Use of the Chew-Low Equation in Strong Coupling*

C. J. GOEBEL
University of Rochester, Rochester, New York
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In charged scalar fixed-point-source theory, the isobar energies and scattering amplitudes in the strong-coupling limit are derived by extending the Serber-Lee solution with the aid of perturbation arguments.

WE shall show in this note that in charged scalar fixed-point-source theory, all the strong-coupling ($\mathfrak{g}\rightarrow\infty$) results (isobar energies and scattering amplitudes)[1] can be deduced from the "reciprocal" Chew-Low equation[2] plus some perturbation ("Feynman diagram") arguments. By the reciprocal Chew-Low equation we mean the equation for the reciprocal of the scattering amplitude, due to Chew and Low[2] and used by Serber and Lee[3] to solve the charged scalar theory in the one-meson approximation; Castillejo, Dalitz, and Dyson[4] noted its use to derive the isobar energy in this approximation.

As was pointed out by Lee and Serber,[3] the Chew-Low equation is very simple in the charged scalar theory, because the crossing relation is simply $f^+(\omega)=f^-(-\omega)$ [here $f^+(\omega)$ is the scattering amplitude of a positive meson, defined on the ω plane cut from $-\infty$ to $-\mu$ and from μ to ∞; we shall henceforth write just f for f^+]. Hence there is just one Chew-Low equation, identical to the causal dispersion relation, namely

$$f(\omega_p)=f^B(\omega_p)+\frac{1}{\pi}\int\frac{d\omega\,\kappa}{\omega-\omega_p-i\epsilon}\frac{\sigma_T(\omega)}{4\pi},\qquad(1)$$

where $\sigma_T(\omega)\equiv\sigma_T^+(\omega)=\sigma_T^-(-\omega)$; the range of integration is where κ is real, namely from $-\infty$ to $-\mu$ plus μ to ∞. It is to be noted that from the nature of the equation, the only singularities of f are those of f^B. If the coupling is so weak that there are no (stable) isobars, then the only singularity of f is at $\omega=0$, the "Yukawa bound state" pole. Then f^B is just the Born approximation (lowest order) scattering amplitude, the renormalized coupling constant being used. For scattering on a proton,

$$f^B=f^{BA}=2\beta_c/\omega,\quad \beta_c=\mathfrak{g}_c^2/4\pi.\qquad(2)$$

Following Chew and Low,[2] we now introduce the function

$$g(\omega)=\frac{2\beta_c}{\omega f(\omega)};$$

we note that

$$\mathrm{Im}g=-\frac{2\beta_c\,\mathrm{Im}f}{\omega|f|^2}=-2\beta_c\frac{\kappa}{\omega}\frac{\sigma_T}{\sigma_{\mathrm{el}}}.\qquad(3)$$

If f has no zeros, then g has no poles and can be written in the form

$$g(\omega_p)=1+\frac{\omega_p}{\pi}\int\frac{d\omega\,\mathrm{Im}g(\omega)}{\omega(\omega-\omega_p-i\epsilon)}\qquad(4)$$

(see Chew and Low,[2] also Castillejo *et al.*,[4] for the argument). Using (3) for Img, we have the key equation

$$g(\omega_p)=1-\frac{2\beta_c}{\pi}\omega_p\int\frac{d\omega\,\kappa}{\omega^2(\omega-\omega_p-i\epsilon)}\cdot\frac{\sigma_T}{\sigma_{\mathrm{el}}},\qquad(5)$$

in which the only unknown is the ratio $\sigma_T/\sigma_{\mathrm{el}}$. If the coupling is so weak that there is no stable isobar, then there is no charge exchange scattering, so that the only contribution to $\sigma_T-\sigma_{\mathrm{el}}$ is meson production. Thus if we neglect meson production (one-meson approximation), Eq. (5) is completely known:

$$g_1(\omega_p)=1-2\beta_c\frac{\mu+ip}{\omega_p},\quad p=\begin{cases}+(\omega^2-\mu^2)^{\frac{1}{2}} & \text{for } |\omega|>\mu\\ +i(\mu^2-\omega^2)^{\frac{1}{2}} & \text{for } |\omega|<\mu\end{cases}\qquad(6)$$

as found by Lee and Serber.[3] (The subscript 1 stands for 1-meson approximation.)

The crucial remark to be made now, is that Eq. (5) for $g(\omega)$ is completely independent of any singularities which f might have,[5] in addition to the "Yukawa" singularity at $\omega=0$. In fact, we note that if $\beta_c>\frac{1}{2}$, g_1 acquires a zero at $\omega=\mu[\beta_c+\frac{1}{4}\beta_c^{-1}]^{-1}\equiv\Delta_1$; hence f_1 has a pole at $\omega=\Delta_1$, and we have

$$f_1^B(\omega)=2\beta_c\left[\frac{1}{\omega}-\frac{\lambda_1}{\omega-\Delta_1}\right],\quad \lambda=\frac{\beta_c^2-\frac{1}{4}}{\beta_c^2+\frac{1}{4}}.\qquad(7)$$

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[1] The latest paper on this subject is A. Pais and R. Serber, Phys. Rev. **105**, 1636 (1957). Our coupling constant differs from theirs; ours is defined by $H_I=\mathfrak{g}\tau_\alpha\phi_\alpha(0)$, $\alpha=1, 2$.

[2] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[3] R. Serber and T. D. Lee, private communication quoted in reference 4; also private communication in reference 2.

[4] Castillejo, Dalitz, and Dyson, Phys. Rev. **101**, 453 (1956).

[5] An equally crucial comment on our use of Eq. (5) is that we *assume* that as the coupling becomes strong f continues to have no zeros; for if this were not so, singular terms, *a priori* unknown, would have to be added to the right-hand side of Eq. (5). We bolster our assumption by an appeal to a principle of "conservation of zeros": as we vary $\mathfrak{g}$, zeros of f can only move around, they cannot be created. As a contrary example suppose that, when taking (7) as the "bound term" of (1), Δ and λ were arbitrary (fixed) numbers unconnected with β_c, i.e., that in *weak* coupling there were particles N_{++} and N_- in addition to p and n. Then in the weak-coupling limit f would have a zero at $\omega=\Delta/(1-\lambda)$, and for stronger coupling f would still have a zero somewhere; this f would then be qualitatively entirely different from f_1, Eq. (6) above.

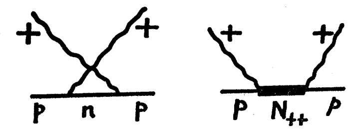

FIG. 1. Scattering diagrams for f_{el}^{BA}.

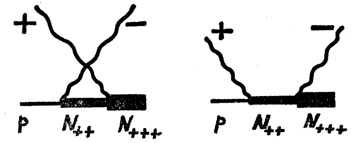

FIG. 2. Scattering diagrams for f_{ex}^{BA}.

That is, in the "one-meson approximation,"[6] if $\beta_c > \frac{1}{2}$ there exists an N_{++} isobar with (excitation) energy Δ_1 and coupling to meson and nucleon of strength $(\lambda_1)^{\frac{1}{2}}$ times the *nucleon*-nucleon-meson coupling.[7] We conclude that when $\beta_c > \frac{1}{2}$ the correct ("one-meson") Chew-Low equation has (7) as its inhomogeneous term; presumably, the original ("one-meson") Chew-Low equation [i.e., with $f^B = 2\beta_c/\omega$] *has no solution* when $\beta_c > \frac{1}{2}$.

We now consider the strong-coupling (s.c.) limit, $\mathfrak{g} \rightarrow \infty$. We take, temporarily, two results from s.c. theory: (a) the cross section for meson production goes as $1/\mathfrak{g}^2$ and can thus be dropped in the s.c. limit, (b) charge exchange scattering equals elastic scattering in the s.c. limit. These facts imply that $\sigma_T/\sigma_{el} = 2$, and thus (5) is again determined:

$$g_{s.c.}(\omega_p) = 1 - 4\beta_c\left(\frac{\mu + ip}{\omega_p}\right). \quad (8)$$

Thus, using (3), we have

$$f_{s.c.}(\omega_p) = \frac{1}{2}\left[\frac{\omega}{4\beta_c} - (\mu + ip)\right]^{-1}. \quad (9)$$

As $\beta_c \rightarrow \infty$, $f_{s.c.} \rightarrow -\frac{1}{2}(\mu + ip)^{-1}$ in agreement with s.c. theory. The zero of $g_{s.c.}$ is at

$$\omega = \mu/2\beta_c \equiv \Delta_{s.c.} (= \tfrac{1}{2}\Delta_1). \quad (10)$$

To compare this isobar energy with s.c. theory, we need to know the renormalization of the coupling constant; fortunately this is trivial, as shown by Lee.[8] The argument can be given verbally: a clothed proton can emit a positive meson only when the core (i.e., the nucleon line) is a proton; the chance of this is the charge renormalization. As the coupling becomes strong, the average number of virtual mesons becomes large, the average charge of the core becomes zero, and the probability that the core is a proton is just $\frac{1}{2}$. Hence $\mathfrak{g}_c = \frac{1}{2}\mathfrak{g}$, $\beta_c = \frac{1}{4}\beta$, and (10) becomes

$$\Delta_{s.c.} = 2\mu/\beta, \quad (10')$$

in agreement with s.c. theory.

Along the same lines as above, we can now treat states of higher charge. Consider the scattering amplitude $f_{++}(\omega)$ of a positive meson on an N_{++} isobar. Using the above results for the energy and the coupling of the N_{++} isobar $[\lambda_{s.c.} = 1 + O(\beta_c^{-2})]$, lowest order perturbation shows that $f_{++}(\omega)$ has a pole term[9] $f_{++}^{BA} = 2\beta_c/(\omega - \Delta)$; thus $g_{++}(\omega)$, defined as $g_{++}(\omega) = f_{++}^{BA}/f_{++}$, satisfies the equation

$$g_{++}(\omega_p) = 1 - \frac{2\beta_c}{\pi}(\omega_p - \Delta)\int \frac{d\omega\,\kappa}{(\omega-\Delta)^2(\omega - \omega_p - i\epsilon)}\cdot\frac{\sigma_T}{\sigma_{el}}. \quad (5')$$

Again putting $\sigma_T/\sigma_{el} = 2$, we have

$$g_{++}(\omega_p) = 1 - 4\beta_c\left(\frac{\bar{\mu} + ip}{\omega_p - \Delta} - \frac{\Delta}{\bar{\mu}}\right),$$

where $\bar{\mu}^2 = \mu^2 - \Delta^2$. The zero of g_{++}, which corresponds to N_{+++}, is found at $\omega_0 = \Delta + \frac{1}{2}\mu/\beta_c = 2\Delta$. Since the zero of energy is here the energy of N_{++}, we have $E_{+++} = 3\Delta$, in agreement with s.c. theory.[10] Also we have $f_{++} \approx f$. We can proceed: calling the energy of the charge Q isobar E_Q, and the amplitude for scattering a positive meson on a charge $Q-1$ isobar f_Q, we have $f_Q^{BA} = 2\beta_c[\omega - (E_{Q-1} - E_{Q-2})]^{-1}$. Then defining $g_Q = f_Q^{BA}/f_Q$, we find that g_Q has a zero at $\omega_0 = E_{Q-1} - E_{Q-2} + \Delta$; thus $E_Q = \omega_0 + E_{Q-1} = 2E_{Q-1} - E_{Q-2} + \Delta$, which implies that $E_Q = \frac{1}{2}Q(Q-1)\Delta$.

In order for our arguments to be independent of s.c. theory, it remains to establish assumptions (a) and (b) above. We sketch the argument, which uses perturbation theory.[11] Using the above results for Δ and λ, we

[6] Actually, when the $Q = 2$ isobar is bound, our ansatz $\sigma_T = \sigma_{el}$ also means neglect of the charge exchange processes $(+,n) \leftrightarrow (-,N_{++})$ and its conjugate $(-p) \leftrightarrow (+,N_-)$. Clearly, when the isobar is almost bound, the corresponding meson production processes $(+,n) \leftrightarrow (-+,p)$ and $(-,p) \leftrightarrow (+--,n)$ will be important; a smooth transition is made as $\mathfrak{g}$ passes the critical "binding" value.

[7] Reference 4, Eqs. (3.4) and (3.5). If $\beta_c < \frac{1}{2}$, Δ_1 would be a zero of g only if p in (6) were equal to $-i(\mu^2 - \omega^2)^{\frac{1}{2}}$. This means that this pole of $f(\omega)$ is found if we go through the cut extending from μ to ∞ in the ω plane, and thus enter the under "physical" Riemann sheet. Note that we have "conservation of poles" as $\mathfrak{g}$ is varied: as $\mathfrak{g}$ is raised, this pole moves out from the under sheet and becomes the "physical" isobar pole. Note also that the fact that the pole has "imaginary coupling" when $\beta_c < \frac{1}{2}$, according to (7), is irrelevant: being unphysical does not make it a ghost, in fact precludes that. It is interesting that as $\mathfrak{g} \rightarrow 0$ this second, non-Yukawa, pole approaches $\omega = 0$, i.e., becomes directly "under" the Yukawa pole, and has *equal* residue; whereas in a nonrenormalizable theory the behavior of an isobar pole as $\mathfrak{g} \rightarrow 0$ is quite different: the pole approaches a pole of the cutoff function v_κ. I am indebted to B. Zumino for an introduction to the concept of unphysical poles.

[8] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

[9] We abbreviate $\Delta_{s.c.}$ by Δ in all that follows.

[10] Note that the difference in energy between the isobar pole and the Yukawa pole system is Δ, exactly as in the charge 2 system. One sees this to be true in every charge state.

[11] The circularity of our argument only precludes the establishment of the uniqueness of the solution; the recourse to perturbation theory may be frowned upon. But this latter, we feel, is an interesting result of the present investigation: namely, that perturbation methods (i.e., "Feynman graphs") can be used successfully in the s.c. region in the charged scalar theory. A

find in the s.c. limit that the lowest approximation to the charge exchange amplitude is $\mu/\omega^2 = -f_{\text{el}}^{\text{BA}}$ (Figs. 1 and 2); and the one-meson production amplitude is of order $\mathfrak{g}^{-1}$. It is then seen that these statements hold in all higher orders of perturbation.

Presumably the above techniques can be used to enter the intermediate coupling region; this is being investigated. A more interesting question is whether the techniques can be used on more complicated theories, for example *symmetric* scalar. Here, as is well known, the complication of the crossing relation prevents the one-meson Chew-Low equation from having a trivial analytic solution[4]; in the s.c. region the essential intractability of such a theory from our viewpoint shows itself in the infinity of isobars coupled in any one charge state.

further example is that in the s.c. limit there is a two-to-one correspondence of charged scalar to neutral scalar self-energy diagrams, and this observation immediately leads to the correct self energy.

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Admissible Solutions of the Covariant Two-Body Problem*

F. L. SCARF, *Department of Physics, University of Washington, Seattle, Washington*

AND

H. UMEZAWA,† *Department of Physics, University of Tokyo, Tokyo, Japan*

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The spectrum of Wick's two-body problem is examined with respect to normalization conditions and the assumptions made in deriving the equation. The problem is considerably simplified by using a formalism based on Sakata-Taketani field operators. It is found that the spurious state amplitudes (suitably normalized) have a behavior inconsistent with the assumptions used by Gell-Mann and Low to derive the bound-state equation. Similar results are found for the two-fermion problem if the vertex operator is $(1+\gamma_5)$, and it is suggested that all spectra of the Bethe-Salpeter equation contain only three quantum numbers.

I. INTRODUCTION

THE covariant many-body equation[1] has been completely solved for one problem only: Wick[2] has derived a Bethe-Salpeter (B.S.) equation for a pair of scalar particles interacting through a scalar massless field and, together with Cutkosky,[3] he has obtained complete solutions in the ladder approximation. Although these solutions are labeled by four quantum numbers (n,l,m,κ), investigations have indicated that only the $\kappa=0$ states correspond to those obtained in nonrelativistic theory.[2–4] It seems plausible that the $\kappa>0$ states are actually spurious[2,5] but no rigorous means of eliminating them has been proposed.

Formal normalization conditions and methods of calculating expectation values for all B.S. amplitudes have been derived in recent years.[6] These conditions are applied here to investigate the origin and significance of the $\kappa>0$ states. It has been found that considerable simplification and clarification of the problem can be attained if this discussion is carried out in a formalism differing from that used by Wick. For instance, we observe that one needs four independent components $(\phi_a,\pi_a,\phi_b,\pi_b)$ to describe a state of two noninteracting scalar particles on a space-like surface, and it is natural to infer that the same number is necessary to describe a bound state. This suggests that knowledge of Wick's one-component amplitude, Φ (on a space-like surface), is insufficient to describe the system, construct expectation values, or determine which states are admissible.

Accordingly, a formalism using two-component Sakata-Taketani (S.T.) field operators[7] is employed to derive a B.S. equation for a four-component amplitude, χ. Since the equation is linear in the two energies $p_0{}^a$, $p_0{}^b$, one obtains a Breit equation of the form $[E-H_0{}^a-H_0{}^b-V^{ab}]\chi=0$ if retardation is neglected. Another consequence of this linearity is a natural reduction to a one-particle problem when $m_b/m_a\rightarrow\infty$.

The solution of the new B.S. equation can be expressed in terms of the Wick-Cutkosky solutions, $\Phi_{n,\kappa}{}^{l,m}$. It is found that $\chi_{n,\kappa}{}^{l,m}(\mathbf{x}, t=0)$ is regular for

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[2] G. C. Wick, Phys. Rev. **96**, 1124 (1954).

[3] R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).

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[7] S. Sakata and M. Taketani, Proc. Math. Phys. Soc. (Japan) **22**, 757 (1940); W. Heitler, Proc. Roy. Irish Acad. **49**, 1 (1943). Although this linear formalism for bosons is not manifestly covariant, it is used instead of the Kemmer equations in order to avoid redundant components.