Dynamical Instability in an Anisotropic Ionized Gas of Low Density*

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(Received November 12, 1957)

It is shown that when the thermal motions of a tenuous ionized gas are sufficiently anisotropic, the gas, and the initially uniform magnetic field which the gas is assumed to contain, become unstable. One mode of instability occurs when the gas pressure is greater parallel to the field than perpendicular, and another mode when the pressure is greater perpendicular than parallel. It is suggested that such instabilities may be of astrophysical interest, particularly with regard to the configuration of the solar dipole field as it is drawn out into interplanetary space by ionized gas from the sun.

I. INTRODUCTION

To the extent that the dynamical properties of an ionized gas satisfy the hydromagnetic equations

$$\rho d\mathbf{v}/dt = -\nabla(p+B^2/8\pi)+(\mathbf{B}\cdot\nabla)\mathbf{B}/4\pi,$$
$$\partial\mathbf{B}/\partial t = \nabla\times(\mathbf{v}\times\mathbf{B}),$$

there are at least three well-known instabilities which may upset a static equilibrium configuration. They are the twist instability, the Taylor instability, and the flute instability. The twist instability was considered first by Lundquist[1] and later by Roberts.[2] The Taylor instability was first described by Kruskal and Schwarzschild,[3] and an excellent discussion of both the Taylor and the flute instabilities appears in a recent article by Rosenbluth and Longmire,[4] where they present a derivation based on the individual-particle treatment for gases of low density, rather than relying (in that case) upon the hydromagnetic equations.

In this paper we shall discuss two further instabilities which may occur in ionized gases of low density, but which do not occur in ordinary hydromagnetic cases. These instabilities arise only in an ionized gas in which the thermal velocities are significantly anisotropic, and they depend explicitly on the anisotropy. We expect anisotropy only in gases which are so tenuous that the time between collisions of the individual particles is long compared to the characteristic dynamical period of the macroscopic mass motions. Therefore it is only in such tenuous gases that we might expect these instabilities to appear.

We shall treat the dynamical properties of an ionized gas of low density with the usual approximations, assuming equal total numbers of electrons and protons, neglecting all particle collisions, and assuming that the cyclotron period and radius of gyration of the ions and electrons are small compared to the characteristic period and scale of the macroscopic fields. Because we shall treat explicitly the possible anisotropy of the thermal motions, it becomes necessary to distinguish between the direction parallel and the direction perpendicular to the magnetic field **B**. We use the subscript s to denote the direction parallel to **B**, and n perpendicular. The ordinary isotropic pressure p is replaced by p_s and p_n, and the mass motion $\mathbf{u}_n$ perpendicular to **B** satisfies the modified hydromagnetic equation[5]

$$\rho d\mathbf{u}_n/dt = -\nabla_n(p_n+B^2/8\pi) + [(\mathbf{B}\cdot\nabla)\mathbf{B}]_n[1+(p_n-p_s)/(B^2/4\pi)](1/4\pi). \quad (1)$$

The magnetic field is in turn related to $\mathbf{u}_n$ by

$$\partial\mathbf{B}/\partial t = \nabla\times(\mathbf{u}_n\times\mathbf{B}), \quad (2)$$

as may be readily seen from the fact that $\mathbf{u}_n$ is the electric drift velocity, $\mathbf{E}=-\mathbf{u}_n\times\mathbf{B}/c$.

Now if the gas pressure p_s parallel to **B** is sufficiently large as compared to the pressure p_n perpendicular to **B**, we see that the coefficient of $(\mathbf{B}\cdot\nabla)\mathbf{B}$ in (1) may become negative. The result is that the hydromagnetic wave equation goes over into an elliptic equation, with resulting instability. This is the first instability in an isotropic gas which we wish to point out.

If now we turn out attention ot the mass motion $\mathbf{u}_s$ parallel to **B**, we find that the situation is somewhat more complicated than with $\mathbf{u}_n$. Instead of an equation of motion of the form of (1), we must solve a Boltzmann equation. We let θ be the angle of pitch; the angle between the magnetic field and the velocity of an individual ion or electron. We define the distribution function $F(s,\theta)$ to be the number of particles per unit volume with angle of pitch θ at some given point which is a distance s along the line of force from the origin. For steady state conditions we find[5,6] that if $F(s,\theta)$ has the form $\sin^\alpha\theta$ at $s=0$, then it has the form

$$F(s,\theta)=N(0)\frac{\Gamma(\alpha+1)}{2^\alpha\Gamma^2[\frac{1}{2}(\alpha+1)]}\left[\frac{B(0)}{B(s)}\right]^{\frac{1}{2}(\alpha-1)}\sin^\alpha\theta \quad (3)$$

elsewhere along the line of force. Here $N(0)$ is the

* Assisted in part by the Office of Scientific Research and the Geophysics Research Directorate, Air Force Cambridge Research Center, Air Research and Development Command, U. S. Air Force.

[1] S. Lundquist, Phys. Rev. **83**, 307 (1951).
[2] P. H. Roberts, Astrophys. J. **124**, 430 (1957).
[3] M. Kruskal and M. Schwarzschild, Proc. Roy. Soc. (London) **A233**, 348 (1954).
[4] M. N. Rosenbluth and C. L. Longmire, Ann. Phys. **1**, 120 (1957).
[5] E. N. Parker, Phys. Rev. **107**, 924 (1957).
[6] K. M. Watson, Phys. Rev. **102**, 19 (1956).

number of particles per unit volume at $s=0$, and $B(s)$ is the field density. We note that the density of particles at s is

$$N(s)=\int_0^{\pi} d\theta F(s,\theta)$$
$$=N(0)[B(0)/B(s)]^{\frac{1}{2}(\alpha-1)}. \quad (4)$$

Isotropy obtains for $\alpha=1$, yielding uniform density. But if the thermal motions are principally perpendicular to $\mathbf{B}$, so that $p_n>p_s$, then $\alpha>1$, and we see that $N(s)$ is largest where $B(s)$ is smallest. The gas pressure in regions of low field density may become sufficiently large, in this steady state case, as further to expand the field there, thereby resulting in dynamical instability.

It is the purpose of this paper to show that at least the instability arising when $p_s>p_n$ is of astrophysical importance. It is to be noted that this instability may be expected to occur after expansion of the gas perpendicular to $\mathbf{B}$, or compression parallel to $\mathbf{B}$, since such processes tend to increase p_s/p_n and to decrease α.

We tentatively suggest (we could at the moment construct only speculative examples) that the instability arising when $p_s<p_n$, as a consequence of expansion parallel and/or compression perpendicular to $\mathbf{B}$, may also prove to be of astrophysical interest.

In general we would expect the instabilities to occur wherever we have anisotropic compression or expansion in a region of sufficiently low density that the collision rate allows p_s to become different from p_n. One thinks of the expanding shells of novae, of the 100 km/sec galactic halo motions,[7] and, locally, of the gas moving radially outward with more or less constant velocity from the sun.[8] In this last case we might hope to account for a disordered magnetic shell around the inner solar system, which seems to be required by the isotropy and form of decay of the increased cosmic-ray intensity from solar flares[9]; the matter is considered briefly in the last section.

II. INSTABILITY WHEN $P_s>P_n$

Consider plane transverse hydromagnetic waves in the uniform magnetic field

$$\mathbf{B}=\mathbf{e}_z B_0,$$

where $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ represent unit vectors along the coordinates axes. We suppose that the waves are of small amplitude and consist of the magnetic field $\mathbf{b}(z,t)$ in the y direction. Then

$$\mathbf{B}=\mathbf{e}_z B_0+\mathbf{e}_y b(z,t). \quad (5)$$

[7] D. S. Heeschen, Astrophys. J. **124**, 662 (1956); *Third Symposium on Cosmical Gas Dynamics, June 24–28, 1957* (Smithsonian Institution, Astrophysical Observatory, Cambridge, Massachusetts).

[8] L. Biermann, Z. Astrophys. **29**, 274 (1951); Z. Naturforsch. **7a**, 127 (1952); Observatory **77**, 109 (1957).

[9] Meyer, Parker, and Simpson, Phys. Rev. **104**, 768 (1956).

The field density, B, is B_0 plus terms second order in ϵ; $\epsilon=b/B_0$. Thus, since we shall neglect all terms second order in ϵ, we have that p_n and p_s are unperturbed by the wave and its variations in magnetic pressure. What is more, $d/dt\cong\partial/\partial t+O^2(\epsilon)$, and (1) reduces to $\mathbf{u}_n=\mathbf{e}_y u_n$, where

$$\partial u_n/\partial t=[B_0(1+\xi)/4\pi\rho]\partial b/\partial z; \quad (6)$$

to the order considered ξ is a dimensionless constant, $4\pi(p_n-p_s)/B_0^2$. (2) reduces to

$$\partial b/\partial t=B_0\partial u_n/\partial y. \quad (7)$$

We eliminate u_n between (6) and (7) in the usual manner to obtain the wave equation

$$\partial^2 b/\partial t^2-[B_0^2(1+\xi)/4\pi\rho]\partial^2 b/\partial z^2=0. \quad (8)$$

The velocity of propagation of the wave is $(1+\xi)^{\frac{1}{2}}$ times the usual hydromagnetic velocity, $B_0/(4\pi\rho)^{\frac{1}{2}}$. Thus, in the limit as $B\rightarrow 0$ the hydromagnetic wave velocity becomes $[(p_n-p_s)/\rho]^{\frac{1}{2}}$, and may be comparable to the speed of sound.

The physical explanation for the change in the velocity of propagation is straightforward. The thermal motion perpendicular to $\mathbf{B}$, which produces p_n and which tends to make ξ and the velocity of propagation large, results in a drift of ions, perpendicular to $\mathbf{B}$ and perpendicular to gradients in B. The resulting current density is in the opposite direction[5] to the current density $(c/4\pi)\nabla\times\mathbf{B}$ required by ordinary hydromagnetic theory. Thus in Maxwell's equation,

$$4\pi\mathbf{i}+\partial\mathbf{E}/\partial t=c\nabla\times\mathbf{B}, \quad (9)$$

the curl of $\mathbf{B}$ tends not to be balanced immediately by $\mathbf{i}$, with the result that $\partial\mathbf{E}/\partial t\neq 0$. The resulting electric field means that the electric drift and $\partial\mathbf{B}/\partial t$, which is equal to $-c\nabla\times\mathbf{E}$, must be larger than would otherwise be the case, resulting in more rapid propagation of the wave.

On the other hand, the thermal motion of the ions parallel to $\mathbf{B}$, which produces p_s and which tends to make ξ and the velocity of propagation smaller, results in an opposite drift of the ions perpendicular to $\mathbf{B}$. The drift arises from the fact that it yields a Lorentz force on the ions which is a necessary back reaction against the centrifugal force as the ions move along curving magnetic lines of force; the drift produces a current in the same direction as $\nabla\times\mathbf{B}$. Thus $4\pi\mathbf{i}$ and $c\nabla\times\mathbf{B}$ may be balanced more quickly in (9), and $\partial\mathbf{E}/\partial t$ may be smaller than in the usual hydromagnetic case; the wave need not propagate as rapidly.

But now consider the interesting situation[10] when $p_s>p_n$ and when B_0 is sufficiently small that $\xi<-1$.

[10] Professor Chandrasekhar has kindly pointed out that the solution to this problem can be obtained as a special case of the general stability of the pinch; Watson, Kaufman, and Chandrasekhar, Proc. Roy. Soc. (London) (to be published).

Then in place of (8) we have

$$\partial^2 b/\partial t^2 + [B_0^2|1+\xi|/4\pi\rho]\partial^2 b/\partial z^2 = 0. \quad (10)$$

If

$$b(z,t) = b_0 \cos kz \exp(t/\tau), \quad (11)$$

then

$$\tau = \pm(4\pi\rho)^{\frac{1}{2}}/B_0 k|1+\xi|^{\frac{1}{2}}. \quad (12)$$

Any irregularities in the field grow exponentially in time with a characteristic period of the order of the time required for a hydromagnetic wave to propagate the length of the irregularity, assuming $|1+\xi|$ to be of the order of unity. We note that we never have instability in the limit of large B_0, because the field is then stiff enough to resist whatever upsetting effects the gas may produce. In the limit as B_0 goes to zero we have

$$\tau \sim [(4\pi)^{\frac{1}{2}}/k][\rho/(p_s - p_n)]^{\frac{1}{2}}. \quad (12a)$$

We see that τ is independent of the density, since ρ/p is just the thermal velocity. We note also that $0<\tau<+\infty$ so long as $p_s>p_n$; the difference between p_s and p_n may be as small as we like and we will still have instability if we wait long enough.

We may understand the physical essence of the instability by noting that it is analogous to the case of a train of beads sliding on a string. The centrifugal force of the beads as they slide around a wave in the string tends to increase the amplitude of the wave.

IV. DISCUSSION

With our brief presentation of the instabilities arising when $p_n \neq p_s$, let us now consider where and when such anisotropy might arise. In the first place, we have neglected to distinguish carefully between the electron and the ion component of the gas. The electron thermal velocity will be rather larger than the ion thermal velocity, with the result that the electron collision rate will be very much higher. We know of no circumstances, except perhaps in a shock front, where the characteristic dynamical time of the mass motions is expected to be less than the electron collision time. Therefore, we do not expect the electron pressure to deviate significantly from isotropy.

It seems, however, that instances may occur where the *ion* collision rate is sufficiently low that anisotropy may occur. Consider, for instance, the general outward flow of gas from the sun, with radial velocities at the orbit of the earth of $v=1000$ km/sec.[8] The temperature of the gas we take to be about 4×10^5 °K at the orbit of the earth,[11] so that the ion thermal velocity is of the order of 100 km/sec. If the collision cross section is 10^{-15} cm^2, then the mean free path is 10^{12} cm for a density of $N=10^3$ ions/cm^3. The time between collisions is $\sim 10^5$ sec. Now the density is dropping off as $1/r^2$ (where r is the distance from the sun) with a characteristic time of $r/2v$ or 0.75×10^5 sec. Thus, at the earth the collision and dynamical times are about equal. Beyond the earth the collision time increases as r^2 and the characteristic time of expansion as r. Thus, at the orbit of the earth and beyond, the anisotropic expansion of the radial flow of gas from the sun may result in anisotropic ion pressures, $p_s \neq p_n$.

Since the lines of force of the solar dipole field will be drawn out into radial lines by continued outflow of solar gas, the general field density will fall off as $1/r^2$; 1 gauss on the sun[12] yields about 2×10^{-5} gauss at the earth. Then $B^2/8\pi \cong 10^{-11}$ erg/cm^3, whereas the gas pressure, NkT is about 0.6×10^{-7} erg/cm^3. Thus the field density is relatively small and we have instability whenever the total pressure is anisotropic by 2×10^{-4} or more. Since the electron pressure is probably never very anisotropic, the ion pressure anisotropy must exceed about 4×10^{-4} to produce instability.

The expansion of the gas moving outward from the sun is perpendicular to the radial direction, and at the orbit of earth is perpendicular to the nearly radial (if $v\cong1000$ km/sec) lines of force from the sun. Thus we would expect that p_n will become somewhat less than p_s. By the time the orbit of the Earth is reached (through not much inside the orbit) we might expect the instability described by (12) to occur. The asymptotic form (12a) is applicable. Suppose that $(p_s-p_n)/p_n \cong 0.1$ and $k=2\pi/\lambda$, where λ is the scale of an irregularity. Since $p/\rho \cong \frac{1}{3}w^2$, where w is the ion thermal velocity, we have $\tau = 3\times10^{-7}\lambda$ sec. In the time that the gas moves one astronomical unit at 1000 km/sec (1.5×10^5 sec), an inhomogeneity with a scale as large as 5×10^6 km can increase its amplitude by a factor of e; smaller inhomogeneities increase proportionately more rapidly.

Therefore, beyond the orbit of the earth we might expect, on the above theoretical grounds, to find inhomogeneities in **B** in all directions from the sun in which there is corpuscular emission.[8] The observation of the slow decay of cosmic rays from solar flares suggests[9] that such a heliocentric region of disordered magnetic field does actually exist.

[11] S. Chapman, Smithsonian Astrophys. Contrib. **2** (1957).

[12] H. W. Babcock and H. D. Babcock, Astrophys. J. **121**, 349 (1955).