proposed on simple theoretical grounds to limit the possible β couplings. It is universal, it is symmetric, it produces two-component neutrinos, it conserves leptons, it preserves invariance under CP and T, and it is the simplest possibility from a certain point of view (that of two-component wave functions emphasized in this paper).

These theoretical arguments seem to the authors to be strong enough to suggest that the disagreement with the He^6 recoil experiment and with some other less accurate experiments indicates that these experiments are wrong. The $\pi\rightarrow e+\bar{\nu}$ problem may have a more subtle solution.

After all, the theory also has a number of successes. It yields the rate of μ decay to 2% and the asymmetry in direction in the $\pi\rightarrow\mu\rightarrow e$ chain. For β decay, it agrees with the recoil experiments[19] in A^{35} indicating a vector coupling, the absence of Fierz terms distorting the allowed spectra, and the more recent electron spin polarization[4] measurements in β decay.

[19] Herrmansfeldt, Maxson, Stähelin, and Allen, Phys. Rev. **107**, 641 (1957).

Besides the various experiments which this theory suggests be done or rechecked, there are a number of directions indicated for theoretical study. First it is suggested that all the various theories, such as meson theory, be recast in the form with the two-component wave functions to see if new possibilities of coupling, etc., are suggested. Second, it may be fruitful to analyze further the idea that the vector part of the weak coupling is not renormalized; to see if a set of couplings could be arranged so that the axial part is also not renormalized; and to study the meaning of the transformation groups which are involved. Finally, attempts to understand the strange particle decays should be made assuming that they are related to this universal interaction of definite form.

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Dispersion Relations for Dirac Potential Scattering

N. N. Khuri* and S. B. Treiman
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
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Dispersion relations for scattering of a Dirac particle by a potential are shown to hold for a broad class of potentials. In contrast to the field theoretic case, the derivation here makes no use of the concept of causality but is instead based directly on the analytic properties of the Fredholm solution of the scattering integral equation. It is shown that the scattering amplitude, considered as a function of energy and momentum transfer, can be extended to a function analytic in the complex energy plane, for real momentum transfer. The dispersion relations then follow in the standard way from Cauchy's theorem. The final results involve one "subtraction." It is also shown that the analytic continuation into the unphysical region for nonforward scattering can be carried out by means of a partial wave expansion.

I. INTRODUCTION

It has recently been shown[1] that, under certain broad conditions, dispersion relations of the type so much discussed for relativistic field theories[2] also hold in ordinary nonrelativistic quantum mechanics for scattering of a particle by a potential. The treatment of this problem is quite straightforward and explicit; in contrast to the field theoretic case, one can show explicitly that the dispersion relations involve no "subtractions" and that the scattering amplitude can be analytically continued into the unphysical region for nonforward scattering by means of a partial wave expansion. In this sense, nonrelativistic quantum mechanics provides a complete and simple model of a system for which dispersion relations are valid. It has already been used as a basis for investigating to what extent the dispersion relations, taken together with the unitarity of the S-matrix, constitute a self-contained formulation of scattering theory.[3]

In the present paper, the discussion of dispersion relations in ordinary quantum mechanics is extended to the case of scattering of a Dirac particle by a potential. Using arguments similar to those employed for the Schrödinger case,[1] one again finds that dispersion relations hold for a broad class of potentials. The restrictions on the potentials are now somewhat more severe; and in the present case one finds that the dispersion

* Lockheed Fellow, 1956–1957.
[1] N. N. Khuri, Phys. Rev. **107**, 1148 (1957).
[2] Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1337 (1957). For a complete list of references see R. H. Capps and G. Takeda, Phys. Rev. **103**, 1877 (1956).
[3] S. Gasiorowicz and M. Ruderman (to be published).

relations involve one, but only one, "subtraction." Again, however, it turns out that the analytical continuation into the unphysical region for nonforward scattering can be effected by a partial wave expansion.

The Dirac case represents an especially interesting model, for it is the analog in ordinary quantum mechanics of the field-theoretic discussion of pion-nucleon scattering.[4] The final dispersion relations are similar in form for the two cases. Both involve the scattering amplitude for particle and antiparticle. Despite the similarities, however, it must once again be emphasized that the discussion of dispersion relations for field theories invokes the concept of microscopic causality; in the ordinary quantum mechanical case this concept seems to play no explicit role and is in fact not even formulated—at least in the present treatment.

The plan of the paper is as follows. In Sec. II the theory of Dirac potential scattering is formulated in the usual way in terms of a scattering integral equation; and the formal solution is obtained by the Fredholm method. The scattering matrix T, an operator in spinor space, is defined in the standard way and is conceived as a function of the two variables: energy E and momentum transfer τ. In Sec. III the T matrix is extended to a function of complex energy, the momentum transfer being kept real. After a discussion of the branch cuts, it is shown that T can be extended to a function analytic in the complex E plane, with poles on the real axis corresponding to bound states. The behavior of the T matrix for large $|E|$ is discussed, as well as the behavior under charge conjugation. In Sec. IV the analyticity of T in the complex E plane is used in the familiar way, via the Cauchy integral theorem, to obtain the sought-for dispersion relations. It is also shown in this section that the analytic continuation into the unphysical region for nonforward scattering can be effected by a partial wave expansion. Finally, in Sec. V the contribution to the dispersion relations coming from bound states is discussed, and a simple example is worked out in detail.

II. DIRAC SCATTERING THEORY

A

We consider the scattering of a Dirac particle of total energy E in a central field V. The Dirac equation reads[5]

$$(E-H)\psi=V\psi, \tag{1}$$

where ψ is a 4-component spinor wave function and

$$H=-i\boldsymbol{\alpha}\cdot\nabla+\beta m \tag{2}$$

is the free-particle Hamiltonian; $\boldsymbol{\alpha}$ and β are the usual Dirac matrices. We shall denote by ϕ the solutions of the free-particle equation

$$(E-H)\phi=0. \tag{3}$$

The solutions ϕ we shall take to be plane waves, so that they are characterized by the energy and momentum eigenvalues as well as by spin. To describe the scattering of a particle with a certain initial momentum and spin (corresponding free-particle wave function ϕ), we look for a solution of (1) which has the asymptotic behavior

$$\psi \xrightarrow[r\to\infty]{} \phi+\frac{1}{r}e^{ikr}v, \tag{4}$$

with $k=+(E^2-m^2)^{\frac{1}{2}}$. This outgoing wave boundary condition is automatically incorporated in the integral equation formulation of (1), which in operator notation can be written

$$\psi=\phi+\lim_{\epsilon\to+0}(E-H+i\epsilon)^{-1}V\psi. \tag{5}$$

The formal solution of this equation is given by[6]

$$\psi=\phi+(E-H-V+i\epsilon)^{-1}V\phi, \tag{6}$$

where the limiting process $\epsilon\to+0$ is henceforth always understood. It should be noted at this point that we are discussing the scattering of a *particle* ($E\geqslant m$). The scattering of antiparticles will be dealt with later.

We now observe that

$$\begin{aligned}(E&-H-V+i\epsilon)^{-1}V\\ &=(E-H-V+i\epsilon)^{-1}(E+H-V+i\epsilon)^{-1}(E+H-V)V\\ &=(E^2-H^2-U+i\epsilon)^{-1}V[E-V+H+(H,V)],\end{aligned} \tag{7}$$

where

$$(H,V)=HV-VH=-i(\nabla V)\cdot\boldsymbol{\alpha}, \tag{8}$$

and

$$U=2EV-V^2-i(\nabla V)\cdot\boldsymbol{\alpha}. \tag{9}$$

Since ϕ is supposed to satisfy (3), (6) may be written

$$\psi=\phi+\frac{1}{E^2-H^2-U+i\epsilon}U\phi; \tag{10}$$

and, reversing the arguments which lead from (5) to (6), we see that ψ satisfies the integral equation

$$\psi=\phi+\frac{1}{E^2-H^2+i\epsilon}U\psi. \tag{11}$$

It is this form of the integral equation which is a convenient basis for obtaining an explicit solution by the Fredholm method.

Let us now introduce the following notation for the various Green's functions:

$$g_0=\frac{1}{E^2-H^2+i\epsilon};\quad g=\frac{1}{E^2-H^2-U+i\epsilon}; \tag{12}$$

and

$$K=g_0U;\quad G=gU. \tag{13}$$

[4] See Chew *et al.*, reference 2.

[5] $\hbar=c=1$.

[6] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

We then have

$$\psi=\phi+K\psi, \tag{14}$$

and

$$\psi=\phi+G\phi. \tag{15}$$

The kernel functions G and K are evidently related by the integral equation

$$G=K+KG=K+GK. \tag{16}$$

Written out explicitly, the operator equation (5) reads

$$\psi_\mu(\mathbf{x})=\phi_\mu(\mathbf{x})-\frac{1}{4\pi}\int(E-i\boldsymbol{\alpha}\cdot\nabla_x+\beta m)_{\mu\nu}\times\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}V(\mathbf{y})\psi_\nu(\mathbf{y})d^3y, \tag{5'}$$

where the Greek subscripts are spinor indices. Equation (11), written in full, becomes

$$\psi_\mu(\mathbf{x})=\phi_\mu(\mathbf{x})-\frac{1}{4\pi}\int\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}U_{\mu\nu}(\mathbf{y})\psi_\nu(\mathbf{y})d^3y. \tag{11'}$$

This just expresses the well-known fact that the Green's function g_0 is

$$g_0(\mathbf{x},\mathbf{y})=-\frac{1}{4\pi}\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}. \tag{12'}$$

The arguments which led from (5) to (11) are just equivalent to going from (5′) to (11′) by carrying out a partial integration and making use of the fact that ψ satisfies the Dirac equation.

B

In order to proceed with the derivation of the dispersion relations, it will be necessary to have an explicit representation for the solution of our scattering equation. We shall obtain this by the method of Fredholm.[7] This method cannot, however, be applied directly to the integral equation (11′), since the Green's function $g_0(\mathbf{x},\mathbf{y})$ is singular at $\mathbf{x}-\mathbf{y}=0$. We therefore iterate (11′) once, obtaining

$$\psi=\phi+K\phi+K^2\psi. \tag{17}$$

Let

$$F=\phi+K\phi;\quad Q=K^2. \tag{18}$$

Written out explicitly,

$$Q_{\mu\nu}(\mathbf{x},\mathbf{y})=\int g_0(\mathbf{x},\mathbf{x}')U_{\mu\rho}(\mathbf{x}')g_0(\mathbf{x}',\mathbf{y})U_{\rho\nu}(\mathbf{y})d^3x'. \tag{19}$$

The Fredholm solution of (17) may now be obtained in the standard way and is well-defined. One finds

$$\psi_\mu(\mathbf{x})=F_\mu(\mathbf{x})+\int\frac{\Delta_{\mu\nu}(k;\mathbf{x},\mathbf{x}')}{\Box(k)}F_\nu(\mathbf{x}')d^3x'; \tag{20}$$

where

$$\Delta_{\mu\nu}(k;\mathbf{x},\mathbf{x}')=Q_{\mu\nu}(\mathbf{x},\mathbf{x}')+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{\rho_1\cdots\rho_n}\int d^3x_1\cdots d^3x_n\times B_{\mu\nu}^{(n)}(\mathbf{x},\mathbf{x}';\mathbf{x}_1\cdots\mathbf{x}_n;\rho_1\cdots\rho_n);$$

$$\Box(k)=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{\rho_1\cdots\rho_n}\int d^3x_1\cdots d^3x_n\times D^{(n)}(\mathbf{x}_1\cdots\mathbf{x}_n;\rho_1\cdots\rho_n);$$

$$B_{\mu\nu}^{(n)}=\begin{vmatrix} Q_{\mu\nu}(\mathbf{x},\mathbf{x}') & Q_{\mu\rho_1}(\mathbf{x},\mathbf{x}_1) & \cdots & Q_{\mu\rho_n}(\mathbf{x},\mathbf{x}_n)\\ Q_{\rho_1\nu}(\mathbf{x}_1,\mathbf{x}') & \cdots & & \\ \vdots & & & \\ Q_{\rho_n\nu}(\mathbf{x}_n,\mathbf{x}') & \cdots & & \end{vmatrix};$$

$$D^{(n)}=\begin{vmatrix} Q_{\rho_1\rho_1}(\mathbf{x}_1,\mathbf{x}_1) & Q_{\rho_1\rho_2}(\mathbf{x}_1,\mathbf{x}_2) & \cdots & Q_{\rho_1\rho_n}(\mathbf{x}_1,\mathbf{x}_n)\\ Q_{\rho_2\rho_1}(\mathbf{x}_2,\mathbf{x}_1) & \cdots & & \\ \vdots & & & \\ Q_{\rho_n\rho_1}(\mathbf{x}_n,\mathbf{x}_1) & \cdots & & \end{vmatrix}.$$

Using an operator notation (Δ is an integral operator in coordinate space as well as a spinor operator; $\Box$ is just a number) we may write

$$\psi=F+(\Delta/\Box)F. \tag{20'}$$

Comparing this with (15) we obtain, finally,

$$G=K+(\Delta/\Box)+(\Delta/\Box)K. \tag{21}$$

This is the result which we require for the discussion of dispersion relations. The question of the convergence of the series involved in the Fredholm solution will be deferred till later.

C

The matrix element which describes the scattering of a particle from an initial state i to a final state f is given by

$$M_{fi}=(\phi_f,V\psi_i)=(\phi_f,V\phi_i)+(\phi_f,VG\phi_i). \tag{22}$$

For later reference we note that this can also be written

$$\begin{aligned}M_{fi}&=(1/2E)(\phi_f,U\psi_i)\\&=(1/2E)\{(\phi_f,U\phi_i)+(\phi_f,UG\phi_i)\}.\end{aligned} \tag{23}$$

We take the free particle solutions ϕ to be plane waves, so that for a particle of momentum $\mathbf{k}$ we have

$$\phi_{\mathbf{k}}=u(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{24}$$

where $u(\mathbf{k})$ is a 4-component spinor normalized to

$$u^\dagger u=1. \tag{25}$$

The spinor u also carries a spin label, which we shall, however, not explicitly write. Let us now define the

[7] The present work was motivated by the interesting application of the Fredholm theory to scattering problems discussed by R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951).

T matrix by

$$M(\mathbf{k}_f,\mathbf{k}_i)=u^\dagger(\mathbf{k}_f)T(\mathbf{k}_f,\mathbf{k}_i)u(\mathbf{k}_i). \tag{26}$$

It is an operator in spinor space. From (21) and (22) we see that

$$T(\mathbf{k}_f,\mathbf{k}_i)=V(\mathbf{k}_f-\mathbf{k}_i)+\int e^{-i\mathbf{k}_f\cdot\mathbf{x}}V(\mathbf{x})G(\mathbf{x},\mathbf{y})e^{i\mathbf{k}_i\cdot\mathbf{y}}d^3xd^3y, \tag{27}$$

where

$$V(\mathbf{k}_f-\mathbf{k}_i)=\int e^{-i(\mathbf{k}_f-\mathbf{k}_i)\cdot\mathbf{x}}V(\mathbf{x})d^3x. \tag{28}$$

The spinors appearing in (26) satisfy the Dirac equation

$$(\boldsymbol{\alpha}\cdot\mathbf{k}+\beta m-E)u=0. \tag{29}$$

It is easy to show that by use of this equation the T matrix as it appears in (26) can always be reduced to the form

$$T=a+\beta b. \tag{30}$$

Alternate ways of representing the T matrix are possible and will in fact be more convenient at a later state. For the purposes of Sec. III, however, the above form is the most useful one. Since the potential V is spherical it now follows that the functions a and b can depend only on the energy and on the magnitude of the momentum transfer.

Let us define

$$\boldsymbol{\tau}=\mathbf{k}_f-\mathbf{k}_i;\quad \boldsymbol{\pi}=\tfrac{1}{2}(\mathbf{k}_f+\mathbf{k}_i); \tag{31}$$

and note that $\boldsymbol{\tau}\cdot\boldsymbol{\pi}=0$. The scattering angle θ is related to the momentum transfer τ by

$$\cos\theta=1-\tau^2/2k^2. \tag{32}$$

We also introduce the variables

$$\mathbf{R}=\tfrac{1}{2}(\mathbf{x}+\mathbf{y}),\ \mathbf{r}=\mathbf{x}-\mathbf{y}. \tag{33}$$

The T matrix, which is a function of τ and $E=(k^2+m^2)^{\frac{1}{2}}$, can now be written

$$T(E,\tau)=V(\tau)+\int \exp[-i(E^2-m^2-\tau^2/4)^{\frac{1}{2}}\mathbf{n}\cdot\mathbf{r}\times\exp(-i\boldsymbol{\tau}\cdot\mathbf{R})V\left(\mathbf{R}+\frac{\mathbf{r}}{2}\right)\times G\left(E;\mathbf{R}+\frac{\mathbf{r}}{2},\mathbf{R}-\frac{\mathbf{r}}{2}\right)d^3rd^3R, \tag{34}$$

where

$$\mathbf{n}=\boldsymbol{\pi}/\pi.$$

III. EXTENSION TO THE COMPLEX ENERGY PLANE

A

Up to this point the T matrix, as well as the kernel function G, are defined and have meaning only for physical values of the variables E and τ: E and τ both real, and $E>+(m^2+\tau^2/4)^{\frac{1}{2}}$. We now want to extend G and T to functions of the complex variable E, τ being kept real. We will find that for a certain class of potentials the extended function T is analytic in the complex E plane, with singularities on the real axis between $-m$ and m.

Since the kernel function G was originally expressed in terms of $k=(E^2-m^2)^{\frac{1}{2}}$ it will first be necessary to make cuts in the complex E plane in order to define G as a function of complex E. The cuts run from $m\to\infty$ and $-m\to-\infty$; and we choose the Riemannian sheet such that the imaginary part of k is always non-negative: $\mathrm{Im}k=\kappa\geqslant 0$. Just above the real energy axis, the real part of k is positive for $E>m$ and negative for $E<m$. The converse holds just below the real energy axis. On the real axis for $-m<E<m$, k is pure imaginary. Similar cuts are chosen to define $(E^2-m^2-\tau^2/4)^{\frac{1}{2}}$ for complex E. We now define

$$G_\pm(E)=\lim_{\epsilon\to+0}G(E\pm i\epsilon)(E\text{ real}), \tag{35}$$

and

$$T_\pm(E)=\lim_{\epsilon\to+0}T(E\pm i\epsilon)(E\text{ real}). \tag{36}$$

The T matrix for physical scattering corresponds to $T_+(E)$ for $E>(m^2+\tau^2/4)^{\frac{1}{2}}$. It is clear from what has been said that

$$G_+(E)=G_-(E)(-m<E<m); \tag{37}$$

and

$$T_+(E)=T_-(E)(-m<E<m). \tag{38}$$

B

Before proceeding to the main task of this section we want to establish here certain symmetry properties of the T matrix which are essential to the derivation of the dispersion relations. At this stage it is useful to avoid commitment to the particular representation of Eq. (30), so we simply write $T=T(E,\mathbf{k}_f,\mathbf{k}_i)$.

1

Define

$$\bar{T}(E,\mathbf{k}_f,\mathbf{k}_i)=\beta T^\dagger(E,\mathbf{k}_f,\mathbf{k}_i)\beta, \tag{39}$$

where $T^\dagger$ is the Hermitean conjugate of T. We will show that

$$\bar{T}_+(E,\mathbf{k}_f,\mathbf{k}_i)=T_-(E,\mathbf{k}_i,\mathbf{k}_f). \tag{40}$$

In the representation of Eq. (30) this implies

$$a_+^*(E,\tau)=a_-(E,\tau),\quad b_+^*(E,\tau)=b_-(E,\tau). \tag{41}$$

To establish this result, let us first introduce the following notation:

$$w_f=e^{i\mathbf{k}_f\cdot\mathbf{r}},\quad w_i=e^{i\mathbf{k}_i\cdot\mathbf{r}}. \tag{42}$$

From (23) and (26) we then have

$$2ET_\pm(E,\mathbf{k}_f,\mathbf{k}_i)=(w_f,Uw_i)+(W_f,UG_\pm w_i); \tag{43}$$

and from (13) and (16)

$$UG_{\pm}=Ug_{0\pm}U+UG_{\pm}g_{0\pm}U,$$
$$=Ug_{0\pm}U+Ug_{0\pm}UG_{\pm}. \quad (44)$$

Just as in (39), define

$$\bar{G}=\beta G^{\dagger}\beta, \quad (45)$$

where $G^{\dagger}$ is the Hermitean conjugate of G, which, it must be remembered, is an integral as well as a spinor operator. Thus

$$G_{\mu\nu}{}^{\dagger}(\mathbf{x},\mathbf{y})=G_{\nu\mu}{}^{*}(\mathbf{y},\mathbf{x}).$$

Define $\bar{U}$ and $\bar{g}_0$ in the same way. It is evident that

$$\bar{U}=U\,;\ \bar{g}_{0\pm}=g_{0\mp}. \quad (46)$$

From (43) we see that

$$2E\bar{T}_{+}(E,\mathbf{k}_f,\mathbf{k}_i)=(w_i,Uw_f)+(w_i,\bar{G}_{+}\bar{U}w_f). \quad (47)$$

But from (44)

$$\bar{G}_{+}\bar{U}=Ug_{0-}U+Ug_{0-}\bar{G}_{+}\bar{U}=UG_{-}. \quad (48)$$

Equation (40) is thus proved.

2

There remains to find relations which connect $T(E)$ with $T(-E)$. It is at this stage that we are led into a discussion of the scattering of antiparticles. As is well known, an antiparticle of energy E and momentum $\mathbf{k}$ is described by a charge conjugate spinor u^c given by

$$u^c(E,\mathbf{k})=Cu^{*}(-E,\,-\mathbf{k}), \quad (49)$$

where the charge conjugation matrix can be taken to be

$$C=-i\beta\alpha_2,$$

in the usual representation for Dirac matrices. It has the properties

$$\left.\begin{aligned}C\alpha_jC&=\tilde{\alpha}_j,\ j=1,2,3\\ C\beta C&=-\tilde{\beta},\end{aligned}\right. \quad (50)$$

$$C=C^{-1}=C^{\dagger}; \quad (51)$$

where the tilde symbol denotes the transposition operation. The charge conjugate spinor $u^c(E,\mathbf{k})$ satisfies (29). In analogy with (26), the matrix element for antiparticle scattering is written

$$M^c(E,\mathbf{k}_f,\mathbf{k}_i)=u^{c\dagger}(E,\mathbf{k}_f)T_{+}{}^{c}(E,\mathbf{k}_f,\mathbf{k}_i)u^c(E,\mathbf{k}_i). \quad (52)$$

From (49) it follows that

$$M^c(E,\mathbf{k}_f,\mathbf{k}_i)=u^{\dagger}(-E,\,-\mathbf{k}_i)\times C\tilde{T}_{+}{}^{c}(E,\mathbf{k}_f,\mathbf{k}_i)Cu(-E,\,-\mathbf{k}_f). \quad (53)$$

But we also have

$$M(-E,\,-\mathbf{k}_i,\,-\mathbf{k}_f)=u^{\dagger}(-E,\,-\mathbf{k}_i)\times T_{-}(-E,\,-\mathbf{k}_i,\,-\mathbf{k}_f)u(-E,\,-\mathbf{k}_f), \quad (54)$$

where we write $T_{-}(-E)$ rather than $T_{+}(-E)$ in order that $(E^2-m^2)^{\frac{1}{2}}$ shall have zero phase in T^c for $E>m$; i.e., so that $T_{+}{}^{c}$ will correspond to an out-going wave matrix. Thus

$$T_{+}{}^{c}(E,\mathbf{k}_f,\mathbf{k}_i)=C\tilde{T}_{-}(-E,\,-\mathbf{k}_i,\,-\mathbf{k}_f)C. \quad (55)$$

If we represent T^c in a form analogous to Eq. (30),

$$T^c=a^c+\beta b^c, \quad (56)$$

then (41) and (55) imply

$$a_{\pm}{}^{c}(E,\tau)=a_{\pm}{}^{*}(-E,\,\tau),$$
$$b_{\pm}{}^{c}(E,\tau)=-b_{\pm}{}^{*}(-E,\,\tau). \quad (57)$$

C

We now turn to our main task, which is to show that the matrix T can indeed be extended to a function analytic in the full complex E plane, the momentum transfer τ being real. We must first show that the Fredholm series in (21) and (22) converge to an analytic function of E and that the kernel function G is analytic. We must then show that in (34) the integral which defines T converges to an analytic function. Finally, in order to employ Cauchy's theorem at a later stage, we must study the behavior of T for $|E|\rightarrow\infty$.

The results may be stated in advance. It will turn out that the kernel function G can be extended to an analytic function of E if there exists a spherically symmetric function $F(r)$ such that

$$F(r)\geqslant|V^2-i\boldsymbol{\alpha}\cdot\nabla V|, \quad (58.a)$$

$$F(r)\geqslant 2m|V|, \quad (58.b)$$

$$F(r)\leqslant M'/r^2,\ M'<\infty, \quad (58.c)$$

$$\int_0^\infty F(r)rdr\leqslant M'',\ M''<\infty. \quad (58.d)$$

Furthermore, for fixed momentum transfer τ, the matrix T can be extended to an analytic function of E if

$$\int_0^\infty e^{(\tau/2)r}F(r)r^2dr\leqslant M'''<\infty. \quad (59)$$

The proof of these assertions follows almost exactly as in reference 1. Thus, although the results are of central importance for this paper, we need indicate the derivation only in brief outline. We do this much, in fact, only to call attention to the slight technical differences between the Dirac and Schrödinger cases.

For convenience we use (21) to rewrite (34) in the following form:

$$T(E,\tau)=V(\tau)+T_2(E,\tau)+T_3(E,\tau)/\square+T_4(E,\tau)/\square, \quad (60)$$

where

$$T_j(E,\tau)=\int\exp[-i(E^2-m^2-\tau/4)^{\frac{1}{2}}\mathbf{n}\cdot\mathbf{r}]\exp(-i\boldsymbol{\tau}\cdot\mathbf{R})\times N_j\left(E;\mathbf{R}+\frac{\mathbf{r}}{2},\mathbf{R}-\frac{\mathbf{r}}{2}\right)d^3rd^3R; \quad (61)$$

and, in matrix notation,

$$N_2 = VK, \quad N_3 = V\Delta, \quad N_4 = V\Delta K. \tag{62}$$

We now use the theorem, stated in reference 1, which in effect says that for the T_j to be analytic in an open region Γ, where Γ is bounded by a closed curve B in the complex energy plane, the integrands in (61) must be analytic in Γ and continuous in the closed region $\Gamma+B$. Furthermore, for all E on B the integrand must be bounded by an integral function of $\mathbf{r}$ and $\mathbf{R}$. In our case B is the curve shown in Fig. 1; the semicircles have large but finite radius E_B, and the horizontal segments approach the real axis in the limit.

That T_2 meets the above requirements for analyticity is trivially shown. From (12′) and (62) one finds the bound

$$|N_2(E,\mathbf{x},\mathbf{y})| \leqslant \frac{|V(\mathbf{x})|}{4\pi}\left(\frac{E_B}{m}+1\right)\frac{e^{-\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}F(y) \tag{63}$$

for all E on B. This is integrable in (61). To establish the analyticity of T_3 and T_4 we must study the properties of $\Delta(E,\mathbf{x},\mathbf{y})$. We shall prove below that for all E in Γ, Δ is analytic and satisfies the inequality

$$e^{\kappa|\mathbf{x}-\mathbf{y}|}|\Delta(E,\mathbf{x},\mathbf{y})| \leqslant C_{E_B}\frac{F(y)}{|\mathbf{y}|}, \tag{64}$$

where C_{E_B} is a constant which depends on E_B. From this result we find, as in reference 1,

$$|T_{3,4}(E,\tau)| \leqslant C_{E_B}' \int e^{\frac{1}{2}\tau|\mathbf{x}-\mathbf{y}|}F(\mathbf{x})F(\mathbf{y})\frac{1}{|\mathbf{y}|}d^3x d^3y. \tag{65}$$

The integral exists for any τ for which (59) holds, hence $T_{3,4}$ are analytic in Γ.

We are left with the task of proving our assertions about $\Delta(E,\mathbf{x},\mathbf{y})$. Once again, the procedure is similar to that in reference 1. First, one shows that for potentials satisfying the conditions (58), the kernel Q is bounded, as follows:

$$|Q(E,\mathbf{x},\mathbf{y})| \leqslant N\left(\frac{|E|}{m}+1\right)^2 e^{-\kappa|\mathbf{x}-\mathbf{y}|}\frac{F(y)}{4\pi|\mathbf{y}|}, \tag{66}$$

where N is a finite constant. Using this one then shows that the series defining Δ is a series of analytic functions, and invoking Hadamard's lemma to obtain upper bounds on the Fredholm determinants one finds:

$$\Delta(E,\mathbf{x},\mathbf{y}) \leqslant N\left(\frac{|E|}{m}+1\right)^2\frac{F(y)}{4\pi|\mathbf{y}|} \times\left\{1+\sum_{n=1}^{\infty}\frac{(n+1)^{\frac{1}{2}(n+1)}}{n!}4^n M''^n\left(\frac{|E|}{m}+1\right)^{2n}N^n\right\}. \tag{67}$$

This differs from the Schrödinger case by the factors $(|E|/m+1)$ and by the factor 4^n, the latter coming from the summation over spinor indices. The above series converges for any finite $|E|$. Hence, the analyticity of Δ in the region Γ is proved. Furthermore, for all $|E| \leqslant E_B$ we see that

$$|\Delta| \leqslant C_{E_B}''\frac{F(y)}{|\mathbf{y}|}, \tag{68}$$

where C_{E_B}'' is a constant which depends on E_B. To prove the inequality (64) it suffices to establish, in addition to (68), the following two limits, for E in Γ:

$$\begin{aligned}\lim_{|\mathbf{x}|\to\infty} e^{\kappa|\mathbf{x}-\mathbf{y}|}|\Delta(E,\mathbf{x},\mathbf{y})| &= 0,\\ \lim_{|\mathbf{y}|\to\infty} e^{\kappa|\mathbf{x}-\mathbf{y}|}|\Delta(E,\mathbf{x},\mathbf{y})| &= 0.\end{aligned} \tag{69}$$

These are proved by writing the series defining Δ and noting, by use of the triangle inequality, that every term in the series when multiplied by $e^{\kappa|\mathbf{x}-\mathbf{y}|}$ vanishes as $|\mathbf{x}|$ or $|\mathbf{y}|$ tends to ∞. Since we have shown that the series defining Δ converges *uniformly* for all x and y (E finite), the results (69) follow.

Finally, we note that the analyticity of the Fredholm denominator $\square$ can be proved by similar methods; in contrast to the Schrödinger case, however, it no longer holds true that $\square\to 1$ as $|E|\to\infty$; but this is of no importance. What concerns us here is that the zeros of $\square$ will give rise to poles for the matrix T.

Now just as in the Schrödinger case, the Fredholm resolvent kernel G as it has been defined is not an irreducible fraction. It can however be replaced, using the Poincaré method, by $\Delta'/\square'$, an irreducible fraction; Δ' and $\square'$ have no zeros in common. Δ' and $\square'$ are defined in the same way as Δ and $\square$, but with elements of K replacing elements of Q in the Fredholm determinants and with zeros along the diagonals. The analyticity properties of the Fredholm resolvent kernel are unaffected by the Poincaré procedure.

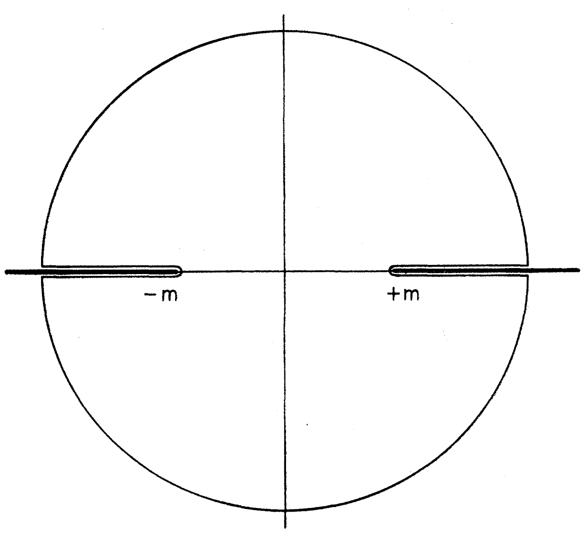

FIG. 1. Contour in complex energy plane.

The zeros E_j of $\square'$ correspond to energies for which the homogeneous equation

$$\psi_j = K\psi_j$$

has at least one solution. It is clear from the definition of K that such solutions satisfy the second order equation

$$(H-V+E_j)(H+V-E_j)\psi_j=0. \tag{70}$$

It is easy to show that the eigenvalues E_j just coincide with the eigenvalues of the first order Dirac equation; i.e., the second order Eq. (70) introduces no spurious eigenvalues. It is also easy to show that the eigenvalues all lie between $-m$ and m. Thus the Fredholm denominator $\square'$ has zeros on the real axis between $-m$ and m, the zeros corresponding to bound states.

D

We have so far shown that $T(E,\tau)$ is regular in Γ, except for poles in the interval $-m<E<m$ on the real axis. Of course we can choose E_B as large as we please, as long as it is finite. Hence T is analytic in the whole finite E plane, the cuts excluded. But before we can derive dispersion relations, we still have to show that T has no essential singularity at infinity. In fact, we claim that for the class of potentials under consideration T has the asymptotic behavior

$$T/E\to 0,\ |E|\to\infty. \tag{71}$$

The derivation of this result is outlined in the Appendix, where we employ methods taken from some work of Schiff.[8]

IV. DISPERSION RELATIONS

A

For practical purposes it will now be convenient to represent T in a manner which differs from Eq. (30). We write

$$T(E,\boldsymbol{\pi},\boldsymbol{\tau})=A(E,\tau)+i\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\times\boldsymbol{\tau}B(E,\tau), \tag{72}$$

where $\boldsymbol{\sigma}$ is the usual 4×4 spin matrix. Similarly,

$$T^c(E,\boldsymbol{\pi},\boldsymbol{\tau})=A^c(E,\tau)+i\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\times\boldsymbol{\tau}B^c(E,\tau). \tag{73}$$

From (40) and (55) it follows that

$$\begin{aligned}&A_+(E,\tau)=A_-^*(E,\tau),\ A_\pm^c(E,\tau)=A_\pm^*(-E,\tau),\\&B_+(E,\tau)=B_-^*(E,\tau),\ B_\pm^c(E,\tau)=B_\pm^*(-E,\tau);\end{aligned} \tag{74}$$

and from (38) we have for $|E|<m$,

$$\begin{aligned}A_+(E,\tau)=A_-(E,\tau)=A_+^c(-E,\ \tau)=A_-^c(-E,\ \tau),\\B_+(E,\tau)=B_-(E,\tau)=B_+^c(-E,\ \tau)=B_-^c(-E,\ \tau).\end{aligned} \tag{75}$$

[8] L. I. Schiff, Phys. Rev. 103, 443 (1956).

Let us define the linear combinations

$$\begin{aligned}F_1(E,\tau)&=A(E,\tau)+A^c(E,\tau),\\F_2(E,\tau)&=A(E,\tau)-A^c(E,\tau),\\F_3(E,\tau)&=B(E,\tau)+B^c(E,\tau),\\F_4(E,\tau)&=B(E,\tau)-B^c(E,\tau).\end{aligned} \tag{76}$$

The corresponding boundary values have the simple properties

$$F_+(E,\tau)=F_-^*(E,\tau); \tag{77}$$

and

$$F_{1,3\pm}(E,\tau)=F_{1,3\pm}^*(-E,\ \tau), \tag{78}$$

$$F_{2,4\pm}(E,\tau)=-F_{2,4\pm}^*(-E,\ \tau). \tag{79}$$

B

We have seen in Sec. III that the matrix T (and also T^c) has the asymptotic behavior

$$T/E\to 0,\quad |E|\to\infty. \tag{80}$$

Evidently this also describes the asymptotic behavior of the amplitude A (and A^c). For the amplitude B (and B^c), on the other hand, this implies

$$E^2B/E\to 0,\quad |E|\to\infty. \tag{81}$$

Now all of our amplitudes, as we have seen, are analytic functions of energy in the cut E plane, with poles corresponding to bound states lying on the real axis between $-m$ and m. Let $E_j(j=1,2,\cdots)$ denote the singular points for the T matrix. From our charge conjugation condition (55) we know that T^c then has singularities at the points

$$E_j^c=-E_j. \tag{82}$$

Let R_{jA} and R_{jB} denote respectively the residues of the amplitudes A and B at the singular points E_j. From (75) it follows that the residues of the charge conjugate amplitudes, at the poles E_j^c, are related to these by

$$R_{jA}{}^c=-R_{jA},\quad R_{jB}{}^c=-R_{jB}. \tag{83}$$

We are now ready to apply Cauchy's theorem to our amplitudes $F(E',\tau)$, choosing the contour in the E' plane shown in Fig. 1 (the semicircles have infinite radius). Consider first the amplitudes F_3 and F_4; and let E be a point which in the limit approaches the real axis from above, with $E>(m^2+\tau^2/4)^{\frac{1}{2}}$. By Cauchy's theorem the integral

$$\frac{1}{2\pi i}\oint\frac{F_{3,4}(E',\tau)}{E'-E}dE' \tag{84}$$

is equal to the sum of residues of the integrand; and from (80) and (81) it is evident that the integral receives zero contribution from the semicircles. Invoking (77), (78), (79), (82), and (83), we then find, carrying out the integrations,

$$\mathrm{Re}B(E,\tau)+\mathrm{Re}B^c(E,\tau) =\sum_j R_{jB}\left\{\frac{1}{E-E_j}-\frac{1}{E+E_j}\right\}+\frac{2}{\pi}\int_m^\infty \frac{E'}{E'^2-E^2} \times\{\mathrm{Im}B(E',\tau)+\mathrm{Im}B^c(E',\tau)\}dE'; \quad (85)$$

$$\mathrm{Re}B(E,\tau)-\mathrm{Re}B^c(E,\tau) =\sum_j R_{jB}\left\{\frac{1}{E-E_j}+\frac{1}{E+E_j}\right\}+\frac{2}{\pi}\int_m^\infty \frac{E}{E'^2-E^2} \times\{\mathrm{Im}B(E',\tau)-\mathrm{Im}B^c(E',\tau)\}dE'. \quad (86)$$

The amplitudes which appear in these equations are in fact the physical amplitudes B_+ and $B_+{}^c$—we henceforth drop the subscripts. The integrals are principle value integrals.

For the amplitudes F_1 and F_2 we cannot write similar dispersion relations. The asymptotic behavior here is governed by Eq. (80), and in this case the contributions from the Cauchy integral (84) over the infinite semicircles do not vanish. We instead form the integral

$$\frac{1}{2\pi i}\oint \frac{F_{1,2}(E',\tau)}{E'(E'-E)}dE'. \quad (87)$$

The extra factor E' in the denominator now guarantees that the semicircles make no contribution; but it introduces a new singularity at $E'=0$ (which we assume does not coincide with any of the natural, bound-state singularities). Proceeding as before, we then find

$$\mathrm{Re}A(E,\tau)+\mathrm{Re}A^c(E,\tau) =2A(0,\tau)+\sum_j\left(\frac{E}{E_j}\right)R_{jA}\left\{\frac{1}{E-E_j}+\frac{1}{E+E_j}\right\} +\frac{2}{\pi}E\int_m^\infty\left(\frac{E}{E'}\right)\left(\frac{1}{E'^2-E^2}\right) \times\{\mathrm{Im}A(E',\tau)+\mathrm{Im}A^c(E',\tau)\}dE', \quad (88)$$

$$\mathrm{Re}A(E,\tau)-\mathrm{Re}A^c(E,\tau) =\sum_j\left(\frac{E}{E_j}\right)R_{jA}\left\{\frac{1}{E-E_j}-\frac{1}{E+E_j}\right\}+\frac{2}{\pi}E\int_m^\infty\frac{1}{E'^2-E^2} \times\{\mathrm{Im}A(E',\tau)-\mathrm{Im}A^c(E',\tau)\}dE'. \quad (89)$$

Once again, the amplitudes which appear here are in fact A_+ and $A_+{}^c$. We have used the fact that $A(0,\tau)=A^c(0,\tau)$ in arriving at these results.

Finally, solving these equations for the individual amplitudes, one obtains the dispersion relations

$$\mathrm{Re}A(E,\tau) =A(0,\tau)+\sum_j\left(\frac{E}{E_j}\right)\frac{R_{jA}(\tau)}{E-E_j}+\frac{1}{\pi}\int_m^\infty\left(\frac{E}{E'}\right) \times\left\{\frac{\mathrm{Im}A(E',\tau)}{E'-E}-\frac{\mathrm{Im}A^c(E',\tau)}{E'+E}\right\}dE', \quad (90)$$

$$\mathrm{Re}A^c(E,\tau) =A(0,\tau)+\sum_j\left(\frac{E}{E_j}\right)\frac{R_{jA}(\tau)}{E+E_j}+\frac{1}{\pi}\int_m^\infty\left(\frac{E}{E'}\right) \times\left\{\frac{\mathrm{Im}A^c(E',\tau)}{E'-E}-\frac{\mathrm{Im}A(E',\tau)}{E'+E}\right\}dE', \quad (91)$$

$$\mathrm{Re}B(E,\tau) =\sum_j\frac{R_{jB}(\tau)}{E-E_j}+\frac{1}{\pi}\int_m^\infty \times\left\{\frac{\mathrm{Im}B(E',\tau)}{E'-E}+\frac{\mathrm{Im}B^c(E',\tau)}{E'+E}\right\}dE', \quad (92)$$

$$\mathrm{Re}B^c(E,\tau) =-\sum_j\frac{R_{jB}(\tau)}{E+E_j}+\frac{1}{\pi}\int_m^\infty \times\left\{\frac{\mathrm{Im}B^c(E',\tau)}{E'-E}+\frac{\mathrm{Im}B(E',\tau)}{E'+E}\right\}dE'. \quad (93)$$

We remark again that principal value integrations are always understood. Note also that for nonzero momentum transfer τ the dispersion relations involve integration over an unphysical region running from m to $(m^2+\tau^2/4)^{\frac{1}{2}}$. We shall return to this point shortly.

C

The dispersion relations take on an especially useful form for forward scattering, since in this case one can invoke the well-known optical theorem. In general, the differential scattering cross section (initial spin state i, final spin state f) is given by

$$\frac{d\sigma_{fi}}{d\Omega}=\frac{2\pi}{v}|M_{fi}|^2\frac{dN}{dE}=\frac{E^2}{(2\pi)^2}|M_{fi}|^2, \quad (94)$$

where dN/dE is the density of final states and v is the velocity. For forward scattering the matrix M is diagonal in spin space and is just equal to the amplitude $A(E,0)$. The optical theorem tells us that

$$\mathrm{Im}M(E,0)=\mathrm{Im}A(E,0)=-\frac{1}{2}\frac{k}{E}\sigma(E), \quad (95)$$

where $\sigma(E)$ is the total cross section. Analogous results, of course, hold for antiparticle scattering. For forward scattering the dispersion relations can therefore be written

$$\begin{aligned}\mathrm{Re}M(E,0)&\\ &=A(0,0)+\sum_j\left(\frac{E}{E_j}\right)\frac{R_{jA}(0)}{E-E_j}\\ &\quad-\frac{E}{2\pi}\int_m^\infty\left(\frac{k'}{E'^2}\right)\left\{\frac{\sigma(E')}{E'-E}-\frac{\sigma^c(E')}{E'+E}\right\}dE',\end{aligned}\tag{96}$$

$$\begin{aligned}\mathrm{Re}M^c(E,0)&\\ &=A(0,0)+\sum_j\left(\frac{E}{E_j}\right)\frac{R_{jA}(0)}{E+E_j}\\ &\quad-\frac{E}{2\pi}\int_m^\infty\left(\frac{k'}{E'^2}\right)\left\{\frac{\sigma^c(E')}{E'-E}-\frac{\sigma(E')}{E'+E}\right\}dE';\end{aligned}\tag{97}$$

and, in particular,

$$\begin{aligned}\mathrm{Re}M(E,0)-\mathrm{Re}M^c(E,0)&\\ &=\sum_j\left(\frac{E}{E_j}\right)R_{jA}(0)\left\{\frac{1}{E-E_j}-\frac{1}{E+E_j}\right\}\\ &\quad-\frac{E}{\pi}\int_m^\infty\frac{k'}{E'(E'^2-E^2)}\{\sigma(E')-\sigma^c(E')\}dE'.\end{aligned}\tag{98}$$

The "subtraction" constant $A(0,0)$ does not appear in this last equation, which makes it therefore especially interesting. Also, we see that

$$\mathrm{Re}M(E,0)\to\mathrm{Re}M^c(E,0)\,(E\to\infty).\tag{99}$$

D

For physical scattering the energy and momentum transfer satisfy the inequality $E>(m^2+\tau^2/4)^{\frac{1}{2}}$. For nonforward scattering the dispersion relations therefore involve integration over an unphysical region, and the question arises how the analytic continuation into this region can in practice be effected. As in the Schrödinger case, and using the same methods, we now show that the continuation can be carried out by a partial wave expansion.

For this purpose we introduce the conventional scattering amplitude $f(\theta)$.[9] This is a 2×2 operator in ordinary Pauli spin space. The differential scattering cross section (initial spin state i, final spin state f) is given by

$$d\sigma_{fi}/d\Omega=|f_{fi}(\theta)|^2.\tag{100}$$

The total scattering amplitude can now be expanded in the well-known way in terms of partial wave amplitudes, the latter being labeled by the parity and total angular momentum quantum numbers, $(-1)^l$ and j respectively. Let δ_{l+} and δ_{l-} be the respective phase shifts for $j=l+\frac{1}{2}$ and $j=l-\frac{1}{2}$, with parity $(-1)^l$; and let f_{l+} and f_{l-} be the corresponding partial wave amplitudes:

$$f_{l\pm}=\frac{1}{2ik}[\exp(2i\delta_{l\pm})-1].\tag{101}$$

The partial wave expansion for $f(\theta)$ is given by[9]

$$\begin{aligned}f(\theta)=\sum_{l=0}^\infty\{(l+1)f_{l+}+lf_{l-}\}P_l(\cos\theta)&\\ +\frac{i\boldsymbol{\sigma}\cdot\mathbf{k}_f\times\mathbf{k}_i}{k^2}\sum_{l=1}^\infty\left(\frac{l}{1-\cos^2\theta}\right)(f_{l+}-f_{l-})&\\ \times\{\cos\theta P_l(\cos\theta)-P_{l-1}(\cos\theta)\}.&\end{aligned}\tag{102}$$

Analogous results hold for antiparticle scattering. The P_l here are Legendre polynomials, and $\boldsymbol{\sigma}$ is the 2×2 Pauli spin matrix.

Using these results, we could, of course, write our amplitudes A and B as partial wave expansions, but there is no need to do this. Instead, what we have to consider is the following. Suppose we treat the scattering amplitude f as a function of k and τ, writing $\cos\theta=1-\tau^2/2k^2$. For $k^2<\tau^2/4$ the argument of the Legendre polynomials becomes less than -1; as $k\to0$, it approaches $-\infty$. The question is: do the above series converge for all $k^2<\tau^2/4$?

Carter[10] has rigorously shown that for sufficiently large l (k fixed), the phase shifts are bounded—to within a constant of order unity—by the Born approximation:

$$\begin{aligned}|\delta_{l+}|&\leqslant C_+\left\{(E+m)\int_0^\infty J_{l+\frac{1}{2}}{}^2(kr)|V|rdr\right.\\ &\qquad\left.+(E-m)\int_0^\infty J_{l+\frac{3}{2}}{}^2(kr)|V|rdr\right\},\\ |\delta_{l-}|&\leqslant C_-\left\{(E+m)\int_0^\infty J_{l+\frac{1}{2}}{}^2(kr)|V|rdr\right.\\ &\qquad\left.+(E-m)\int_0^\infty J_{l-\frac{1}{2}}{}^2(kr)|V|rdr\right\},\end{aligned}\tag{103}$$

where the J_ν are Bessel functions. Once these bounds are known, one can proceed exactly as in reference 1. We shall not reproduce the argument in detail. Essentially, one shows that for momentum transfers τ such that

$$\int_0^\infty e^{\tau r}r^2F(r)dr<\infty,\tag{104}$$

the series in (102) will converge for all $k\leqslant\tau/2$. Notice

[9] J. Lepore, Phys. Rev. 79, 137 (1950).

[10] D. S. Carter, thesis, Princeton, 1952 (unpublished).

that (104) is a more stringent restriction than that required to derive the dispersion relations, Eq. (59).

V. BOUND STATE CONTRIBUTIONS

Our final task is to discuss the bound state contributions to the dispersion relations; i.e., to show how the residues R_{jA} and R_{jB} can be computed from information about the bound states.

From (6) and (22) we have

$$T_{fi}=(w_f,Vw_i)+\left(w_f,\ V\frac{1}{E-H-V+i\epsilon}Vw_i\right). \quad (105)$$

Let us introduce a complete set of solutions of the Dirac equation. This includes scattering solutions ψ_s, with continuous label s, and discrete bound state solutions ψ_j, with $-m<E_j<m$. Using the completeness relation

$$\sum_j \psi_j\psi_j^\dagger+\int ds\psi_s\psi_s^\dagger=1, \quad (106)$$

we write

$$T_{fi}=V(\tau)+\sum_j \frac{1}{E-E_j}(w_f,V\psi_j)(\psi_j,Vw_i)+\text{continuum}. \quad (107)$$

We are interested only in the bound state contributions. From the Dirac equation itself we have

$$V\psi_j=(E_j-H)\psi_j; \quad (108)$$

and thus,

$$T_{fi}=\sum_j \frac{1}{E-E_j}(E_j-E)^2(w_f,\psi_j)(\psi_j,w_i)+\cdots. \quad (109)$$

The residue of T_{fi} at the bound state energy E_j is evidently given by

$$R_T=\lim_{E\to E_j}(E_j-E)^2(w_f,\psi_j)(\psi_j,w_i). \quad (110)$$

Despite appearances, this limit does not vanish; as we shall presently see, the matrix elements have appropriate singularities.

The procedure is best illustrated now by means of a simple example.[11] Suppose there is a bound $S_{\frac{1}{2}}$ state (even parity, total angular momentum $\frac{1}{2}$). Let W denote the binding energy: $W=m-E_j$. The bound state wave function has the form (for magnetic quantum number m)

$$\psi^m=\frac{1}{r}\begin{pmatrix} g(r)\chi^m \\ if(r)(\boldsymbol{\sigma}\cdot\mathbf{r}/r)\chi^m\end{pmatrix}, \quad (111)$$

where χ^m is a two-component Pauli spinor. A sum over magnetic quantum numbers is implied in (110). The radial functions satisfy the equations

$$\frac{dg}{dr}-\frac{g}{r}+(2m-W-V)f=0,$$
$$\frac{df}{dr}+\frac{f}{r}+(W+V)g=0. \quad (112)$$

Asymptotically, the radial functions have the following behavior

$$g\underset{r\to\infty}{\longrightarrow}Ne^{-\kappa r};\ f\underset{r\to\infty}{\longrightarrow}\rho Ne^{-\kappa r}, \quad (113)$$

where

$$\kappa=[W(2m-W)]^{\frac{1}{2}},\quad \rho=[W/(2m-W)]^{\frac{1}{2}}; \quad (114)$$

and N is the normalization constant. In general, let us now write

$$g=N[e^{-\kappa r}-u(r)],$$
$$f=\rho N[e^{-\kappa r}-v(r)]; \quad (115)$$

where we only have to know about u and v that: $e^{\kappa r}u\to e^{\kappa r}v\to 0$, as $r\to\infty$; and $u(0)=v(0)=1$. The functions u and v do not contribute singular terms in the matrix elements of (110) and hence do not contribute to the residue. On the other hand, we find

$$\int\frac{1}{r}e^{-\kappa r}e^{i\mathbf{k}\cdot\mathbf{r}}d^3r=\frac{4\pi}{E^2-E_j^2}, \quad (116)$$

$$\int\frac{1}{r^2}e^{-\kappa r}e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\sigma}\cdot\mathbf{r}d^3r=\frac{4\pi}{E^2-E_j^2}\left(\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{k}\right)\left(\frac{\kappa}{ik}\right). \quad (117)$$

We now evaluate the singular part of the spinor matrix $(w_f,\psi_j)(\psi_j,w_i)$, then sum over magnetic quantum numbers, use the Dirac equation to express the result in the standard form of Eq. (72), and finally, pass to the limit $E\to E_j$. There results the following expressions for the residues $R_A(\tau)$ and $R_B(\tau)$:

$$R_A(\tau)=\frac{(2\pi)^2}{(m-W)m(2m-W)^2}\times\{2m(2m-W)+\tfrac{1}{2}\tau^2\}|N|^2; \quad (118)$$

$$R_B(\tau)=\frac{(2\pi)^2}{(m-W)m(2m-W)^2}|N|^2. \quad (119)$$

The final results thus depend on two parameters which characterize the bound state: W, the binding energy, and N, the normalization constant defined in (115). The expressions become somewhat more transparent if we replace N by another parameter, a characteristic length r_0 defined by

$$\tfrac{1}{2}(1+\rho^2)r_0=\int_0^\infty\left\{(1+\rho^2)e^{-2\kappa r}-\frac{1}{|N|^2}(g^2+f^2)\right\}dr. \quad (120)$$

[11] A similar calculation was first done for nucleon-nucleon scattering, by Goldberger, Nambu, and Oehme, Ann. Phys. (to be published).

One finds

$$|N|^2=2\kappa\left(\frac{1}{1+\rho^2}\right)\left(\frac{1}{1-\kappa r_0}\right). \qquad (121)$$

For small binding energy r_0 plays the role of effective range of the potential.

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APPENDIX

We shall derive here the result expressed in Eq. (71). For this purpose it is convenient to introduce a strength parameter λ which multiplies U and which at the end is set equal to unity. Using (16) and (43), let us first write down the Born series for the matrix T, leaving aside for the moment the question of convergence. For very large $|E|$ we can replace $(E^2-m^2-\tau^2/4)^{\frac{1}{2}}$ by k; so for large $|E|$ the Born series reads

$$T(E,\tau)=\frac{\lambda}{2E}U(E,\tau)+\sum_{n=2}^{\infty}T_n(E,\tau), \qquad (A1)$$

where $U(E,\tau)$ is the Fourier transform of U and

$$T_{n+1}=\frac{\lambda}{2E}\int\exp\left[-i\frac{\boldsymbol{\tau}}{2}\cdot(\mathbf{x}+\mathbf{y})\right]\times\exp[-ik\mathbf{n}\cdot(\mathbf{x}-\mathbf{y})]U(\mathbf{x})K_n(\mathbf{x},\mathbf{y})d^3xd^3y. \qquad (A2)$$

From the recurrence formula

$$K_n(\mathbf{x},\mathbf{y})=-\frac{\lambda}{4\pi}\int\frac{e^{ik|\mathbf{x}-\mathbf{z}|}}{|\mathbf{x}-\mathbf{z}|}U(\mathbf{z})K_{n-1}(\mathbf{z},\mathbf{y})d^3z, \qquad (A3)$$

with

$$K_1(\mathbf{x},\mathbf{y})\equiv K(\mathbf{x},\mathbf{y})=-\frac{\lambda}{4\pi}\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}U(y), \qquad (A4)$$

we can rewrite T_{n+1} as follows, for $n>1$:

$$T_{n+1}=\frac{\lambda}{2E}\int\exp\left[-i\frac{\boldsymbol{\tau}}{2}\cdot(\mathbf{x}+\mathbf{y})\right]\exp[-ik\mathbf{n}\cdot(\mathbf{x}-\mathbf{y})]\times U(\mathbf{x})K_{n-1}(\mathbf{x},\mathbf{y})R(\mathbf{y})d^3xd^3y; \qquad (A5)$$

where

$$R(\mathbf{x})=-\frac{\lambda}{4\pi}\int\exp\left(-i\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{z}\right)\exp(-ik\mathbf{n}\cdot\mathbf{z})\times U(\mathbf{x}+\mathbf{z})\frac{1}{z}e^{ikz}d^3z. \qquad (A6)$$

The crucial point is now this. For potentials which satisfy the Eqs. (58) one can show explicitly that, for large $|E|$, $|R(\mathbf{x})|$ is bounded by a finite constant; i.e., for any fixed $\eta>0$

$$R(\mathbf{x})<|\lambda|C, \quad |E|\geqslant m+\eta. \qquad (A7)$$

The factor U contains the troublesome term $2EV$, which would appear to lead to a divergence in (A6). But one can carry out a partial integration[8] which brings in a factor k in the denominator, so the integral (A6) is in fact bounded as $|E|\to\infty$, as asserted in the foregoing. We shall not go into the details here.[12] Using (A7) then, and proceeding by induction, we find

$$T_{n+1}<\left|\frac{\lambda}{2E}\right||\lambda C|^{n-1}\int\left|\exp\left[-i\frac{\boldsymbol{\tau}}{2}\cdot(\mathbf{x}+\mathbf{y})\right]\times\exp[-ik\mathbf{n}\cdot(\mathbf{x}-\mathbf{y})]U(\mathbf{x})K(\mathbf{x},\mathbf{y})\right|d^3xd^3y. \qquad (A8)$$

The foregoing integral is bounded by a finite constant times $|E|^2$. We therefore conclude that $|T_{n+1}/E|$ is bounded, and that the Born series converges uniformly and absolutely if $|\lambda C|<1$. Thus $|T/E|$ is bounded if $|\lambda C|<1$.

But we now assert that for $|E|\to\infty$, T/E in fact vanishes. It is evident that T_{n+1}/E vanishes when $\kappa\equiv\mathrm{Im}k\to\infty$, for in this case the integrand in (A8) contains a damping factor $\exp\{-\kappa[|\mathbf{x}-\mathbf{y}|-\mathbf{n}\cdot(\mathbf{x}-\mathbf{y})]\}$ which vanishes almost everywhere for $\kappa\to\infty$. On the other hand, for $\mathrm{Re}k\to\infty$, the integrand of (A2) oscillates very rapidly almost everywhere because of the factor $\exp[-ik\mathbf{n}\cdot(\mathbf{x}-\mathbf{y})]$. If we divide both sides of (A2) by E, the integral of the absolute value of the integrand on the right-hand side will be finite. Hence, we conclude using the Riemann Lebesgue lemma that T_{n+1}/E vanishes as $\mathrm{Re}E\to\infty$.

This completes the proof that (71) holds true, at least when the Born series converges uniformly—and we know that for sufficiently small values of the strength parameter, $|\lambda C|<1$, the Born series does converge. But now we note that considered as a function of λ, T is a meromorphic function of λ for any fixed E; T was defined as a ratio of two entire functions of λ. For real λ and $|E|>m$, T has no poles. It is therefore clear that if T/E vanishes as $|E|\to\infty$ for $|\lambda C|<1$ it will vanish for all finite real values of λ.

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[12] For a detailed proof of the results of this appendix see N. N. Khuri, thesis, Princeton, 1957 (unpublished).