view of these remarks, the relation of Eq. (20), derived by Ui, can be justified. For the "first-order" approximation in the surface potential, the expression reduces to

$$T_{ba}=\langle\phi_b|V_b^c|G_a^{(+)}\rangle+\langle G_b^{(-)}|V_b^s|G_a^{(+)}\rangle. \qquad (21)$$

The second, or surface, matrix element is therefore to be evaluated in the distorted-wave Born approximation, as is disclosed by examination of Eqs. (16) and (18). An identical result has been obtained by Tobocman whose derivation, not in the algebraic formalism, is rather more transparent.

IV. CONCLUSIONS

While the foregoing is predominantly formal, and quite removed from a *general* theory of nuclear reactions, deuteron-induced or otherwise, many of its aspects can be regarded as having relevance to physical evidence as disclosed by experiment. Specific to stripping, one is inevitably faced with the problem of eliminating the core contribution to the transition matrix element of Eq. (21), or at least making it very small.[5] In this connection it is requisite that theories which explicitly contain the fine-structure expansion of Wigner or Kapur-Peierls for interior nuclear wave functions be utilized with care. One must, if the distorted waves arise from complex potentials,[6] not only be restricted to incident energy uncertainties covering many resonance levels, but also to the computation of explicit averages of the scattering amplitudes over this energy interval. Indeed this raises the well-known question of level-width correlations, which up to now is not answered in any satisfactory way.

Despite the inherent difficulties associated with the formulation, we are carrying out computations of cross sections for $D(d,p)H^3$ and (α,n) reactions, and as well, polarizations for the former; these being based upon Eq. (21) in one or more of its specific forms.

[5] A. M. Lane, Revs. Modern Phys. **29**, 191 (1957).
[6] W. Tobocman and M. Kalos, Phys. Rev. **97**, 132 (1955).

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Bounds on Scattering Phase Shifts. I*†

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This paper investigates the method of obtaining bounds on scattering phases developed by T. Kato. Circumstances under which it is possible to obtain rather good bounds on the phase shifts by simple procedures are investigated. Useful formulas and techniques are presented and illustrative examples given.

1. INTRODUCTION

THE use of variational methods for calculating phase shifts in scattering problems has proved very fruitful. A variety of variational formulations has been developed and it has been shown that with an intelligent choice of the trial wave function these formulations give good approximations to the phase shifts.

Unfortunately, with most of these methods it is not generally possible to determine whether the approximation to the phase shift is above or below the correct value. This is a very serious defect, since, in the first place, the variational expressions are in some cases extremely sensitive to the choice of trial function and, in the second place, the inclusion of more free parameters in the trial function does not guarantee improved results.

Fortunately, Kato[1] found a variational formulation which gives both upper and lower bounds to the phase shifts under rather general conditions. He applied the method to a particular problem in which the method gave very close bounds to the phase shift with a simple trial wave function and a modest amount of computation.

The Kato formulation is not without its limitations. Trial wave functions that give good results in other variational methods may give very poor bounds in the Kato method or may even cause the integrals to diverge. However, the obvious superiority of the Kato method over other methods in cases where it does work well certainly justifies further investigation.

Some of the difficulties involved in the application of the Kato method will be investigated here, and some circumstances in which the method gives good bounds by means of simple trial functions and modest calculational efforts will be presented. Before we speak more

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[1] T. Kato, Progr. Theoret. Phys. (Kyoto) **6**, 295 (1950); **6**, 394 (1951); Phys. Rev. **80**, 475 (1950).

explicitly of these matters, however, it will be convenient to give a brief introduction to the formulation.

2. KATO METHOD

In Kato's paper[1] the formalism is developed for the case of zero angular momentum. However, as Kato pointed out, the extension to arbitrary angular momentum is formally trivial, and since this paper will be chiefly concerned with cases where $L>0$, we will present the formalism for the case of arbitrary L.

The operator $\mathfrak{L}_L$ is defined by

$$\mathfrak{L}_L=\frac{d^2}{dr^2}+k^2-\frac{L(L+1)}{r^2}+W(r). \tag{1}$$

The "potential" $W(r)$ stands for $-2m\hbar^{-2}V(r)$, where $V(r)$ is the potential. The problem is to obtain bounds on the phase shift $\bar{\eta}_L$ due to $W(r)$. In all that follows the exact quantities will be distinguished from trial quantities by being barred.

The wave functions are normalized by the condition that their asymptotic form be as follows:

$$u_{\theta L}(kr)\underset{r\to\infty}{\longrightarrow}\cos(kr-\tfrac{1}{2}L\pi+\theta)+\cot(\eta_L-\theta)\sin(kr-\tfrac{1}{2}L\pi+\theta). \tag{2}$$

The normalization constant, θ, lies between 0 and π but is otherwise arbitrary.

The quantity $P(\bar{\eta}_L)$ is defined by

$$P(\bar{\eta}_L)=k\cot(\bar{\eta}_L-\theta)-k\cot(\eta_L-\theta)+\int u_{\theta L}\mathfrak{L}_L u_{\theta L}dr. \tag{3}$$

Then it can be shown that the following inequality holds:

$$-\alpha_{\theta L}^{-1}\int\rho^{-1}(\mathfrak{L}_L u_{\theta L})^2dr\leq P(\bar{\eta}_L)\leq\beta_{\theta L}^{-1}\int\rho^{-1}(\mathfrak{L}_L u_{\theta L})^2dr. \tag{4}$$

Here ρ is some non-negative weight factor and $\alpha_{\theta L}$ and $\beta_{\theta L}$ are certain eigenvalues of the differential equation

$$\mathfrak{L}_L\phi_{nL}+\mu_n\rho\phi_{nL}=0,\quad 0\leq r\leq\infty. \tag{5}$$

The eigenfunctions ϕ_{nL} and their corresponding eigenvalues μ_n are determined by the boundary conditions that the ϕ_{nL} vanish at the origin and have the asymptotic phase shift, $\delta_L(\mu)$ given by the equation

$$\delta_L(\mu)=\theta+n\pi,\quad n=0,\pm1,\cdots. \tag{6}$$

Then $\alpha_{\theta L}$ is the smallest positive eigenvalue and $(-\beta_{\theta L})$ the smallest (in absolute value) negative eigenvalue. The values of $\alpha_{\theta L}$ and $\beta_{\theta L}$ need not be determined exactly. Even quite crude lower bounds on $\alpha_{\theta L}$ and $\beta_{\theta L}$ will suffice to give close bounds on the phase shifts, provided the integral $\int\rho^{-1}(\mathfrak{L}_L u_{\theta L})^2dr$, which vanishes for the exact wave function, can be made sufficiently small.

The conditions on the choice of $\rho(r)$ are that it be non-negative for all r, and that it behave in such a way that the phase shift in Eq. (6) is well defined for all values of μ.

We are now in a position to state the matters that will be covered in the following sections. In Sec. 3 the case $\rho=W\geq0$ will be considered and formulas for bounds on $\alpha_{\theta L}$ and $\beta_{\theta L}$ will be given. In Sec. 4 the circumstances under which $\beta_{\theta L}$ can be put equal to infinity will be investigated. This is a particularly important special case since the calculation of one of the bounds on $\bar{\eta}_L$ is greatly simplified. In Sec. 5 the choice of br^{-2} for ρ will be investigated and the advantages that attend this choice will be pointed out. Illustrative examples will be given in Secs. 4 and 5.

3. BOUNDS ON $\alpha_{\theta L}$ AND $\beta_{\theta L}$ WHEN $\rho(r)=|W(r)|$

If the "potential" $W(r)$ is of constant sign, then a permissible and rather natural choice for ρ is $\rho=|W(r)|$. For this case there are a few simple formulas for lower bounds on $\alpha_{\theta L}$ and $\beta_{\theta L}$. To simplify the discussion only the case where $W(r)$ is non-negative so that $\rho=W(r)$ will be presented.

The eigenvalue equation (5) now takes the form

$$\frac{d^2}{dr^2}\phi_{nL}+k^2\phi_{nL}-\frac{L(L+1)}{r^2}\phi_{nL}+(1+\mu_n)W\phi_{nL}=0. \tag{7}$$

Apart from the unique case $\mu=-1$, $\theta=0$ there will, in general, be no analytic, closed-form solutions for any values of μ and θ. The possibility of finding lower bounds on $\alpha_{\theta L}$ and $\beta_{\theta L}$ arises from the two circumstances that, first, there are "potentials" for which Eq. (7) is solvable, and second, that there is a theorem which states that the phase shift increases monotonically with increasing "potential."[2]

Now, for almost any non-negative $W(r)$ met with in practice, the inequality

$$W(r)\leq b/r^2 \tag{8}$$

holds for some value of b. The Lth phase shift of the equation

$$\frac{d^2}{dr^2}\phi_{nL}+k^2\phi_{nL}-\frac{L(L+1)}{r^2}\phi_{nL}+(1+\mu')\frac{b}{r^2}\phi_{nL}=0 \tag{9}$$

is known for any values of μ' and b such that

$$L(L+1)-(1+\mu')b>0. \tag{10}$$

The Lth phase shift is given by $-(\nu-L)(\pi/2)$ where ν is determined by the equation

$$\nu(\nu+1)=L(L+1)-(1+\mu')b. \tag{11}$$

[2] This monotonicity theorem is mentioned by Kato and is easily proved by the calculus of variations.

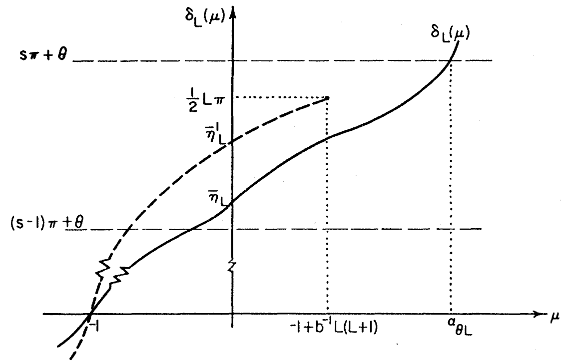

FIG. 1. Determination of lower bound on $\alpha_{\theta L}$ under the assumptions that $0\le W\le br^{-2}$, that $b<L(L+1)$, that $(s-1)\pi+\theta<\bar{\eta}_L$, and that $\frac{1}{2}L\pi<s\pi+\theta$. The dashed curve represents the (solvable) phase shift associated with Eq. (9). Equation (9) has no meaning beyond $\mu'=-1+b^{-1}L(L+1)$, which is a lower bound on $\alpha_{\theta L}$. [If $\bar{\eta}_L'<s\pi+\theta$ but $\frac{1}{2}L\pi>s\pi+\theta$, the dashed curve crosses the $s\pi+\theta$ line. The point of crossing then serves as a lower bound on $\alpha_{\theta L}$, and we have Eq. (14). If $b>L(L+1)$, the dashed curve does not even cross the vertical axis and $\alpha_{\theta L}$ cannot be determined by this method. These last two situations are not shown in the figure.]

Now, because of the monotonicity theorem, the value of $(1+\mu')$ in Eq. (9) which produces a specified phase shift is smaller in absolute value than the value of $(1+\mu)$ in Eq. (7) which produces the same phase shift. For zero phase shift, μ and μ' are both equal to -1. For the same negative phase shift, $\mu<\mu'<-1$. For the same positive phase shift, $\mu>\mu'>-1$.

By considering first the β eigenvalue, it can easily be seen that if $\bar{\eta}_L>\pi$, then $\beta_{\theta L}<1$; since $\mu=0$ corresponds to a phase shift greater than π, and $\mu=-1$ corresponds to zero phase shift in (7), a phase shift θ must be obtained for some value of μ between 0 and -1. An upper bound on $\beta_{\theta L}$ is useless, though, for the inequality given by (4) is not preserved if it is used.

If, however, $\bar{\eta}_L<\pi$ and, if it is possible to show that there is a θ such that $\bar{\eta}_L<\theta<\pi$, then the phase shift in Eq. (7) for $-1<\mu<0$, is between 0 and $\bar{\eta}_L$ and thus never equal to $\theta(\text{mod. }\pi)$. To attain a phase shift $\theta(\text{mod. }\pi)$ for negative μ, μ must be less than -1. The phase shift for the first negative eigenvalue will be $(\theta-\pi)$. The value of μ' which produces a phase shift $(\theta-\pi)$ in Eq. (9) is obtained by solving for ν from $(\theta-\pi)=-\frac{1}{2}(\nu-L)\pi$, substituting into Eq. (11) and solving for μ'. The value of $-\mu'$ so obtained serves as a lower bound on $\beta_{\theta L}$. In this way it is found that

$$\beta_{\theta L}\ge 1+b^{-1}(2-2\theta\pi^{-1})(2L+3-2\theta\pi^{-1}). \qquad (12)$$

This is a generalization of a result obtained by Kato for $L=0$.[1] For large L and/or small b, it can be a considerable improvement upon the result $\beta_{\theta L}\ge 1$ which follows from $\bar{\eta}_L<\theta<\pi$.

Turning now to the bound on $\alpha_{\theta L}$, it is seen that the eigenvalue equation with the "potential" br^{-2} has no well-defined phase shift for any positive eigenvalues unless $b<L(L+1)$. Thus one cannot obtain any bound by this procedure for the case $L=0$. Moreover, because of this pecular character of the "potential" br^{-2} having no admissible solutions for negative ν, if $(\bar{\eta}_L-\frac{1}{2}L\pi)>0$, the present method is not applicable. Furthermore, if $0>(\bar{\eta}_L-\frac{1}{2}L\pi)>-\pi$, then the best lower bound on $\alpha_{\theta L}$ for any θ such that $\theta<(\pi-\frac{1}{2}L\pi+\bar{\eta}_L)$ is given by that value of μ' in Eq. (9) for which the phase shift, $-\frac{1}{2}(\nu-L)\pi$, is equal to $\frac{1}{2}L\pi$. This corresponds to $\nu=0$, and from Eq. (11) it is found that

$$\alpha_{\theta L}\ge -1+b^{-1}L(L+1). \qquad (13)$$

However, if $(\bar{\eta}_L-\frac{1}{2}L\pi)$ is less than $-\pi$ and s is the largest integer for which $\bar{\eta}_L>[(s-1)\pi+\theta]$, then $\alpha_{\theta L}$ is the eigenvalue of Eq. (7) for which the phase shift is $(s\pi+\theta)$. The value of μ' that produces such a phase shift in (9) serves as a lower bound on $\alpha_{\theta L}$. Using Eq. (11) to obtain μ', one readily finds that

$$\alpha_{\theta L}\ge -1+b^{-1}(2s+2\theta\pi^{-1})(2L+1-2s-2\theta\pi^{-1}) \qquad (14)$$

(see Fig. 1).

Bounds on $\alpha_{\theta L}$ and $\beta_{\theta L}$ for cases in which W is nonpositive and $\rho=-W$ are readily obtained in the same fashion.

4. CONDITIONS UNDER WHICH $\beta_{\theta L}$ GOES TO INFINITY

Since it is often a very tedious matter to evaluate the integral $\int(\mathfrak{L}_L u_{\theta L})^2\rho^{-1}dr$ in the error term, it greatly simplifies matters in obtaining one bound on $\bar{\eta}_L$ if one can cause the term to vanish by showing that $\beta_{\theta L}$ goes to infinity. From inequality (4) one then has

$$k\cot(\bar{\eta}_L-\theta)\le k\cot(\eta_L-\theta)-\int u_{\theta L}\mathfrak{L}_L u_{\theta L}dr. \qquad (4')$$

It was pointed out by Kato that if $W(r)$ vanishes for $r>a$ and if $\bar{\eta}_0<\theta<\pi$ and $ka<(\pi-\theta)$, then it can be shown, by considering the phase shift due to a "potential" that is perfectly repulsive for $r<a$, that $\beta_{\theta 0}\to\infty$.

More generally the result $\beta_{\theta L}\to\infty$ can be obtained under the following conditions. If the "potential" $W(r)$ is solvable for $r>a$ so that one can form a trial function $u_{\theta L}$ for which $\mathfrak{L}_L u_{\theta L}\equiv 0$, $r>a$, and if the phase shift $\bar{\eta}_L^\infty(W)$, due to a "potential" that is infinitely negative for $r<a$ and equal to $W(r)$ for $r>a$, differs from $\bar{\eta}_L$ by less than π, then it is possible to pick θ so that $\beta_{\theta L}\to\infty$.

This result comes about as follows. Since $\mathfrak{L}_L u_{\theta L}\equiv 0$ for $r>a$, $\rho(r)$ can be chosen arbitrarily small for $r>a$ without affecting the value of the integral, $\int(\mathfrak{L}_L u_{\theta L})^2\rho^{-1}dr$. In the associated eigenvalue Eq. (5), the contribution of ρ for $r>a$ to the phase shift can be made arbitrarily small for any value of μ by choosing ρ for $r>a$ sufficiently small. Hence, for any negative value of μ, no matter how large in magnitude, it is possible to choose ρ so that the phase shift is less negative than $\bar{\eta}_L^\infty(W)$. If θ is chosen so that there is no value θ(mod. π) between $\bar{\eta}_L$ and $\bar{\eta}_L^\infty(W)$, then $\beta_{\theta L}$ can be put equal to infinity. (Alternatively, the problem can be reformulated in terms of the region $0 \leq r \leq a$. Note that the trial function must be continous and have a continuous first derivative at $r=a$.)

Now of course the "potentials" with which one ordinarily deals neither vanish identically nor are solvable beyond some point. However, it frequently happens that the "potential" beyond some $r=a$ can be closely approximated by some solvable "potential." If a solvable "potential" $U(r)$ is such that $U(r) \leq W(r)$ for $r>a$, then a lower bound on the phase shift that results from substituting $U(r)$ for $W(r)$ for $r>a$ is also a lower bound on $\bar{\eta}_L$. If the change in phase shift caused by the substitution is small then a good lower bound on the new phase shift is also a good lower bound on $\bar{\eta}_L$. Furthermore, and this is the significant feature, the substitution may enable one to set $\beta_{\theta L}$ equal to infinity.

To illustrate this procedure, consider now the case of a Yukawa-type "potential" with values for the parameters involved taken from a paper by Mowrer[3]:

$$W(r) = W_0(r_0/r) \exp(-r/r_0),$$
$$W_0 r_0^2 = 5.85, \quad kr_0 = 0.72.$$

A lower bound on $\bar{\eta}_L$ will be found.

Since $W(r)>0$ one can choose $U(r) \equiv 0$. In order to take the cutoff point, a, as far out as possible while still preserving the feature $\beta_{\theta L} \to \infty$ one must choose θ as small as possible, subject to the condition $\theta > \bar{\eta}_L$. It is therefore necessary to have a fairly close upper bound on $\bar{\eta}_2$ before the method can be reasonably applied. It will be shown in Sec. 4 that $\bar{\eta}_2$ is less than 8°. A value for a/r_0 of 70/9 leads to a value for $\bar{\eta}_2^\infty(W)$ of (8.5°–180°) so that θ may be taken as 8°, using such a cutoff point. The part of the potential discarded is obviously insignificant. From the simple trial function[4]

$$u_{\theta 2} = A(r/a)^3 + B(r/a)^4 + C(r/a)^5, \quad r<a$$
$$u_{\theta 2} = [\sin(\eta_2 - \theta)]^{-1}[-(\sin\eta_2) krn_2(kr) + (\cos\eta_2) krj_2(kr)], \quad r>a, \tag{15}$$

substituted into inequality (4′), one obtains the result $\bar{\eta}_2 > 6.2°$. The value for $\bar{\eta}_2$ calculated by Mowrer by numerical integration is 6.3°. It will be shown in the Appendix that the additional phase shift due to the discarded "potential" tail is much less than 0.1° so that the difference between Mowrer's value and the value calculated here is not due to cutting off the "potential," and the result could be improved by a more complicated trial function.

If the "potential" tail is not positive, then a lower bound on the phase shift due to the truncated "potential" is not necessarily a lower bound on $\bar{\eta}_L$. However, the sum of a lower bound on the phase shift due to the truncated "potential" and a lower bound on the additional phase shift due to the "potential" tail is a lower bound on the phase shift $\bar{\eta}_L$ due to $W(r)$. This is the simplest procedure for utilizing the result $\beta_{\theta L} \to \infty$ when the tail of $W(r)$ is not positive.

However, one can, if one chooses, follow the original procedure of substituting a $U(r)$ for $W(r)$ for $r>a$, where $U(r) \leq W(r)$, by taking $U = -b/r^2$. The analytic solutions for this potential are known for arbitrary L. The solutions for a particular value of b must be obtained from the series expansion for $j_G(kr)$ and $n_G(kr)$ as functions of the index G.

As an example of this, consider the repulsive exponential "potential"

$$W(r) = -W_0 \exp(-r/r_0),$$

where $W_0 r_0^2 = 1$, $kr_0 = 0.25$. The case $L=0$ is treated because this has an analytic solution[5] with which the approximate result can be compared.

For $r>\pi/k$ one makes the substitution $-br^{-2}$ for W where b is given by

$$b = (\pi/k)^2 W_0 \exp(-\pi/kr_0).$$

Then $\beta_{\theta 0}$ goes to infinity if θ is chosen so that (1) $(\theta - \pi)$ is greater than the phase shift that results from the new "potential" and (2) $(\theta - 2\pi)$ is less than the phase shift due to the "potential" that is infinitely negative for $r<\pi/k$ and is $-b/r^2$ for $r>\pi/k$. A value of θ equal to $(\pi - 0.01)$ is satisfactory.

For $r>\pi/k$ the trial function must be of the form

$$u_{\theta 0} = [\sin(\eta - \theta)]^{-1}[krj_G(kr)\cos(\eta + \tfrac{1}{2}G\pi) - krn_G(kr)\sin(\eta + \tfrac{1}{2}G\pi)], \quad r > \frac{\pi}{k}, \tag{16}$$

where $G(G+1) = b$. For $r<\pi/k$ the trial function

$$u_{\theta 0} = A(kr/\pi) + B(kr/\pi)^2 + C(kr/\pi)^3 + D(kr/\pi)^4 \tag{17}$$

was used. Substituting this trial function into inequality (4′), one finds that $\bar{\eta}_0 \geq -0.2765$. The exact calculation gives the result $\bar{\eta}_0 = -0.2759$.

5. CHOICE[6] OF br^{-2} FOR ρ

Since the potential br^{-2} is solvable for all L and for all values of b such that $b \leq L(L+1)$, it seems a rather obvious choice to select br^{-2} as the weight function, ρ. However, there is the difficulty here, that with this choice for ρ, the associated eigenvalue equation will not have eigenfunctions with well-defined phase shifts for large positive values of μ. Since the inequality (4) was based on the assumption that the ϕ_{nL} formed a complete set, the proof breaks down.

This difficulty can be avoided as follows. Consider br^{-2} as the limit case of the "potential" $br^{-1}(r+\epsilon)^{-1}$ as

[3] L. Mowrer, Phys. Rev. 99, 1065 (1955).

[4] For definition of $j_L(kr)$ and $n_L(kr)$, see P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 622.

[5] See reference 4, Vol. II, p. 1670.

[6] The authors are indebted to Professor Kato for a very useful suggestion on this matter.

ϵ goes to zero. With $\rho=br^{-1}(r+\epsilon)^{-1}$, the associated eigenvalue equation has eigenfunctions with well-defined phase shifts for all values of μ. In addition, for those values of μ for which the limit case br^{-2} has solutions with well-defined phase shifts, the values of μ resulting in a given phase shift with $\rho=br^{-1}(r+\epsilon)^{-1}$ can be made to differ by arbitrarily small amounts from the corresponding values of μ with $\rho=br^{-2}$ by taking ϵ sufficiently small.

The advantages of this choice of ρ over $\rho=W$ are, first, that it can be used in cases where W is not of constant sign and hence not suitable as a choice for ρ, second, that it usually leads to better bounds for the same trial function and third, that it makes possible the use of a wider class of trial functions.

The fact that the choice $\rho=br^{-2}$ usually leads to better bounds for the same trial function than $\rho=W$ can be seen as follows. Since bounds on $\alpha_{\theta L}$ and $\beta_{\theta L}$ are ordinarily obtained by comparing W with br^{-2} the bounds on $\alpha_{\theta L}$ and $\beta_{\theta L}$ are much the same for the two cases.[7] On the other hand, the integral $\int(\mathcal{L}_L u_{\theta L})^2\rho^{-1}dr$ will be smaller for $\rho=br^{-2}$ than for $\rho=W$ since b will be chosen so that $br^{-2}\geq W(r)$. Hence for the same trial function, $\rho=br^{-2}$ will usually produce better bounds.

The fact that the choice $\rho=br^{-2}$ makes possible the use of a wider class of trial functions arises as follows. Since most $W(r)$ fall off exponentially with large r, the integral $\int(\mathcal{L}_L u_{\theta L})^2\rho^{-1}dr$ will not converge for $\rho=W$ unless $\mathcal{L}_L u_{\theta L}$ vanishes at least exponentially. Because of this, the asymptotic form of the trial function must, to this same order of exactness, be equal to

$$[\sin(\eta_L-\theta)]^{-1}[-(\sin\eta_L)krn_L(kr)+(\cos\eta_L)krj_L(kr)].$$

With $\rho=br^{-2}$, on the other hand, the integral will converge if $\mathcal{L}_L u_{\theta L}$ vanishes as r^{-2}, as it does for many simple trial functions. In the following example a simple trial function will be used with $\rho=br^{-2}$ that cannot be used with $\rho=W$.

Returning now to the Yukawa-type "potential" used in the previous section,[3] we shall obtain an upper bound on $\bar{\eta}_2$ with the choice br^{-2} for ρ. The trial function used is

$$u_{\theta L}=[\sin(\eta_2-\theta)]^{-1}[(\cos\eta)krj_2(kr)-(\sin\eta)krj_3(kr)]. \quad (18)$$

The integrands that result are all quadratically integrable over the range 0 to ∞. They can also all be done analytically. It is possible to add exponentials to improve the trial function without losing these features.

A lower bound on $\alpha_{\theta 2}$ for values of $\theta>\bar{\eta}_2$ is obtained by using br^{-2} as comparison potential. By arguments similar to those used in Sec. 3, it is shown that

$$\alpha_{\theta 2}\geq b^{-1}(2\theta\pi^{-1})(5-2\theta\pi^{-1})-1. \quad (19)$$

The value of θ is varied by steps to obtain the best bound. In this way the following result was obtained:

$$\bar{\eta}_2<6.6°.$$

Mowrer's value for $\bar{\eta}_2$ is 6.3°.

APPENDIX

Our purpose here is to show how the additional phase shift, $\Delta\bar{\eta}_L$, due to a "potential" tail, may be bounded if the phase shift $\bar{\eta}_L{}^c$ due to the inner part of the "potential" is known to a good degree of accuracy. This additional phase shift is assumed to be small compared to the total phase shift so that rough bounds are sufficient.

The procedure for finding bounds on $\Delta\bar{\eta}_L$ is, with a few necessary changes, quite the same as the procedure for finding phase shifts already outlined. However for this case, in the associated eigenvalue problem one takes $\rho=0$ for r less than the cutoff point, a. With such a choice of ρ one must take for the trial function in the region $r<a$ the exact wave function. This wave function is of course unknown but one formally takes the exact wave function for the truncated potential as the new trial function over the whole range of r. Then, since $\mathcal{L}_L u_{\theta L}$ is zero for $r<a$ and $Wu_{\theta L}$ for $r>a$, the bounds on $\Delta\bar{\eta}_L$ are given by the inequalities

$$-\alpha_{\theta L}{}^{-1}\int_a^\infty(u_{\theta L}W)^2\rho^{-1}dr\leq k\cot(\Delta\bar{\eta}_L-\theta)+k\cot\theta+\int_a^\infty u_{\theta L}{}^2Wdr\leq\beta_{\theta L}{}^{-1}\int_a^\infty(u_{\theta L}W)^2\rho^{-1}dr, \quad (A1)$$

where the $\alpha_{\theta L}$ and $\beta_{\theta L}$ are eigenvalues of the new associated eigenvalue problem. Thus for $r>a$ the exact wave function for the truncated potential is

$$u_{\theta L}=[\sin(\bar{\eta}_L{}^c-\theta)]^{-1}[-(\sin\bar{\eta}_L{}^c)krn_L(kr)+(\cos\bar{\eta}_L{}^c)krj_L(kr)]. \quad (A2)$$

The function is not exactly determined since $\bar{\eta}_L{}^c$ is not exactly known but if $\bar{\eta}_L{}^c$ is closely bounded this defect is not serious.

This method has been used to get an upper bound on the phase shift due to the "potential" tail of the Yukawa type "potential" in Sec. 4. The asymptotic form, $u_{\theta L}=\sin(kr-\frac{1}{2}L\pi+\bar{\eta}_L{}^c)/\sin(\bar{\eta}_L{}^c-\theta)$, and $\rho=W$ were used in evaluating the integrals. In this way it was shown that $\Delta\bar{\eta}_2$ is less than 0.004°.

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[7] More exactly, it can be shown that using a comparison "potential" br^{-2} results in the same lower bound on $\alpha_{\theta L}$ for the two cases, but, for $\beta_{\theta L}$, the bound on the $\beta_{\theta L}$ associated with $\rho=W$ is greater by 1 than the bound on the $\beta_{\theta L}$ associated with $\rho=br^{-2}$.