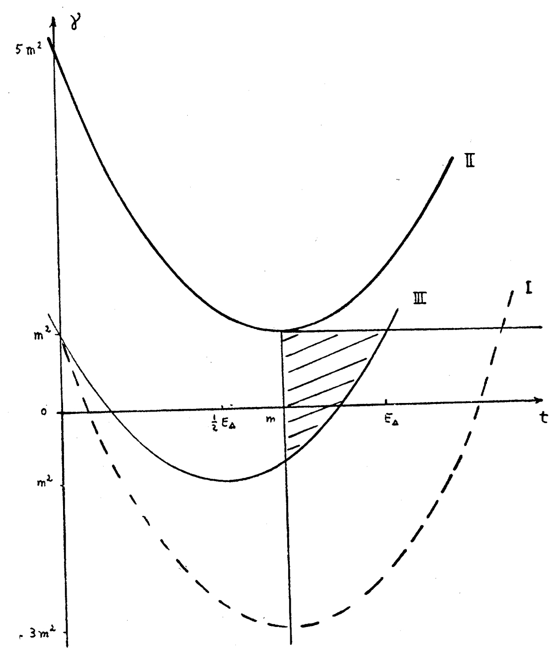

FIG. 1. This figure is drawn for $\Delta^2=m^2$. The region of interest is given by $t\geq m$, $\gamma\leq m^2$. The corresponding values of (q_1,q_2) lie in the region of analyticity given by the semitube method (shaded region) or the edge of the wedge theorem (unshaded region).

[13] K. Symanzik, Phys. Rev. **105**, 743 (1957). This paper contains further references.

$\rho^2(t,\gamma)=0$. In the region defined by $\rho^2(t,\gamma)>-\bar{\delta}^2$, which is given by

$$[t-\tfrac{1}{2}(E_\Delta{}^2-\bar{\delta}^2)^{\frac{1}{2}}]^2>\tfrac{1}{4}(\Delta^2-\bar{\delta}^2+\gamma),$$

we always have the corresponding values of q_1, q_2 lying in N, while for other values of t, γ this is not true. This latter region is completely contained within the region $m\leq t<E_\Delta$, $\gamma\leq m^2$.

The ideal solution of the problem would be to obtain the envelope of holomorphy of the domain $[W\cup D(t,\Delta^2)\times W\cup D(t,\Delta^2)]$, where

$$D(t,\Delta^2)=[q:\mathrm{Im}q=0,\ (\mathrm{Re}q_0+t)^2-(\mathrm{Re}\mathbf{q})^2<4m^2,\ (\mathrm{Re}q_0-t)^2-(\mathrm{Re}\mathbf{q})^2<4m^2,\ m\leq t<E_\Delta].$$

There are several methods of obtaining this region of holomorphy, for example by means of the continuity theorem,[14] or by the general method of Bremermann[7]; we shall not attempt to solve here the general problem.

For $\Delta^2<3m^2$ the region $D(t,\Delta^2)$ is as in Fig. 2, while for $\Delta^2>3m^2$ we also have the other topological possibility for $D(t,\Delta^2)$, as shown in Fig. 3. We shall restrict ourselves here to $\Delta^2<3m^2$. Then, for $m\leq t<E_\Delta$, we have

$$D(t,\Delta^2)\supset D'=[q:\mathrm{Im}q=0,\ |\mathrm{Re}q_0|<\eta(t),\ |\mathrm{Re}\mathbf{q}|<\infty],$$

where $\eta(t)=(2m-t)$. For this case we apply lemma 3 of the Appendix to $g(q_1,q_2,q_3)$, where E of that theorem is now the subspace D' of R_4. Then the lemma shows that $g(q_1,q_2,q_3)$ is analytic if $(q_1,q_2)\epsilon(W\cup N')\times(W\cup N')$ for each $q_3{}^2=t^2$, and $m\leq t<E_\Delta$, where N' is a semitube neighborhood of D'.

We now write $(W\cup N')$ to exhibit its property of being a semitube,

$$W\cup N'=[q:q_0\epsilon B,\ |\mathrm{Im}\mathbf{q}|<v(q_0),\ |\mathrm{Re}\mathbf{q}|<\infty],$$

where $B=[q_0\text{-plane}]-[q_0:\mathrm{Im}q_0=0,\ |\mathrm{Re}q_0|\geq\eta(t)]$

$$v(q_0)=\sup[|\mathrm{Im}q_0|,v^*(q_0)]$$

and $v^*(q_0)=\frac{1}{4}[\eta(t)-|\mathrm{Re}q_0|]$, according to Lemma 3 of the Appendix.

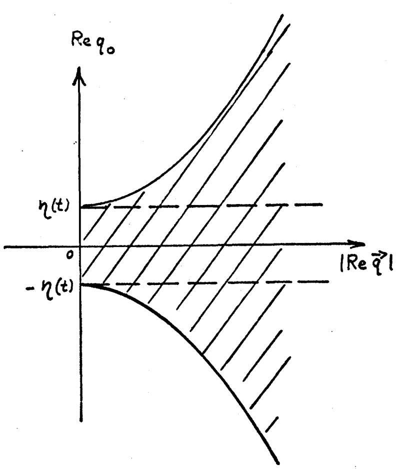

FIG. 2. The set $D(t,\Delta^2)$ for $\Delta^2<3m^2$.

[14] See reference 4, p. 49, Satz 17; also reference 6.

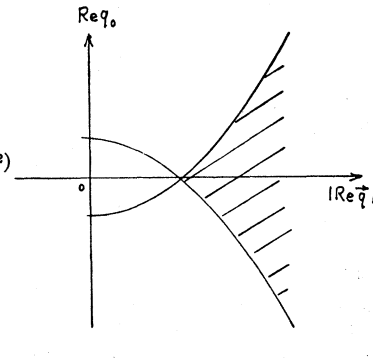

FIG. 3. The set $D(t,\Delta^2)$ for $\Delta^2>3m^2$, $t>2m$.

One can prove on the basis of the continuity theorem that the envelope of holomorphy of this semitube will be

$$E(W\cup N')=[q:q_0\epsilon B,\ |\mathrm{Im}\mathbf{q}|<V(q_0),\ |\mathrm{Re}\mathbf{q}|<\infty],$$

where $V(q_0)$ is the *smallest* superharmonic majorant of $v(q_0)$.[6]

We now prove that $V(q_0)=\tilde{V}(q_0)$, where $\tilde{V}(q_0)$ is defined to be the function $\mathrm{Im}[q_0{}^2-\eta^2(t)]^{\frac{1}{2}}$. For $\tilde{V}(q_0)$ is harmonic in B, and it may be seen by a straightforward calculation that $\tilde{V}(q_0)\geq v(q_0)$. Hence $\tilde{V}(q_0)$ is a superharmonic majorant of $v(q_0)$, and so

$$\tilde{V}(q_0)\geq V(q_0).$$

To obtain the reverse inequality, we note that $\tilde{V}(q_0)$ satisfies the same boundary conditions on the cut and asymptotically for $q_0\to\infty$ as does $v(q_0)$, and we have $V(q_0)\geq v(q_0)$. Since the Dirichlet problem for such a "boundary" has a unique solution, then this is $\tilde{V}(q_0)$, and hence by the definition of superharmonic functions[15] $V(q_0)\geq\tilde{V}(q_0)$ for all $q_0\epsilon B$.

The envelope of holomorphy of $(W\cup N')\times(W\cup N')$ is then $E(W\cup N')\times E(W\cup N')$, which we shall denote by H. At the points of interest [Eq. (3.16)] we have q_{10}, $q_{20}\epsilon B$, provided

$$(m^2-\gamma)/4t<\eta(t). \tag{3.17}$$

The curve of equality in (3.17) is drawn in Fig. 1, the required region being inside the parabola I.

Then the points of interest lie in H if $|\mathrm{Im}q_i|<|\mathrm{Im}(q_{i0}{}^2-\eta^2(t))^{\frac{1}{2}}|$, $(i=1,2)$, which implies

$$|\mathrm{Im}\rho(t,\gamma)|<\left|\mathrm{Im}\left[\left(\frac{m^2-\gamma}{4t}\right)^2-\eta^2(t)\right]^{\frac{1}{2}}\right|. \tag{3.18}$$

The curve of equality of (3.18) is also drawn in Fig. 1, the required region being outside the parabola II.

In the region $t\geq m$, $\gamma\leq m^2$ we see from these curves that for $\Delta^2<m^2$ the part of this region not covered by the edge of the wedge theorem [the part for which $\rho(t,\gamma)$ is pure imaginary] is now contained in the region inside I and outside II. Hence the corresponding points q_1, q_2 lie inside H.

This proves our Theorem 1.

[15] For the definition and discussion of superharmonic functions see F. Riesz, Acta Math. 48, 329 (1926); also references 4 and 6.

4. EXTENSION TO OTHER CASES

Our discussion in the previous two sections has given a proof of the dispersion relations for the scattering of bosons with equal mass, provided $\Delta^2<m^2$. We know that this specific restriction is merely due to our limited analytic continuation, because, by employing the semitube method, we have not made sufficient use of those regions of the domain E, which are important for larger values of Δ^2. By the methods of Bremermann[7,16] it is possible to make better use of E, and further calculations show that we can obtain $\Delta_{\max}=2m^2$.

Let us now discuss how the proof may be extended to other cases of more physical interest. One can easily see that the essential changes are due to different spectral conditions. Therefore it is useful to formulate the assumptions of Theorem 1 in a more general form using adjustable parameters.

Instead of the conditions (3.11), we now require that the Fourier transforms $f_{ij}(p_1,p_2,p_3,p_4)$ satisfy the following relations:

$$\begin{aligned} &f_{rj}-f_{aj}\equiv 0 \quad \text{for} \quad p_1^2<a^2 \quad \text{and} \quad p_3^2<b^2, \quad j=r,a;\\ &f_{ir}-f_{ia}\equiv 0 \quad \text{for} \quad p_2^2<a^2 \quad \text{and} \quad p_4^2<b^2, \quad i=r,a;\\ &\qquad f_{ij}\equiv 0 \quad \text{for} \quad (p_1+p_3)^2<\kappa^2 \quad \text{or} \quad p_{10}+p_{30}<0,\\ &\qquad\qquad i=r\ a; \quad j=r,a. \end{aligned} \tag{4.1}$$

Then there is a function $\chi(z_1,z_2,z_3,z_4,z_5;z_6)$ with the properties (1), (2), and (3) of Theorem 1, except that we are now interested in values of γ, with $-R<\gamma\leq\gamma_{\text{phys}}$. These values of γ give values of q_1, q_2 which are in the region of analyticity given by the semitube method, provided $\Delta^2<\Delta_{\max}^2$. Furthermore, we have in (3) $\chi\equiv 0$ for $z_6<\kappa^2$.

First we consider the *dispersion relations for pion-nucleon scattering*. The discussions of Secs. 2 and 3 can be readily applied to this case, and from the spectral conditions we find

$$a=m+\mu, \quad b=3\mu, \quad \kappa^2=(m+\mu)^2.$$

The physical value for γ is $\gamma_{\text{phys}}=\mu^2$, and by means of the edge of the wedge and the semitube theorems we find $\Delta_{\max}^2=2\mu^2$. *Since there are no other restrictions on Δ^2 than those connected with the analytic continuation in the proof of Theorem 1*, we have given a derivation of the pion-nucleon dispersion formulas for $\Delta^2<2\mu^2$. As we have discussed in the equal-mass case, this limit may be removed by further analytic continuation.§

We turn now to the problem of deriving *dispersion relations for nucleon-nucleon scattering*. Again the considerations of Secs. 2 and 3 go through straightforwardly and a brief discussion corresponding to those of Sec. 1 has been given by Goldberger, Nambu, and one of us (R. O.).[1] For the parameters in Theorem 1, we have

$$a=b=m+\mu, \quad \kappa^2=(2\mu)^2, \quad \gamma_{\text{phys}}=m^2.$$

One finds, that even for $\Delta^2=0$ the points (q_1,q_2) with (γ,t) values

$$\gamma=m^2,\ \tfrac{1}{2}\{(m+\mu)-[(m-\mu)^2-2\mu^2]^{\frac{1}{2}}\}<t<\tfrac{1}{2}\{(m+\mu)+[(m-\mu)^2-2\mu^2]^{\frac{1}{2}}\} \tag{4.2}$$

[q_1, q_2 are given in forms of γ, t by Eq. (3.16)], do not lie in the region of analytic continuation of the absorptive part. Since the function χ becomes zero only for $t<\frac{1}{2}\kappa=\mu$, we cannot assure the validity of the nucleon-nucleon relations. The troublesome region of t given in Eq. (4.2) vanishes only for

$$\mu>(\sqrt{2}-1)m, \tag{4.3}$$

which is much larger than the experimental mass ratio. The limitation (4.3) is *not* due to our use of the semitube method, because one can easily give examples of functions $g(q_1,q_2,q_3)=\chi(z_1,z_2,z_3,z_4,z_5;z_6)$ which are analytic in $z_1, \cdots, z_5$ in the region corresponding to $(q_1,q_2)\epsilon[W\cup D(t,0)]\times[W\cup D(t,0)]$, but have singularities at the points $z_1=z_2=m^2$, $z_3=z_4=\gamma$ and $z_5=0$ if $\gamma=2(m+\mu-t)^2+2t^2-m^2$. For $\gamma=m^2$ this gives $t=\frac{1}{2}\{(m+\mu)\pm[(m-\mu)^2-2\mu^2]^{\frac{1}{2}}\}$, which is in the range $t\geq\mu$. The functions[17]

$$g(q_1,q_2,t)=\frac{\rho(t)}{[\alpha^2-q_1^2]^n[\alpha^2-q_2^2]^n}, \tag{4.4}$$

with $\rho(t)=0$ for $t<\mu$, $\rho(t)>0$ for $t\geq\mu$, $n\geq 1$ and $\alpha^2=(m+\mu-t)^2$ have such properties. For the corre-

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[16] Hans J. Bremermann, Trans. Am. Math. Soc. **82**, 17 (1956).

§ *Note added in proof.*—We have constructed examples[17] of functions $\chi(z_1\cdots z_5;z_6)$, which have the properties described in the text and are such that for $z_1=z_2=m^2$, $z_3=z_4=\gamma$, $z_5=-4\Delta^2$, $z_6=4t^2$ they have singularities for $\Delta^2\geq G(t,\gamma;a,b)$, where $a\geq b$ and

$$G(t,\gamma;a,b)=\left[\left(t^2+\tfrac{1}{2}ab-\frac{m^2+\gamma}{4}\right)^2-\left(\frac{a+b}{2}t-\frac{a-b}{2}\frac{m^2-\gamma}{4t}\right)^2\right]\left[t^2-\left(\frac{a-b}{2}\right)^2\right]^{-1}.$$

In the equal mass case we have $a=b=2m$, $\gamma=m^2$, $t\geq m$ and find $G_{\min}=2m^2$ for $t=\sqrt{3}m/\sqrt{2}$. *Hence our limit $\Delta_{\max}^2=2m^2$ cannot be improved.* For pion-nucleon scattering the parameters are $a=m+\mu$, $b=3\mu$, $\gamma=\mu^2$, $t\geq\frac{1}{2}(m+\mu)$. The minimum of G is at $t=\frac{1}{2}(m+\mu)$ [μ/m=experimental mass ratio], and we obtain $G_{\min}(\pi)=\Delta_{\max}^2=(8\mu^2/3)(2m+\mu/2m-\mu)$. In order to prove that the points $z_1\cdots z_5$ given above are contained in the region of holomorphy for all $t\geq\frac{1}{2}(m+\mu)$ and $2\mu^2<\Delta^2<G_{\min}(\pi)$, we go back to the vectors q_1, q_2 (see Sec. 3) and use a general representation for functions which are analytic in $W\cup N$, where N is some complex neighborhood of the real set $S=[q\colon (q_0+t)^2-\mathbf{q}^2<a^2,\ (q_0-t)^2-\mathbf{q}^2<b^2]$. [A proof for such representations has been given by F. J. Dyson, Phys. Rev. (to be published); see also reference 17 and L. Garding and A. Wightman (to be published).] We find that we have analyticity for $\Delta^2<G(t,\gamma;a,b)$, where G is the same quantity we obtain from our examples. Hence *the dispersion relations for pion-nucleon scattering can be proved for momentum transfers 2Δ in the center-of-mass system, which are smaller than $4\mu(\frac{2}{3})^{\frac{1}{2}}[(2m+\mu)/(2m-\mu)]^{\frac{1}{2}}$.* At present it is not known whether a discussion of the complete envelope of holomorphy of the four-body Green's function (a function of six *complex* variables) will lead to an improvement of our limit. Results corresponding to those described above have been obtained also by H. Lehmann (private communication).

[17] These examples were inspired by the paper of R. Jost and H. Lehmann, Nuovo cimento **10**, 1598 (1957). We are indebted to Professor Lehmann for bringing this paper to our attention.

sponding $\chi(z_1,\cdots;z_6)$ we find, using Eq. (3.16),

$$\chi(z_1,\cdots,z_5;4t^2)=\frac{\rho(t)}{[2t^2-2(m+\mu)t+(m+\mu)^2-\frac{1}{2}(z_1+z_3)]^n[2t^2-2(m+\mu)t+(m+\mu)^2-\frac{1}{2}(z_2+z_4)]^n}. \quad (4.5)$$

These examples show that those points $z_1,\cdots,z_5$, which give rise to the limitation (4.3), lie on the envelope of holomorphy. Using only causality and spectral conditions we cannot hope to continue beyond these points. But there are certain symmetry properties of the four-body Green's function which we have not explored. We cannot exclude the possibility that these permit a further continuation in the relevant region.

The problem of proving a dispersion formula for the *meson-nucleon vertex function* $F[(k-p)^2,k^2,p^2]=\langle k|j(0)|p\rangle(4k_0p_0)^{\frac{1}{2}}$, is intimately related to the problem of deriving nucleon-nucleon dispersion relations. In essence we need only disregard the second four-vector variable q_2 in the discussions of Sec. 3. Then we can prove the representation $[k^2<(m+\mu)^2, p^2<(m+\mu)^2]$

$$F(z,k^2,p^2)=\frac{z^N}{\pi}\int_{(3\mu)^2}^{\infty}d\sigma^2\frac{\rho(\sigma^2,k^2,p^2)}{(\sigma^2-z)\sigma^{2N}}+\sum_{n=0}^{N-1}C_n(k^2,p^2)z^n \quad (4.6)$$

for $k^2+p^2<(m+\mu)^2$, and the requirement $k^2=p^2=m^2$ leads again to the condition $\mu>(\sqrt{2}-1)m$. In Eq. (4.5) we have omitted spin and isotopic spin variables, which are unimportant for the analytic properties of the vertex. The lower limit of the σ^2 integral corresponds to the case of pseudoscalar mesons; for scalar mesons it is $(2\mu)^2$. An example corresponding to Eq. (4.4) shows that we cannot improve this limit. However, in view of the symmetry conditions, an extension of the region of analyticity of $F(z_1,z_2,z_3)$ at the relevant points cannot be excluded. In any case, Jost[18] has given an example for $F(z_1,z_2,z_3)$, which satisfies spectral and causality conditions and is completely symmetric in all three variables. This example shows that, even including the symmetry conditions, one cannot derive the representation (4.6) for $F(z,m^2,m^2)$ if $\mu<(2/\sqrt{3}-1)m$. This value is above the experimental mass ratio. At present it is not clear, what further conditions one has to impose in order to assure the validity of the dispersion formula (4.6) for the pion-nucleon vertex. It has been shown by Nambu[19] that this relation holds in perturbation theory in every finite order.

Finally we would like to mention that our methods may also be applied to the case of *dispersion relations for K-meson-nucleon scattering*. Here we have again an unphysical continuum due to states of one Λ particle and one or more pions. The parameters of Theorem 1 are

$$a=M_K+2\mu,\quad b=m+\mu,\quad c=M_\Lambda+\mu,\quad \gamma_{\text{phys}}=m^2,$$

and for $\gamma=m^2$ there is a small region near $t=\frac{1}{2}(M_\Lambda+\mu)$, for which the corresponding points (q_1,q_2) are not included in the envelope of holomorphy.

ACKNOWLEDGMENTS

We would like to express our gratitude to Professor Robert Oppenheimer for his kind hospitality at the Institute for Advanced Study. One of us (J.G.T.) would like to thank the Commonwealth Fund of New York for a grant.

APPENDIX

In this Appendix we prove the "edge of the wedge" theorem, which has been extensively used in Sec. 3. It is convenient to prove first Lemma 1.

Lemma 1.—Let the function $f(z_0,z_1)$ of two complex variables (z_0,z_1) be given as the Fourier transform of two tempered distributions $\tilde{f}_r$ and $\tilde{f}_a$ such that it is analytic in the "wedge" W,

$$W=[(z_0,z_1):|y_1|<|y_0|,\ |x_0|<\infty,\ |x_1|<\infty].$$

For a given domain E in the (x_0,x_1) plane, we say that a sequence of pairs of complex numbers (z_{0n},z_{1n}) is an "E-limiting sequence" if it satisfies the following conditions,

(1) $\lim_n y_{0n}=\lim_n y_{1n}=0$,

(2) $\lim_n(x_{0n},x_{1n})\epsilon E$,

(3) there is a number $c>1$, and independent of n, so that for all n $|y_{0n}|>c|y_{1n}|$. Then we assume that $f(z_0,z_1)$ has the limiting property that for any E-limiting sequence the limit $\lim_n f(z_{0n},z_{1n})$ exists and is independent of the particular sequence, only depending on the limit point.

We wish to prove that $f(z_0,z_1)$ is also analytic in some neighborhood N of the set

$$S=[(z_0,z_1):y_0=y_1=0,\ (x_0,x_1)\epsilon E].$$

[18] R. Jost (private communication). Professor Jost was so kind to permit us to quote his example. We write it in the form

$$F_J(z_1,z_2,z_3)=f(z_1,z_2,z_3)+f^*(z_1^*,z_2^*,z_3^*),$$

where

$$f(z_1,z_2,z_3)=[(1+a^2-z_1m^{-2})^{\frac{1}{2}}+(1+a^2-z_2m^{-2})^{\frac{1}{2}}+(1+a^2-z_3m^{-2})^{\frac{1}{2}}-b-ic]^{-1},\quad (a>0,b>0,c>0).$$

The conditions on the parameters are

$$b>2a,\quad 1+a^2-(b+c)^2>0.$$

Since we may take c as small as we please, these two inequalities imply $a^2<\frac{1}{3}$. Taking $m(1+a^2)^{\frac{1}{2}}=m+\mu$, we have to require $\mu>(2/\sqrt{3}-1)m$ in order to avoid a contradiction with a dispersion relation of the form (4.6) for $z_2=z_3=m^2$. The condition $1+a^2-(b+c)^2>0$ is sufficient to assure that F_J is analytic in the domain obtained on the basis of causality, spectrum and symmetry. For the proof the region of analyticity obtained by G. Källén and A. S. Wightman is very useful [*Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, Inc., New York, 1957)].

[19] Y. Nambu, Nuovo cimento (to be published).

To achieve this, we shall prove analyticity in a neighborhood of each point of S. Since the origin of the real coordinates (x_0,x_1) has not been fixed, we may take this origin to be at the particular point of E that we are considering. Thus we wish to prove analyticity at $z_0=z_1=0$, where $(0,0)\epsilon E$.

If we suppose that $f(z_0,z_1)$ is analytic in some neighborhood of $z_0=z_1=0$, then for some r and $|z_0|<r$, $|z_1|<r$, $f(z_0,z_1)$ has the uniformly convergent power series expansion

$$f(z_0,z_1)=\sum_{m,n=0}^{\infty} a_{mn}z_0^m z_1^n. \tag{A1}$$

On the analytic plane Π_α: $z_0=\alpha_0\lambda$, $z_1=\alpha_1\lambda$, with α_0, α_1 real and $|\alpha_1/\alpha_0|<1$, then

$$f(z_0,z_1)=\sum_{m,n} a_{mn}\alpha_0^m\alpha_1^n\lambda^{m+n}$$

$$=\sum_{\rho=0}^{\infty}\lambda^\rho\sum_{n=0}^{\rho} a_{\rho-n,\,n}\alpha_0^{\rho-n}\alpha_1^n=\sum_{\rho=0}^{\infty} c_\rho\lambda^\rho, \tag{A2}$$

where the rearrangement from the second to the third expression in Eq. (A2) is permitted since the power series is absolutely and uniformly convergent. The coefficients c_ρ are obtained in terms of the coefficients a_{mn} as

$$c_\rho=\sum_{n=0}^{\rho} a_{\rho-n,\,n}\alpha_0^{\rho-n}\alpha_1^n. \tag{A3}$$

The method we shall use here is to reverse this procedure and determine the a_{nm} from the c_ρ for different Π_α. We may take $\alpha_0=1$ without loss of generality, and write α in place of α_1. Any analytic plane Π_α with real α satisfying $|\alpha|<1$ lies completely inside W, except for $\mathrm{Im}\lambda=0$. Hence, by our initial assumption, $f(z_0,z_1)$ is analytic in Π_α except possibly on the real λ axis. The limiting property assumed for $f(z_0,z_1)$ is equivalent to requiring that $f(z_0,z_1)$ has the same limit as we approach the real axis in Π_α either from above or below, and similarly for the partial derivatives of $f(z_0,z_1)$, provided $(\lambda,\alpha\lambda)\epsilon E$. Since $(0,0)\epsilon E$, $|\alpha|<1$, and E is open, then $(\lambda,\alpha\lambda)\epsilon E$ for a small open interval of λ near $\lambda=0$. Thus $f(z_0,z_1)$ is analytic in a small neighborhood of the origin in Π_α. Then there is some small positive number d so that we may expand $f(z_0,z_1)$ as in Eq. (A2) for $(z_0,z_1)\epsilon\Pi_\alpha$ and $|\lambda|<d$. We note that d may be chosen to be independent of α for $|\alpha|<1$.

It is immediate that $f(z_0,z_1)$ is bounded on any bounded subset of W which has a positive distance from the boundary of W. In addition, one can show by distribution methods using the Fourier representation that there is a positive number M, depending only on c and d, such that

$$|f(z_0,z_1)|<M(c,d)$$

for $(z_0,z_1)\epsilon\Pi_\alpha$, $|\alpha|<c<1$, $|\lambda|<d$. Then by Cauchy's inequality

$$|c_\rho|<M(c,d)/d^\rho. \tag{A4}$$

For $|\alpha|<c$, the coefficients $c_\rho(\alpha)$ are polynomials in α of order ρ,

$$c_\rho(\alpha)=\sum_{n=0}^{\rho} a_{\rho-n,\,n}\alpha^n. \tag{A5}$$

In order to verify this statement, we have to show that $d^{\rho+1}c_\rho(\alpha)/d\alpha^{\rho+1}=0$ for $\rho=1, 2, \cdots$. The equation for $\rho=0$ is evident, since

$$c_0(\alpha)=\lim_{\lambda\to 0}[f(z_0,z_1)]_{z_0=\lambda,\ z_1=\alpha\lambda}$$

is independent of α. For $\rho=1$, we have then

$$\frac{dc_1(\alpha)}{d\alpha}=\lim_{\lambda\to 0}\left[\frac{\partial f(z_0,z_1)}{\partial z_1}\right]_{z_0=\lambda,\ z_1=\alpha\lambda}, \tag{A6}$$

which again is independent of α. This follows from the Fourier representation and the E-limiting property of $f(z_0,z_1)$. The proof for higher ρ proceeds by induction on ρ.

Equation (A5) may now be solved for the a_{mn} in terms of the $c_\rho(\alpha)$ and α. This requires the use of a number of different analytic planes. The first few equations of (A5) are

$$c_0=a_{00},\quad c_1=a_{10}+a_{01},\quad c_2=a_{20}+a_{11}\alpha+a_{02}\alpha^2.$$

To determine a_{10} and a_{01}, we also have

$$c_1(\alpha')=a_{10}+a_{01}\alpha',$$

and so

$$a_{01}=[c_1(\alpha')-c_1(\alpha)]/(\alpha'-\alpha)$$

$$a_{10}=[\alpha'c_1(\alpha)-\alpha c_1(\alpha')]/(\alpha'-\alpha),$$

provided $\alpha'\neq\alpha$. Evidently such α' and α can be chosen to satisfy $\alpha'\neq\alpha$, $|\alpha'|<c$ and $|\alpha|<c$. In general we may determine the quantities a_{n0}, $a_{n-1,1}$, $\cdots$, $a_{n-r,\,r}$, $\cdots$, $a_{0,\,n}$ in terms of $c_n(\alpha)$ for n analytic planes Π_{α_1}, $\cdots$, Π_{α_n}, provided that the corresponding determinant

$$\prod_{n\geq i\geq j\geq 1}(\alpha_i-\alpha_j)\neq 0.$$

Such $\alpha_1\cdots\alpha_n$ can evidently be chosen in the interval $(-c,+c)$ for any n.

Equation (A4), written in terms of the a_{mn}, now becomes

$$\left|\sum_{n=0}^{\rho} a_{\rho-n,\,n}\alpha^n\right|<\frac{M(c,d)}{d^\rho} \tag{A7}$$

for $\rho=0, 1, 2, \cdots$. We wish to obtain bounds on the coefficients a_{mn} so that on replacing the c_ρ in Eq. (A2) by their expression in terms of the a_{mn} as given in Eq. (A5) the resulting series of Eq. (A2) may be shown to be absolutely convergent for small enough $|z_0|$, $|z_1|$, and so rearranged to give the expansion of Eq. (A1).

Let us consider the polynomial

$$P(x)=\sum_{n=0}^{N} b_n x^n,$$

and we assume that for real x satisfying $|x|\leq c$, then

$$|P(x)|\leq M.$$

We wish to determine bounds on the coefficients b_n. The Cauchy inequalities cannot be used here, since though we may immediately extend $P(x)$ to $P(z)$ for complex z the bound on $P(x)$ may not extend to the inside of the circle $|z|=c$.

We define[20]

$$f(z)=P(z)[z+i(c^2-z^2)^{\frac{1}{2}}]^{-N}.$$

Then $f(z)$ is a bounded analytic function in the whole complex z plane, including infinity, with a cut from $-c$ to $+c$ along the real axis. By the maximum modulus theorem,

$$|f(z)|\leq \operatorname*{Max}_{|x|\leq c}|f(x)|\leq M/c^N.$$

The Cauchy inequality for b_n is thus

$$|b_n|<\frac{1}{2\pi}\left|\int_{|z|=R}\frac{f(z)}{z^{n+1}}[z+i(c^2-z^2)^{\frac{1}{2}}]^N dz\right|$$

$$\leq\frac{M}{R^n c^N}\sup_{|z|=R}|z+i(c^2-z^2)^{\frac{1}{2}}|^N\leq\frac{M}{c^N R^n}[R+(R^2+c^2)^{\frac{1}{2}}]^N.$$

Taking $R=c$, we obtain

$$|b_n|\leq M3^N/c^n;$$

hence we have

$$|a_{\rho-n,\,n}|<M(c,d)3^\rho/d^\rho c^n.$$

Then

$$\left|\sum_{\rho=0}^{\infty} z_0{}^\rho \sum_{n=0}^{\rho} a_{\rho-n,\,n} z_1{}^n z_0{}^{-n}\right|$$

$$\leq\sum_{\rho=0}^{\infty}|z_0|^\rho M(c,d)\frac{3^\rho}{d^\rho}\sum_{n=0}^{\rho}\left|\frac{z_1}{cz_0}\right|^n. \quad \text{(A8)}$$

If $|z_0|<\frac{1}{3}d$, $|z_1|<\frac{1}{3}cd$, the right-hand side of Eq. (A8) is less than

$$M(c,d)\sum_{\rho=0}^{\infty}\rho d^\rho,$$

and so is finite provided $d<1$ (which is assumed). Hence the series $(\sum_{\rho=0}^{\infty} z_0{}^\rho \sum_{n=0}^{\rho} a_{\rho-n,\,n} z_1{}^n z_0{}^{\rho-n})$ is absolutely and uniformly convergent in $|z_0|<\frac{1}{3}d$, $|z_1|<\frac{1}{3}dc$, and so may be rearranged inside this region to give $\sum_{m,\,n=0}^{\infty} a_{mn} z_0{}^m z_1{}^n$.

So it has been shown that the power series expansions on the analytic planes Π_α can be joined together to give a power series expansion, convergent for $|z_0|<\frac{1}{3}d$, $|z_1|<\frac{1}{3}cd$, which equals these separate expansions on the various planes, and so equals $f(z_0,z_1)$ from which we started.

Hence we have proved analyticity of $f(z_0,z_1)$ in a small neighborhood of $z_0=z_1=0$.

We now consider a function f of four complex variables $z=(z_0,\mathbf{z})=(z_0,z_1,z_2,z_3)$ and prove Lemma 2.

Lemma 2.—Let the function $f(z)$ be analytic in the four complex variables z_0, $\mathbf{z}$ in the wedge W in four dimensions,

$$W=[(z_0,\mathbf{z}): |y_0|>|\mathbf{y}|,\ |x_0|<\infty,\ |\mathbf{x}|<\infty],$$

where it is the Fourier transform of two tempered distributions. We assume that $f(z)$ has the limiting property of Lemma 1, extended by replacing $\mathbf{z}_1$ by $\mathbf{z}$. Then we can find a neighborhood N of the set

$$S=[z: y_0=0,\ \mathbf{y}=0,\ x\epsilon E],$$

so that $f(z)$ can be continued analytically throughout $W\cup N$.

The proof of this lemma is obtained by simple extension of the proof of Lemma 1.

We now prove the "edge of the wedge" theorem.

"Edge of the Wedge" Theorem.—Let $f(z,z')$ be a function of 8 complex variables,

$$z=(z_0,z_1,z_2,z_3)=(z_0,\mathbf{z}),\quad z'=(z_0{}',\mathbf{z}').$$

We suppose that $f(z,z')$ is analytic in the double wedge $W\times W$,

$$W=[z: |y_0|>|\mathbf{y}|,\ |x_0|<\infty,\ |\mathbf{x}|<\infty],$$

where it is the Fourier transform[21] of tempered distributions. We further assume that $f(z,z')$ has the limiting property that for any E-limiting sequence of complex numbers $z_n=(z_{0n},\mathbf{z}_n)$, then

$$\lim_n[f(z_n,z')]|_{z'\epsilon W\cup R_4}\quad\text{and}\quad \lim_n[f(z,z'_n)]|_{z\epsilon W\cup R_4}$$

exist, and are independent of the particular sequence, only depending on the limit point.

We wish to prove that there is some neighborhood N of the set

$$S=[z: y=0,\ x\epsilon E],$$

so that $f(z,z')$ may be analytically continued to $(W\cup N)\times(W\cup N)$.

Proof.—We may prove the theorem by applying the method used in the proof of Lemma 1 to the variables z and z' simultaneously. In order to avoid repetition,

[20] Compare M. Riesz, Acta Math. **40**, 43 (1916). During a series of lectures which one of us (R. O.) gave at the University of Maryland in July 1957, Professor Marcel Riesz kindly pointed out that he had done similar calculations in 1916.

[21] As an equivalent assumption we could require that $f(z,z')$ is bounded by a polynominal for $(z,z')\epsilon W\times W$, with (x,x'), restricted to any compact $K\epsilon R_8$. Compare L. Schwartz, Meddelanden fran Lunds Universitets, Supplementband, p. 196 (1952).

we give only a brief sketch of the generalization. We may consider z and z' as two-vectors, since the extension to four-vectors is straightforward.

Consider a point in S, say (0,0), and the analytic planes $\Pi_{\alpha\beta}$ which are given by $z_0=\lambda$, $z_1=\alpha\lambda$; $z_0'=\lambda'$, $z_1'=\beta\lambda'$ with α,β real and $|\alpha|$, $|\beta|<c<1$. As in Lemma 1, we have

$$f(z,z')=\sum_{\rho,\sigma=0}^{\infty} c_{\rho\sigma}(\alpha,\beta)\lambda^{\rho}\lambda'^{\sigma},\ (z,z')\epsilon\Pi_{\alpha\beta},\ |\lambda|,|\lambda'|<d;$$

and

$$c_{\rho\sigma}(\alpha,\beta)=\sum_{n=0}^{\rho}\sum_{m=0}^{\sigma} A_{\rho-n,\,n;\,\sigma-m,\,m}\alpha^n\beta^m. \tag{A9}$$

Equation (A8) follows from the E-limiting properties of $f(z,z')$; it can be solved for the $A_{r,\,n;\,s,\,m}$ by using a sequence of different planes $\Pi_{\alpha\beta}$. As in Lemma 1, we obtain $|f(z,z')|<M(c,d)$ for $(z,z')\epsilon\Pi_{\alpha\beta}$; $|\lambda|$, $|\lambda'|<d$ and find $|A_{\rho-n,\,n;\,\sigma-m,\,m}|<M(c,d)3^{\rho+\sigma}/d^{\rho+\sigma}c^{n+m}$. Consider now the series $\sum_{rn,\,sm}A_{rn,\,sm}z_0{}^r z_1{}^n z_0{}'^s z_1{}'^m$; it is convergent for $|z_0|$, $|z_0'|<\frac{1}{3}d$, $|z_1|$, $|z_1'|<\frac{1}{3}cd$ and equal to $f(z,z')$ on an infinite sequence of planes $\Pi_{\alpha\beta}$. Hence the series represents a continuation of $f(z,z')$. Performing the corresponding construction for all $(z,z')\epsilon S$ we obtain a continuation to $(W\cup N)\times(W\cup N)$.

In the "edge of the wedge theorem" we proved that we have analyticity of $f(z,z')$ at each point of $(W\cup N)\times(W\cup N)$, for some neighborhood N of the set S. The neighborhood N is the union of all $N(X)$ with $X\epsilon E$, where $N(X)$ is some neighborhood of the point $z=X$. The dependence of N upon X is as yet arbitrary, except that $N(X)$ vanishes as X tends to the boundary of E. In Sec. 3 we find it useful to take for E the set $E=[x\colon |x_0|<\eta(t),\ |\mathbf{x}|<\infty]$.

We must construct a semitube from $W\cup N$ in order to apply the semitube method. Since W is already a tube, then it is necessary to construct a semitube H contained in N. Because $\mathbf{x}$ is arbitrary while x_0 is restricted, in E, then we expect H to be a semitube of the form

$$H=[z\colon |\mathbf{x}|<\infty,\ |\mathbf{y}|<v(z_0),\ |x_0|<\eta(t),\ |y_0|<\epsilon]$$

for some small ϵ. The quantity $v(z_0)$ will tend to zero as $|x_0|$ tends to $\eta(t)$. We have $H\subset N$ only if $N(X)$ is independent of $\mathbf{X}$. This independence does not occur for *any* function analytic in S, as is easy to show by simple examples. We cannot conclude that there exists a semitube $H\subset N$ without using analyticity in W.

We wish to show that f is analytic in some semitube neighborhood H of the set S. It is evident from what we have said above that, to do this, we will have to continue f from $W\cup N$ to $W\cup H$. We achieve this continuation by the direct method developed by one of us (H. J. B.).[7]

For simplicity we replace the three-vector by the complex number z_1. By a simple extension our result in terms of z_1 may be immediately generalized to the case $\mathbf{z}$ instead of z_1.

We take the analytic plane $z_1=X_1$. In the z_0 plane the domain of analyticity of our function is the cut plane, with cuts running from $\eta(t)$ to $+\infty$, and $-\eta(t)$ to $-\infty$. In the cut plane we take any point x_0, with $|x_0|<\eta(t)$. Furthermore we take the circle $c(x_0)$ in that plane, with center x_0 and radius r less than $\eta(t)-|x_0|$. We may assume $N(X_1)$ to be the region

$$N(X_1)=[(z_0,z_1)\colon x_1=X_1,\ |y_0|<\epsilon,\ |y_1|<\epsilon,\ |x_0|<\eta(t)-\delta(\epsilon,X_1)],$$

where $\delta(\epsilon,X_1)\rightarrow 0$ as $\epsilon\rightarrow 0$. Then we have that N is the union of the $N(X_1)$ for all X_1. We define $D=W\cup N$, with $N=\cup N(X_1)$, where $N(X_1)$ is given above. The circle $c(x_0)$ is completely in D. The Euclidean distance $\delta_D(z_0)$ of any point z on $c(x_0)$ from the boundary of D is then $\delta_D(z_0)=|y_0|/\sqrt{2}+O(\epsilon)$. We evaluate $h(\epsilon)=(1/2\pi)\int_0^{2\pi}\log\delta_D(re^{i\theta})d\theta$ and find $h(\epsilon)=(r/2\sqrt{2})+O(\epsilon)$. Now the theorem of reference 7 allows us to analytically continue $f(z_0,z_1)$ into the sphere $\|z-Z\|<h(\epsilon)$ about the point Z with $Z_0=0$, $Z_1=X_1$. Since ϵ is arbitrarily small and r is as close to $[\eta(t)-|x_0|]$ as we please, we may continue to the sphere $\|z-Z\|<(1/2\sqrt{2})[\eta(t)-|x_0|]$. In particular, we have analyticity in the set

$$H(X_1)=[z\colon x_1=X_1,\ |y_1|<\tfrac{1}{4}(\eta(t)-|x_0|),\ |y_0|<\tfrac{1}{4}(\eta(t)-|x_0|),\ |x_0|<\eta(t)],$$

and hence

$$H=\cup H(X_1)=[z\colon |x_1|<\infty,\ |y_1|<\tfrac{1}{4}(\eta(t)-|x_0|),\ |y_0|<\tfrac{1}{4}(\eta(t)-|x_0|),\ |x_0|<\eta(t)].$$

H is now a semitube.

Lemma 3.—Let $f(z,z')$ be a function satisfying the conditions of the "edge of the wedge" theorem. Then we can find a neighborhood H of the set $S=[z\colon y_\mu=0,\ |x_0|<\eta(t),\ |\mathbf{x}|<\infty]$ which is of the form

$$H=[z\colon |\mathbf{x}|<\infty,\ |\mathbf{y}|<\tfrac{1}{4}(\eta(t)-|x_0|),\ |y_0|<\tfrac{1}{4}(\eta(t)-|x_0|),\ |x_0|<\eta(t)],$$

and such that $f(z,z')$ may be analytically continued into $(W\cup H)\times(W\cup H)$.

To prove this lemma, we use the above process to continue $f(z,z')$, as a function of z, to $W\cup H$. We can show that $f(z,z')$ is an analytic function of (z,z') together in $(W\cup H)\times(W\cup S)$. We continue on z' now to $W\cup H$, and again we have that $f(z,z')$ is an analytic function of (z,z') together in $(W\cup H)\times(W\cup H)$. This proves the lemma.