Possible Determination of the Spin of Λ^0 from Its Large Decay Angular Asymmetry

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General consideration of the angular distribution of the decay products of a hyperon into a pion and a nucleon is carried out for arbitrary values J of the hyperon spin. Limitations on the magnitude of the asymmetry in the angular distribution are found. These limitations are formulated in terms of certain test functions which when applied to experimental results may lead to an unambiguous determination of the value of the hyperon spin. These considerations and the large "up-down" asymmetry in Λ^0 decay reported in recent literature suggest that the spin of Λ^0 is $\frac{1}{2}$.

I

RECENT experiments[1] have shown that in the decay of (partially) polarized Λ^0 there is a strong asymmetry in the distribution of the momentum of the decay π^-. We wish to point out in the present note that such a strong asymmetry may also serve to rule out high values of spin for Λ^0. No nonrelativistic approximation on either of the decay products is made in this note.

We consider a sample $\mathcal{S}$ of Λ^0's in their rest system, produced in any process or collection of processes, selected in any manner as long as *the selection process does not involve the angular distribution of the decay products of* Λ^0. It is well known that the angular distribution $\mathcal{g}$ of the decay pion from such a sample $\mathcal{S}$ when expanded into spherical harmonics Y_{LM} involves only L values up to $2J$, a conclusion that follows from the law of invariance under space rotation. Such conclusions on the maximum complexity of an angular distribution have been widely used to yield information on the spins of various systems. An additional type of conclusion, which we shall discuss below, is that the coefficients of such an expansion in Y_{LM} satisfy certain inequalities, which in the case of Λ^0 decay can lead also to useful information about the spin of Λ^0. To be more specific, let

$$\tfrac{1}{2}\mathcal{g}(\xi)d\xi, \quad (1\geqq\xi\geqq-1), \tag{1}$$

where

$$\xi=\cos\zeta, \tag{2}$$

be the distribution with respect to $\cos\zeta$. Here ζ is the angle between the decay proton momentum and the z axis, which is any direction fixed in any manner (e.g., with respect to the production process), but independent of the angular distribution of the decay products. As examples, one can quote the following two theorems (proved in the appendix):

Theorem 1

$$\frac{-1}{2J+2}\leqq\langle\xi\rangle\leqq\frac{1}{2J+2}, \tag{3}$$

where

$$J\equiv\text{spin of }\Lambda^0$$

and $\langle\ \rangle\equiv$average over the distribution $\mathcal{g}$.

Theorem 2

If one assumes that $\mathcal{g}$ is linear in ξ;

$$\mathcal{g}(\xi)=1+\alpha\xi, \tag{4}$$

then

$$\frac{-1}{6J}\leqq\langle\xi\rangle\leqq\frac{1}{6J}. \tag{5}$$

The experimental results[1] so far, with a total of $N\sim500$ cases indicate that [with the $+z$ axis chosen along $\mathbf{p}_\Lambda\times\mathbf{p}_{in}$]

$$\langle\xi\rangle\sim0.17. \tag{6}$$

This large value of the average of ξ compared to the maxima given by (3) and (4) lends hope that perhaps one can narrow down in an unambiguous way, *free from assumptions* (such as K spin$=0$), the value of the spin of Λ^0.

In the next section the most general conditions on the distribution function for $J=\frac{3}{2}$ and $J=\frac{5}{2}$ are given. These lead to certain test functions which when applied to the experimental data may give a determination of J. The limit of confidence of such tests is also discussed. In the appendix we give the general conditions on the distribution function for arbitrary J.

II

Since we do not discuss the azimuthal distribution of the decay protons, the sample $\mathcal{S}$ can[2] be considered as an *incoherent* mixture of states each with a definite angular momentum $m(=J, J-1, \cdots$ or $-J)$ in units of $\hbar$ along the z axis. We denote by I_m the statistical weight of the

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[1] F. S. Crawford *et al.*, Phys. Rev. **108**, 1102 (1957); F. Eisler *et al.*, Phys. Rev. **108**, 1353 (1957); L. Leipuner and R. Adair, Phys. Rev. **109**, 1358 (1958).

[2] Any state function ψ can be decomposed into a coherent mixture of such states, but the interference term between the states characterized by m and $m'\neq m$ always have the azimuthal distribution (constant)$\times e^{i(m-m')\phi}+$complex conjugate, which contributes zero when integrated over the azimuthal angle ϕ. If azimuthal distribution is also studied, more inequalities can be derived than those discussed in the present paper. They are however of more complicated forms, involving quadratic or higher rank forms in the averages of the spherical harmonics.

state characterized by m. By definition

$$\sum_{m=-J}^{J} I_m = 1, \tag{7}$$

and

$$I_m \geqq 0. \tag{8}$$

The inequalities that will be derived follow from (8). To illustrate the reasoning involved, let us take as an example the case $J=\frac{3}{2}$. The decay proton and π^- must then be in a mixture of $p_{\frac{3}{2}}$ and $d_{\frac{3}{2}}$ states. If the amplitudes of the p and d waves are respectively A and B where $|A|^2+|B|^2=1$, the final state wave function resulting from a Λ^0 with spin component m is

$$Ap_{\frac{3}{2},m}+Bd_{\frac{3}{2},m}.$$

One therefore obtains for the incoherent mixture that makes up $\mathcal{S}$,

$$\mathcal{J}=\sum_{m=-J}^{J} I_m(Ap_{\frac{3}{2},m}+Bd_{\frac{3}{2},m})^\dagger(Ap_{\frac{3}{2},m}+Bd_{\frac{3}{2},m}). \tag{9}$$

We use the notation $(\ )^\dagger(\ )$ to indicate a matrix multiplication with respect to the proton spin coordinate only. $\mathcal{J}$ is therefore a function of the proton momentum direction.

It is easy to verify that

$$F_{\frac{3}{2},m}\equiv p_{\frac{3}{2},m}{}^\dagger p_{\frac{3}{2},m}=d_{\frac{3}{2},m}{}^\dagger d_{\frac{3}{2},m}, \tag{10}$$

$$G_{\frac{3}{2},m}\equiv d_{\frac{3}{2},m}{}^\dagger p_{\frac{3}{2},m}=p_{\frac{3}{2},m}{}^\dagger d_{\frac{3}{2},m}, \tag{11}$$

$$F_{\frac{3}{2},m}=F_{\frac{3}{2},-m}, \tag{12}$$

$$G_{\frac{3}{2},m}=-G_{\frac{3}{2},-m}. \tag{13}$$

One can therefore write (9) in the form

$$\begin{aligned}\mathcal{J}=(I_{\frac{3}{2}}+I_{-\frac{3}{2}})F_{\frac{3}{2},\frac{3}{2}}+\alpha(I_{\frac{3}{2}}-I_{-\frac{3}{2}})G_{\frac{3}{2},\frac{3}{2}}&\\ +(I_{\frac{1}{2}}+I_{-\frac{1}{2}})F_{\frac{3}{2},\frac{1}{2}}+\alpha(I_{\frac{1}{2}}-I_{-\frac{1}{2}})G_{\frac{3}{2},\frac{1}{2}},&\end{aligned} \tag{14}$$

where

$$\alpha=2\,\mathrm{Re}(A^*B)/(|A|^2+|B|^2) \tag{15}$$

is a real constant between -1 and 1 that characterizes an interference between the two final states of different parities. The functions F and G can be easily calculated from their definitions (10) and (11);

$$\begin{aligned}&F_{\frac{3}{2},\frac{3}{2}}=\tfrac{3}{2}(1-\xi^2), && F_{\frac{3}{2},\frac{1}{2}}=\tfrac{1}{2}(1+3\xi^2),\\ &G_{\frac{3}{2},\frac{3}{2}}=\tfrac{3}{2}(-\xi+\xi^3), && G_{\frac{3}{2},\frac{1}{2}}=\tfrac{1}{2}(5\xi-9\xi^3),\end{aligned} \tag{16}$$

where $\xi=\cos\zeta$ and ζ is the angle between decay proton momentum and the z axis.

From the intensity formula (14) one can compute the various averages (over $\mathcal{J}$) of the Legendre polynomials P_L;

$$P_L(\xi)=\frac{1}{2^L L!}\frac{d^L}{d\xi^L}(\xi^2-1)^L,$$

$$\begin{aligned}\langle P_1\rangle&=-(3/15)(I_{\frac{3}{2}}-I_{-\frac{3}{2}})\alpha-(1/15)(I_{\frac{1}{2}}-I_{-\frac{1}{2}})\alpha,\\ \langle P_2\rangle&=-\tfrac{1}{5}(I_{\frac{3}{2}}+I_{-\frac{3}{2}})+\tfrac{1}{5}(I_{\frac{1}{2}}+I_{-\frac{1}{2}}),\\ \langle P_3\rangle&=(3/35)(I_{\frac{3}{2}}-I_{-\frac{3}{2}})\alpha-(9/35)(I_{\frac{1}{2}}-I_{-\frac{1}{2}})\alpha.\end{aligned} \tag{17}$$

These equations, together with (7), yield the following identities:

$$\begin{aligned}I_{\frac{3}{2}}+I_{-\frac{3}{2}}&=\tfrac{1}{2}-\tfrac{5}{2}\langle P_2\rangle,\\ I_{\frac{1}{2}}+I_{-\frac{1}{2}}&=\tfrac{1}{2}+\tfrac{5}{2}\langle P_2\rangle,\\ \alpha(I_{\frac{3}{2}}-I_{-\frac{3}{2}})&=-(9/2)\langle P_1\rangle+(7/6)\langle P_3\rangle,\\ \alpha(I_{\frac{1}{2}}-I_{-\frac{1}{2}})&=-\tfrac{3}{2}\langle P_1\rangle-\tfrac{7}{2}\langle P_3\rangle.\end{aligned} \tag{18}$$

The I's are all positive. Since $|\alpha|\leqq 1$, one must have

$$(I_m+I_{-m})\geqq|\alpha(I_m-I_{-m})|.$$

Hence

$$\begin{aligned}[\tfrac{1}{2}-\tfrac{5}{2}\langle P_2\rangle]&\geqq|-(9/2)\langle P_1\rangle+(7/6)\langle P_3\rangle|,\\ [\tfrac{1}{2}+\tfrac{5}{2}\langle P_2\rangle]&\geqq|-\tfrac{3}{2}\langle P_1\rangle-\tfrac{7}{2}\langle P_3\rangle|.\end{aligned} \tag{19}$$

Clearly one has also from (14) and (16);

$$\langle P_L\rangle=0 \quad \text{for} \quad L\geqq 4. \tag{20}$$

Equations (19) and (20) are necessary conditions for $J=\frac{3}{2}$, for an infinite sample $\mathcal{S}$ (for which the determination of the averages $\langle P_L\rangle$ is exact). *In the absence of additional knowledge* concerning the sample $\mathcal{S}$ and the quantity α, these equations also insure the possibility of mathematically constructing a sample $\mathcal{S}$ of a kind of particle of spin $\frac{3}{2}$ with a suitable mixing of parities in its decay so that the decay product has the distribution $\mathcal{J}$. Mathematically speaking (19) and (20) therefore give a necessary and sufficient condition for $J=\frac{3}{2}$, provided the azimuthal dependence is not considered.

For higher values of J the generalization of (20) is immediate and is well known; $\langle P_L\rangle=0$ for $L\geqq 2J+1$. The generalization of (19) is given in the appendix. The explicit form for the case $J=\frac{5}{2}$ is listed below

$$\begin{aligned}&[\tfrac{1}{3}-(25/12)\langle P_2\rangle+\tfrac{3}{2}\langle P_4\rangle]\\ &\qquad\geqq|-5\langle P_1\rangle+(35/12)\langle P_3\rangle-(11/30)\langle P_5\rangle|,\\ &[\tfrac{1}{3}+(5/12)\langle P_2\rangle-(9/2)\langle P_4\rangle]\\ &\qquad\geqq|-3\langle P_1\rangle-(49/12)\langle P_3\rangle+(11/6)\langle P_5\rangle|,\\ &[\tfrac{1}{3}+(5/3)\langle P_2\rangle+3\langle P_4\rangle]\\ &\qquad\geqq|-\langle P_1\rangle-(7/3)\langle P_3\rangle-(11/3)\langle P_5\rangle|.\end{aligned} \tag{21}$$

III

To test the inequalities (19) it is convenient to define the following four test functions:

$$\begin{aligned}T_{\frac{3}{2},\frac{3}{2}}&\equiv 9P_1+5P_2-(7/3)P_3,\\ T_{\frac{3}{2},-\frac{3}{2}}&\equiv -9P_1+5P_2+(7/3)P_3,\\ T_{\frac{3}{2},\frac{1}{2}}&\equiv 3P_1-5P_2+7P_3,\\ T_{\frac{3}{2},-\frac{1}{2}}&\equiv -3P_1-5P_2-7P_3.\end{aligned} \tag{22}$$

The inequalities (19) then reduce to

$$\langle T_{\frac{3}{2},m}\rangle\leqq 1, \quad m=\tfrac{3}{2},\cdots,-\tfrac{3}{2}. \tag{23}$$

Similarly, we define

$$T_{\frac{5}{2},\frac{5}{2}}\equiv 15P_1+(25/4)P_2-(35/4)P_3-(9/2)P_4+(11/10)P_5,$$
$$T_{\frac{5}{2},-\frac{5}{2}}\equiv -15P_1+(25/4)P_2+(35/4)P_3-(9/2)P_4-(11/10)P_5,$$
$$T_{\frac{5}{2},\frac{3}{2}}\equiv 9P_1-(5/4)P_2+(49/4)P_3+(27/2)P_4-(11/2)P_5, \quad (24)$$
$$T_{\frac{5}{2},-\frac{3}{2}}\equiv -9P_1-(5/4)P_2-(49/4)P_3+(27/2)P_4+(11/2)P_5,$$
$$T_{\frac{5}{2},\frac{1}{2}}\equiv 3P_1-5P_2+7P_3-9P_4+11P_5,$$
$$T_{\frac{5}{2},-\frac{1}{2}}\equiv -3P_1-5P_2-7P_3-9P_4-11P_5.$$

Eq. (21) now becomes

$$\langle T_{\frac{5}{2},m}\rangle \leqq 1, \quad m=\tfrac{5}{2}, \cdots, -\tfrac{5}{2}. \quad (25)$$

The inequalities (23) and (25), for $J=\frac{3}{2}$ and $\frac{5}{2}$, apply to infinite samples. For any finite sample $\mathcal{S}$ of N cases the determination of the average $\langle T\rangle$ of any function T has the standard statistical uncertainty;

$$\langle T\rangle = \left(\frac{1}{N}\sum T\right) \pm N^{-\frac{1}{2}}\left[\left(\frac{1}{N}\sum T^2\right)-\left(\frac{1}{N}\sum T\right)^2\right], \quad (26)$$

where the sums are extended over the finite sample $\mathcal{S}$. By using the various test functions $T_{J,m}$ in (26), (23), and (25) one obtains tests for various values of J together with confidence limits.

The large experimental[1] value of $\langle P_1\rangle = \langle\xi\rangle \cong 0.17$, together with the large positive coefficients of P_1 in $T_{\frac{3}{2},\frac{3}{2}}$ and $T_{\frac{5}{2},\frac{5}{2}}$ make these test functions the most sensitive ones. Lacking the detailed experimental information, we make the following estimates which, it may be hoped, are not too different from the experimental data. We assume *in calculating the right-hand side* of (26) that the experimental distribution of $N=500$ cases follows a linear law like (4). We assume also $\langle P_1\rangle = 0.17$. The test

$$\langle T_{\frac{5}{2},\frac{5}{2}}\rangle \leqq 1$$

then becomes

$$2.55\pm 0.45 \leqq 1,$$

showing that $J=\frac{5}{2}$ is very improbable. For the test

$$\langle T_{\frac{3}{2},\frac{3}{2}}\rangle \leqq 1$$

one obtains

$$1.53\pm 0.27 \leqq 1,$$

which indicates that $J=\frac{3}{2}$ may also become unlikely.

The discussion in the last paragraph is meant only for orientation purposes. One sees from the estimate that if the sample has a slightly larger value of $\langle\xi\rangle = \langle P_1\rangle$, say $\langle P_1\rangle = 0.20$, one can conclude that $J=\frac{5}{2}$ and $J=\frac{3}{2}$ are both quite impossible. It may therefore be worthwhile to select from the experimental sample those Λ^0 produced at an angle of production θ between, say, 30° and 150° for which the average polarization is likely to be greater than the over-all average, and thereby obtain a sample with a larger value for $\langle\xi\rangle$.

APPENDIX

We give here the formulas for general values of J. Derivation of these formulas follows the same line as for the case $J=\frac{3}{2}$. The distribution function $\mathcal{I}$ is expressible in terms of the diagonal elements I_m of the density matrix of Λ^0;

$$\mathcal{I} = \sum_{m>0} F_{J,m}(I_m+I_{-m}) + \sum_{m>0} G_{J,m}(I_m-I_{-m})\alpha, \quad (A1)$$

where α, as before, is a parity mixing parameter[3] with its value between -1 and 1. The functions F and G are given by

$$F_{J,m} = \sum_L D(J-\tfrac{1}{2}, J-\tfrac{1}{2}; J, m, m; L, 0)P_L, \quad (A2)$$
$$G_{J,m} = \sum_L D(J+\tfrac{1}{2}, J-\tfrac{1}{2}; J, m, m; L, 0)P_L, \quad (A3)$$

where

$$D(l',l;J,m',m;L,M) \equiv (-1)^{m'+(1/2)+M+L}(2J+1)(2L+1)(2l+1)^{\frac{1}{2}}(2l'+1)^{\frac{1}{2}} \times \begin{pmatrix} l & l' & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & J & J \\ M & m' & -m \end{pmatrix} \begin{Bmatrix} L & J & J \\ \frac{1}{2} & l & l' \end{Bmatrix}. \quad (A4)$$

The ( ) and { } symbols are the $3j$ and $6j$ symbols which are symmetrical versions of the Clebsch-Gordan-Wigner coefficients and the Racah coefficients.[4]

The averages $\langle P_L\rangle$ are given by

$$\langle P_L\rangle = \sum_{m>0} S_J(L,m)(I_m+I_{-m}), \text{ (even } L), \quad (A5)$$

$$\langle P_L\rangle = \sum_{m>0} S_J(L,m)(I_m-I_{-m})\alpha, \text{ (odd } L), \quad (A6)$$

where

$$S_J(L,m) = 2(-1)^{m-J-\frac{1}{2}L} \times (J,m;J,-m|J,J;L,0)/U_{J,L}, \quad \text{(even } L), \quad (A7)$$

$$S_J(L,m) = 2(-1)^{m-J-\frac{1}{2}L-\frac{1}{2}} \times (J,m;J,-m|J,J;L,0)/V_{J,L}, \quad \text{(odd } L), \quad (A8)$$

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[3] It is of interest to note that for arbitrary spin value J of Λ^0 the longitudinal polarization of the decay proton from unpolarized Λ^0 is always $-\alpha$. The special case of $J=\frac{1}{2}$ has been discussed recently. T. D. Lee and C. N. Yang, Phys. Rev. **108**, 1645 (1957). See also R. Gatto, University of California Radiation Laboratory Report UCRL-3795 (unpublished).

[4] See, e.g., A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1957).

and

$$U_{J,L}=\left[\frac{(2J+L+1)!(2L+1)}{(2J-L)!}\right]^{\frac{1}{2}} \times\frac{(\frac{1}{2}L)!(\frac{1}{2}L)!(J-\frac{1}{2}L-\frac{1}{2})!}{L!(J+\frac{1}{2}L+\frac{1}{2})!}, \quad \text{(A9)}$$

$$V_{J,L}=\left[\frac{(2J+L+1)!(2L+1)}{(2J-L)!}\right]^{\frac{1}{2}} \times\frac{(\frac{1}{2}L-\frac{1}{2})!(\frac{1}{2}L-\frac{1}{2})!(J-\frac{1}{2}L)!}{L!(J+\frac{1}{2}L)!}. \quad \text{(A10)}$$

The symbols $(J, m; J, -m|J, J; L, 0)$ are the standard[4] Clebsch-Gordan-Wigner coefficients. The inverse of the Eqs. (A5) and (A6) are given by

$$I_m+I_{-m}=\sum_{\text{even } L} Q_J(m,L)\langle P_L\rangle \quad \text{(A11)}$$

and

$$\alpha(I_m-I_{-m})=\sum_{\text{odd } L} Q_J(m,L)\langle P_L\rangle, \quad \text{(A12)}$$

where

$$Q_J(m,L)=(-1)^{m-J-\frac{1}{2}L} \times(J, m; J, -m|J, J; L, 0)U_{J,L}, \quad \text{(even } L), \quad \text{(A13)}$$

$$Q_J(m,L)=(-1)^{m-J-\frac{1}{2}L-\frac{1}{2}} \times(J, m; J, -m|J, J; L, 0)V_{J,L}, \quad \text{(odd } L). \quad \text{(A14)}$$

The test functions are

$$T_{J,m}=-\sum_{L=1}^{2J}(J+\tfrac{1}{2})Q_J(m,L)P_L, \quad (m=J, J-1, \cdots, -J). \quad \text{(A15)}$$

The inequalities are

$$[\langle T_{J,m}\rangle]\leqq(J+\tfrac{1}{2})Q_J(m,0)=1, \quad (m=J, \cdots, -J). \quad \text{(A16)}$$

Equation (A16) together with

$$\langle P_L\rangle=0 \quad \text{for} \quad L\geqq 2J+1 \quad \text{(A17)}$$

give a necessary and sufficient condition for the case of spin J, in the mathematical sense discussed in Sec. II.

To prove Theorem 1, one computes from (A8) that

$$S_J(1,m)=-m/[2J(J+1)].$$

Thus (A6) shows that

$$\langle P_1\rangle=-\sum_{m>0}\alpha m(I_m-I_{-m})/[2J(J+1)]$$
$$=-\sum_m\frac{\alpha m I_m}{2J(J+1)}=-\left[\sum_m\frac{\alpha m I_m}{2J(J+1)}\right]\Big/[\sum_m I_m].$$

I.e., $\langle P_1\rangle$ is equal to a weighed average of $[-\alpha m/2J(J+1)]$ with positive weights. Theorem (1) then follows immediately.

To prove Theorem 2, one computes from (A14) that

$$Q_J(m,1)=-6m/(J+\tfrac{1}{2}).$$

Thus if $\langle P_L\rangle=0$ for $L\geqq 2$, (A15) shows that

$$\langle T_{J,J}\rangle=6J\langle P_1\rangle.$$

(A16) then gives directly Theorem 2.

We remark that the functions $F_{J,\frac{1}{2}}$ and $G_{J,\frac{1}{2}}$ have been used before in recent literature.[5]

[5] R. K. Adair, Phys. Rev. **100**, 1540 (1955); S. B. Treiman, Phys. Rev. **101**, 1216 (1956); T. D. Lee and C. N. Yang, Phys. Rev. **104**, 822 (1956).