Magnetization Curve of the Infinite Cylinder

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The nucleation field for an infinite circular ferromagnetic cylinder, with the field applied along its axis, is rigorously calculated by exploring the whole eigenvalue spectrum of Brown's equations.

The nucleation fields for cylinders with "large" radii are found to be those of the magnetization curling. For "small" radii the nucleation field is found to be somewhat smaller than that for the magnetization buckling. However, the difference is only about 1%, and the exact *mode* of reversal is very close to the buckling. The transition from exact buckling to curling is abrupt. These two modes are shown to be the only modes for nucleation for any radius, the other modes giving higher nucleation fields.

It is shown that the magnetization reversal occurs in one jump, which means that the magnetization curve for the cylinder is a rectangular loop.

I. INTRODUCTION

IT has recently been found that the reversal of magnetization in a ferromagnetic cylinder does not occur by "rotation in unison."[1,2] Much more complicated modes have to be considered for the reversal.[3–7] In particular, for a cylinder whose axis is parallel to the field, the reversal starts when the initially saturated state along the axis ceases to be stable, at a certain negative field intensity. This field is called the "nucleation field" and was shown by Brown[7] to be the smallest eigenvalue of a certain set of partial differential equations. The modes of deviation from the initially saturated state are the associated eigenfunctions.[7]

Frei *et al.* have calculated the nucleation field of an infinite cylinder for three modes, namely: spin rotation in unison, "magnetization curling," and "magnetization buckling."[2] However, it has been pointed out by Brown[8] that of these postulated modes, the buckling is not an eigenfunction of his equations, so that the calculation in this case is only approximate. Because of this, a rigorous solution of Brown's differential equations is undertaken and the eigenvalue spectrum is calculated. This calculation will be given in Part II.

In Part III the behavior after the nucleation is considered. Only the modes which are associated with the most positive nucleation field need be treated in this respect. As will be shown in Part II, these modes are the curling and the "exact buckling" which is the eigenfunction approximated by the magnetization buckling calculated previously.[2] The equations which govern the behavior after nucleation were already given in 1940 by Brown.[9] For curling, these equations reduce to an ordinary nonlinear differential equation with boundary conditions, the numerical solution of which is given in Part III. The case of exact buckling however, leads to a complicated set of nonlinear partial equations; therefore, only a Ritz approximation,[10] equivalent to the approximate buckling, is made. This will be justified in Part III.

II. THE NUCLEATION FIELD

A. Eigenvalue Equations

Consider an infinite circular cylinder of radius R, which does not possess magnetocrystalline anisotropy. Let the field $\mathbf{H}$ be in the direction of the cylinder axis which is chosen as the z axis in a Cartesian coordinate system. Let the direction cosines of the spin be α_x, α_y and α_z, respectively. According to Brown,[7] the nucleation field H_n is the least eigenvalue H of the set of equations:

$$\begin{gathered}-(2A/I_s)\nabla^2\alpha_x+\partial U/\partial x+H\alpha_x=0,\\ -(2A/I_s)\nabla^2\alpha_y+\partial U/\partial y+H\alpha_y=0,\\ \nabla^2 U=4\pi I_s(\partial\alpha_x/\partial x+\partial\alpha_y/\partial y),\\ \text{for}\quad x^2+y^2\leqslant R^2,\end{gathered}\tag{1}$$

and

$$\nabla^2 U=0\quad\text{for}\quad x^2+y^2\geqslant R^2,\tag{2}$$

with the following boundary conditions at $x^2+y^2=R^2$:

$$\begin{gathered}\partial\alpha_x/\partial n=\partial\alpha_y/\partial n=0,\\ U_{\text{in}}=U_{\text{out}},\\ -\partial U_{\text{in}}/\partial n+4\pi I_n=-\partial U_{\text{out}}/\partial n.\end{gathered}\tag{3}$$

Here A is the exchange constant,[2] U is the scalar magnetostatic potential associated with the free poles of the magnetization in the cylinder (and does not include the external field), and n is the normal to the cylinder.

In a cylindrical coordinate system (r,φ,z) in which the axis $\varphi=z=0$ coincides with the x axis, Eqs. (1)

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[1] S. Shtrikman and D. Treves, Bull. Research Council of Israel **6A**, 145 (1957).
[2] Frei, Shtrikman, and Treves, Phys. Rev. **106**, 446 (1957).
[3] E. Kondorskii, Doklady Akad. Nauk S.S.S.R. **82**, 365 (1952); Izvest. Akad. Nauk S.S.S.R. Ser. Fiz. **16**, 398 (1952).
[4] S. V. Vonsovskii, *Sovremnoi Uchenie O Magnetism*, (G.I.T.T.L. Moscow, 1953), pp. 262–268.
[5] W. F. Brown, Jr., Bull. Am. Phys. Soc. Ser. II, **1**, 323 (1956).
[6] W. F. Brown, Jr., Bull. Am. Phys. Soc. Ser. II, **2**, 9 (1957).
[7] W. F. Brown, Jr., Phys. Rev. **105**, 1479 (1957).
[8] W. F. Brown, Jr. (private communication).
[9] W. F. Brown, Jr., Phys. Rev. **58**, 736 (1940).
[10] H. Margenau and G. M. Murphy, *The Mathematics of Physics and Chemistry* (D. van Nostrand Company, Inc., Princeton, 1956), second edition, p. 377.

are transformed into:

$$-2A\left(\cos\varphi\nabla^2\alpha_r-\frac{2\sin\varphi}{r^2}\frac{\partial\alpha_r}{\partial\varphi}-\frac{\cos\varphi}{r^2}\alpha_r\right.$$
$$\left.-\sin\varphi\nabla^2\alpha_\varphi-\frac{2\cos\varphi}{r^2}\frac{\partial\alpha_\varphi}{\partial\varphi}+\frac{\sin\varphi}{r^2}\alpha_\varphi\right)$$
$$+I_s\cos\varphi\partial U/\partial r-I_sr^{-1}\sin\varphi\partial U/\partial\varphi$$
$$+HI_s(\alpha_r\cos\varphi-\alpha_\varphi\sin\varphi)=0, \quad (4a)$$

$$-2A\left(\sin\varphi\nabla^2\alpha_r+\frac{2\cos\varphi}{r^2}\frac{\partial\alpha_r}{\partial\varphi}-\frac{\sin\varphi}{r^2}\alpha_r\right.$$
$$\left.+\cos\varphi\nabla^2\alpha_\varphi-\frac{2\sin\varphi}{r^2}\frac{\partial\alpha_\varphi}{\partial\varphi}-\frac{\cos\varphi}{r^2}\alpha_\varphi\right)$$
$$+I_s\sin\varphi\partial U/\partial r+I_sr^{-1}\cos\varphi\partial U/\partial\varphi$$
$$+HI_s(\alpha_r\sin\varphi+\alpha_\varphi\cos\varphi)=0, \quad (4b)$$

$$\nabla^2U=4\pi I_s\{\partial\alpha_r/\partial r+r^{-1}(\alpha_r+\partial\alpha_\varphi/\partial\varphi)\}, \quad (4c)$$

for $r\leqslant R$. The boundary conditions at $r=R$ become

$$\partial\alpha_r/\partial r=\partial\alpha_\varphi/\partial r=0, \quad (5a)$$
$$U_{\text{in}}=U_{\text{out}}, \quad (5b)$$
$$4\pi I_s\alpha_r=\partial U_{\text{in}}/\partial r-\partial U_{\text{out}}/\partial r. \quad (5c)$$

Here α_r, α_φ, and α_z are the direction cosines of the spin in the cylindrical coordinate system.

Let the notations

$$\begin{array}{lll} t=r/R, & h=H/(2\pi I_s), & R_0=A^{\frac{1}{2}}/I_s, \\ u=U/(2\pi I_sR_0), & S=R/R_0, & p=z/R, \end{array} \quad (6)$$

and the operator

$$\nabla'^2=\frac{\partial^2}{\partial t^2}+\frac{1}{t}\frac{\partial}{\partial t}+\frac{1}{t^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial p^2} \quad (7)$$

be substituted in (4). Multiplying (4a) by $\cos\varphi$ and (4b) by $\sin\varphi$ and adding, yields

$$\nabla'^2\alpha_r-\alpha_rt^{-2}-2t^{-2}\partial\alpha_\varphi/\partial\varphi-\pi S\partial u/\partial t-\pi S^2h\alpha_r=0. \quad (8a)$$

Multiplying (4a) by $-\sin\varphi$ and (4b) by $\cos\varphi$ and adding, one obtains

$$\nabla'^2\alpha_\varphi-\alpha_\varphi t^{-2}+2t^{-2}\partial\alpha_r/\partial\varphi$$
$$-\pi St^{-1}\partial u/\partial\varphi-\pi S^2h\alpha_\varphi=0. \quad (8b)$$

Substituting (6) and (7) in (4c) yields

$$\nabla'^2u-2S\{\partial\alpha_r/\partial t+t^{-1}(\alpha_r+\partial\alpha_\varphi/\partial\varphi)\}=0. \quad (8c)$$

The relations (8) are valid for

$$t\leqslant 1. \quad (8d)$$

Using the same substitution, one obtains

$$\nabla'^2u=0 \quad \text{for} \quad t\geqslant 1. \quad (9)$$

The boundary conditions at $t=1$ are

$$\partial\alpha_r/\partial t=\partial\alpha_\varphi/\partial t=0, \quad (10a)$$
$$u_{\text{in}}=u_{\text{out}}, \quad (10b)$$
$$2S\alpha_r=\partial u_{\text{in}}/\partial t-\partial u_{\text{out}}/\partial t. \quad (10c)$$

The complete regular solution of Eqs. (8) is a linear combination of functions of the type

$$\alpha_r=A_r(t)\cos(kp-p_0)\cos(m\varphi-\varphi_0), \quad (11a)$$
$$\alpha_\varphi=A_\varphi(t)\cos(kp-p_0)\sin(m\varphi-\varphi_0), \quad (11b)$$
$$u=U_t(t)\cos(kp-p_0)\cos(m\varphi-\varphi_0). \quad (11c)$$

Here A_r, A_φ, and U_t are independent of φ and z. The parameters k, p_0, and φ_0 are real constants, while m is an integer.

Substituting (11) in (8), adding and subtracting (8a) and (8b), respectively, one obtains

$$\left\{-k^2+\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}-\frac{(m+1)^2}{t^2}-\pi S^2h\right\}(A_r+A_\varphi)$$
$$+\pi S\left(m\frac{U_t}{t}-\frac{dU_t}{dt}\right)=0, \quad (12a)$$

$$\left\{-k^2+\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}-\frac{(m-1)^2}{t^2}-\pi S^2h\right\}(A_r-A_\varphi)$$
$$-\pi S\left(m\frac{U_t}{t}+\frac{dU_t}{dt}\right)=0, \quad (12b)$$

$$\left\{-k^2+\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}-\frac{m^2}{t^2}\right\}U_t$$
$$-S\left\{(m+1)\frac{A_r+A_\varphi}{t}-(m-1)\frac{A_r-A_\varphi}{t}\right.$$
$$\left.+\frac{d(A_r+A_\varphi)}{dt}+\frac{d(A_r-A_\varphi)}{dt}\right\}=0. \quad (12c)$$

It can be shown by substitution that

$$A_r-A_\varphi=aJ_{m-1}(i\mu t), \quad (13a)$$
$$U_t=bJ_m(i\mu t), \quad (13b)$$
$$A_r+A_\varphi=cJ_{m+1}(i\mu t) \quad (13c)$$

is a solution of (12), if the following equations are satisfied.

$$\begin{array}{c} i\mu\pi Sb+(\mu^2-k^2-\pi S^2h)c=0, \\ (\mu^2-k^2-\pi S^2h)a-i\mu\pi Sb=0, \\ i\mu Sa+(\mu^2-k^2)b-i\mu Sc=0. \end{array} \quad (14)$$

Here J_m is Bessel's function of the first kind and of the mth order. In order that at least one of the constants a, b, and c be nonzero, the determinant of the coefficients in (14) should be zero. This implies for μ either the value

$$\mu_1=(k^2+\pi S^2h)^{\frac{1}{2}}, \tag{15a}$$

in which case one gets by substituting in (14)

$$b_1=0, \quad a_1=c_1, \tag{15b}$$

or one of the two values

$$\mu_{2,3}=[k^2+\pi S^2(\tfrac{1}{2}h+1) \pm S\{2\pi k^2+\pi^2S^2(\tfrac{1}{2}h+1)^2\}^{\frac{1}{2}}]^{\frac{1}{2}}, \tag{16a}$$

in which case

$$c_{2,3}=-a_{2,3}; \quad b_{2,3}=\frac{2iS\mu_{2,3}}{k^2-\mu_{2,3}{}^2}a_{2,3}. \tag{16b}$$

Since there are three values for μ, Eq. (13) is the general regular solution of (12). (The other three solutions are the associated Neumann functions.)

Up to now the calculation has been done for any value of m. However, though it is possible to write the general solution, this is not actually necessary for $m\neq 1$, so that the complications involved in the straightforward solution can be avoided in these cases by some other treatment. Therefore only for the case $m=1$ will the direct method be applied (in Sec. D). In the following two sections (B and C) the cases $m=0$ and $m>1$ will be treated, using different considerations.

B. Case $m=0$

Returning to Eqs. (12) and substituting $m=0$, one sees that the variable A_φ can be separated, yielding

$$\left(-k^2+\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}-\frac{1}{t^2}-\pi S^2h\right)A_\varphi=0, \tag{17}$$

and

$$\left(-k^2+\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}-\frac{1}{t^2}-\pi S^2h\right)A_r-\pi S\frac{dU_t}{dt}=0, \tag{18a}$$

$$\left(-k^2+\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}\right)U_t-2S\left(\frac{dA_r}{dt}+\frac{A_r}{t}\right)=0. \tag{18b}$$

The general regular solution of Eq. (17) is

$$A_\varphi=CJ_1(i\mu_1t), \tag{19}$$

with μ_1 given by (15a).

Using (11a) and the boundary condition (10a), one obtains the eigenvalue equation

$$dJ_1(i\mu_1)/d(i\mu_1)=0. \tag{20}$$

The smallest root of this equation is[2,7]

$$i\mu_1=1.841. \tag{21}$$

Substituting this value in (15a), one obtains

$$h(k,S)=-1.08S^{-2}-k^2/(\pi S^2).$$

The least negative value of h is therefore

$$h_n(S)=-1.08S^{-2}. \tag{22}$$

This is the same result as obtained earlier[2] for the magnetization curling, and in fact the mode associated with (22) is identical to the magnetization curling.

The eigenvalues of (18) are more difficult to evaluate rigorously. However, they can be estimated by the following reasoning.

If the self-magnetostatic energy is neglected, it is certainly easier to reverse the magnetization than when this energy is taken into account. Now, it is seen from the derivation of the basic equations[7] that neglecting the self-magnetostatic energy is equivalent to inserting $U_t=0$ in (18a) and discarding (18b). The solution is then the same as (19), and the nucleation is therefore given by (22). Since assumptions were made to facilitate the reversal, the actual nucleation field for this mode is certainly more negative than that given by (22).

One can therefore conclude that for $m=0$ the mode for nucleation is the magnetization curling and that the nucleation field is given by (22).

C. Case $m>1$

By a similar reasoning to the one that led to the estimation of the eigenvalues of (18), it will be shown here that the magnetization reversal is more difficult for $m>1$ than for curling, for any value of S. To see this, the self magnetostatic energy is neglected again, so that in (12) one has to write $U_t=0$ and discard (12c). The solution is then

$$A_r+A_\varphi=aJ_{m+1}(i\mu_1t), \tag{23a}$$

$$A_r-A_\varphi=bJ_{m-1}(i\mu_1t), \tag{23b}$$

with μ_1 given by (15a). The boundary condition (10a), with (11a) and (11b), then leads to one or both of the following equations for the nucleation field,

$$k^2+\pi S^2h_n=-X_{m+1}{}^2, \tag{24a}$$

$$k^2+\pi S^2h_n=-X_{m-1}{}^2, \tag{24b}$$

where X_m is the smallest solution of

$$J_m'(X)=0. \tag{25}$$

Since $m>1$, and since X_m is a monotonous ascending function[11] of m, it is seen by comparing (24) and (22) that the nucleation field for the case discussed here cannot be more positive than for curling. Therefore when the magnetostatic energy is not neglected, the nucleation for $m>1$ is more difficult than by magnetization curling.

[11] G. Petieu, *La Théorie des Fonctions de Bessel* (Centre National de la Recherche Scientifique, Paris, 1955), p. 467.

D. Case $m=1$

1. General Case

As has been remarked before, in this case one needs the general solution of (12), and the eigenvalues have to be evaluated numerically, using the boundary conditions (10).

According to Sec. A, the general regular solution of (12) for $m=1$ is

$$A_r - A_\varphi = \sum_{n=1}^{3} a_n J_0(i\mu_n t), \tag{26a}$$

$$U_t = 2iS \sum_{n=2}^{3} \frac{a_n \mu_n}{k^2-\mu_n^2} J_1(i\mu_n t), \tag{26b}$$

$$A_r + A_\varphi = \sum_{n=1}^{3} (-1)^{\frac{1}{2}n(n-1)} a_n J_2(i\mu_n t), \tag{26c}$$

where the μ_n are given by (15a) and (16a). If all the μ_n are different from zero, Eqs. (26) involve 3 arbitrary constants a_1, a_2 and a_3. [The other 3 independent solutions are the Neumann functions which are infinite at $t=0$ and are therefore not included in (26).] Here it will be assumed that all the μ_n are different and nonzero, the other cases being treated later.

Equation (26b) describes the potential for $t\leqslant 1$. The potential for $t\geqslant 1$ is the solution of (9):

$$u = CH_1^{(1)}(ikt)\cos(kp-p_0)\cos(m\varphi-\varphi_0). \tag{27}$$

Here $H_1^{(1)}$ is the Hankel function of the first kind and order. The constant C is evaluated from the boundary condition (10b) which gives, when one uses (11c), (26b), and (27),

$$C = \frac{2iS}{H_1^{(1)}(ik)} \sum_{n=2}^{3} \frac{a_n\mu_n}{k^2-\mu_n^2} J_1(i\mu_n). \tag{28}$$

In a similar way, one obtains from the boundary condition (10c)

$$\frac{2J_1(i\mu_1)}{i\mu_1}a_1 + \sum_{n=2}^{3} \frac{k^2 a_n}{k^2-\mu_n^2}\left\{[J_0(i\mu_n)-J_2(i\mu_n)] - \frac{\mu_n}{k}\frac{J_1(i\mu_n)}{H_1^{(1)}(ik)}[H_0^{(1)}(ik)-H_2^{(1)}(ik)]\right\}=0. \tag{29}$$

Similarly one obtains from the boundary conditions (10a)

$$\sum_{n=1}^{3} i\mu_n J_1(i\mu_n)a_n = 0, \tag{30a}$$

$$\sum_{n=1}^{3} (-1)^{\frac{1}{2}n(n-1)} i\mu_n\{J_1(i\mu_n)-J_3(i\mu_n)\}a_n = 0. \tag{30b}$$

Since at least one of the a_n should be different from zero, the determinant of the coefficients of (29) and (30) should be zero. This implies that

$$\begin{vmatrix} \dfrac{2J_1(i\mu_1)}{i\mu_1} & \dfrac{k^2}{k^2-\mu_2^2}\Omega_1(\mu_2) & \dfrac{k^2}{k^2-\mu_3^3}\Omega_1(\mu_3) \\ i\mu_1 J_1(i\mu_1) & i\mu_2 J_1(i\mu_2) & i\mu_3 J_1(i\mu_3) \\ \Omega_2(\mu_1) & -\Omega_2(\mu_2) & -\Omega_2(\mu_3) \end{vmatrix} = 0, \tag{31}$$

where

$$\Omega_1(\mu_n) = J_0(i\mu_n) - J_2(i\mu_n) - \frac{\mu_n}{k}\frac{J_1(i\mu_n)}{H_1^{(1)}(ik)}\{H_0^{(1)}(ik)-H_2^{(1)}(ik)\}, \tag{32a}$$

$$\Omega_2(\mu_n) = i\mu_n\{J_1(i\mu_n)-J_3(i\mu_n)\}. \tag{32b}$$

For every value of S the nucleation field h_n is the least negative value of h satisfying (31). A numerical solution of this equation was therefore undertaken using tabulated Bessel functions.[12] [The necessary transformations of the functions involved in (31) for the use of these tables can be found in ordinary text-books.[13]]

In a previous paper[2] the nucleation field for the magnetization buckling was calculated. This mode is not an eigenfunction[8] of Eq. (1), but it is an approximation to the case $m=1$ (with the assumption $A_r=A_\varphi$ $=$const). The search for zeros of (31) could therefore start with the buckling solutions, which proved to be a good approximation. In fact, for $S=1$ and $S=2$ the magnitude of the exact nucleation field was found to be only about 1% smaller than the approximate solution. For this reason it was not found worthwhile to compute h_n for other values of S. The approximate solution was believed to be close enough in the "interesting region"[2] $S<1.1$. For $S>1.1$ the exact buckling is not important since the values of h_n are close to those of the approximate buckling which are more negative than those of the curling. (In particular, for large values of S the exact buckling nucleation field need not be calculated, since it is certainly more negative than that of the curling, because of the magnetostatic energy involved in the buckling mode.) These facts also suggest that the exact buckling eigenfunction should be approximately independent of t, and that $A_r \approx A_\varphi$. Actually for $S=1$ it was found that A_r and A_φ were different by less than 1%. The function A_r (for $S=1$) is plotted *vs* t in Fig. 1, and it can be seen that it does not vary much.

[12] *Annals of the Computation Laboratory of Harvard University* (Harvard University Press, New York, 1947), Vols. III and IV; *British Association for the Advancement of Science, Mathematical Tables* (Cambridge University Press, New York, 1950), Vols. VI and X.

[13] See, for example, N. W. Maclachlan, *Bessel Functions for Engineers* (Clarendon Press, Oxford, 1954).

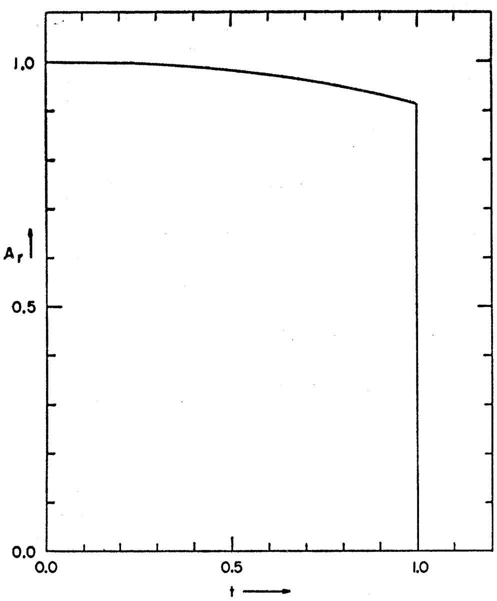

FIG. 1. The reduced radial component of the direction cosine A_r as a function of the reduced radial distance t for the case $R=R_0$, and for the "exact buckling" mode. It is seen that the mode does not differ much from the approximate buckling for which $A_r=$const.

2. *Special Cases*

In the general treatment of Sec. 1, it was assumed that Eq. (26) is the general solution of (12). This is not the case whenever at least one of the μ_n is zero or at least two of them are equal. These special cases will therefore be treated separately here.

2.1. Case $\mu_1=0$.—From (15a) and (16a) it is seen that in this case μ_2 is also zero. In this case the general regular solution of (12) is

$$A_r-A_\varphi=2\pi S^2a_1t^2+a_2-a_3J_0(i\mu t), \tag{33a}$$

$$U_t=4Sa_1t-\frac{2iS\mu_3}{k^2-\mu_3^2}a_3J_1(i\mu t), \tag{33b}$$

$$A_r+A_\varphi=-(k^2+\pi S^2)a_1t^2+a_3J_2(i\mu t), \tag{33c}$$

with

$$\mu=(k^2+2\pi S^2)^{\frac{1}{2}}. \tag{33d}$$

(The nonregular solutions in this case, besides the Neumann function, are $A_r+A_\varphi=a_4t^{-2}$; $A_r-A_\varphi=a_5\ln t$; $U_t=0$.)

Using the boundary conditions (9a), one obtains

$$\begin{aligned} 4\pi S^2a_1+i\mu J_1(i\mu)a_3&=0,\\ -2(k^2+\pi S^2)a_1+\tfrac{1}{2}i\mu\{J_1(i\mu)-J_3(i\mu)\}a_3&=0. \end{aligned} \tag{34}$$

Suppose first that $a_1=a_3=0$. In this case (33) yields $A_r=\frac{1}{2}a_2$, which is in disagreement with $U_t=0$ unless $a_2=0$. Therefore either a_1 or a_3 must differ from zero, so that the determinant of the coefficients in (34) must be zero. This yields

$$(k^2+2\pi S^2)i\mu J_1(i\mu)=\pi S^2i\mu J_3(i\mu). \tag{35}$$

According to the power series expansion for Bessel functions,[14] $J_1(i\mu)$ has the opposite sign to that of $J_3(i\mu)$ for real μ (which is the case here). This means that (35) does not have any root except $\mu=0$ which is of no interest.

2.2. Case $\mu_2=0$.—From (16a) it is seen that either $\mu_1=0$, which has been treated in the former paragraph, or $k=0$. In the latter case the regular solution of (12) is

$$A_r-A_\varphi=a_1J_0(i\mu_1t)+a_2+a_3J_0(i\mu_3t), \tag{36a}$$

$$U_t=-\tfrac{1}{2}Sha_2t-(2iSa_3/\mu_3)J_1(i\mu_3t), \tag{36b}$$

$$A_r+A_\varphi=a_1J_2(i\mu_1t)-a_3J_2(i\mu_3t), \tag{36c}$$

with

$$\mu_1=(\pi S^2h)^{\frac{1}{2}},\quad \mu_3=\{\pi S^2(h+2)\}^{\frac{1}{2}}. \tag{36d}$$

From the boundary condition (10a) one obtains two linear homogeneous equations in a_1 and a_3. If at least one of them is not zero, the determinant of the coefficients is zero, which yields

$$i\mu_1J_1(i\mu_1)\{J_2(i\mu_3)-i\mu_3J_1(i\mu_3)\}=-i\mu_3J_1(i\mu_3)J_2(i\mu_1),$$

i.e.,

$$1-\frac{J_2(i\mu_3)}{i\mu_3J_1(i\mu_3)}=\frac{J_2(i\mu_1)}{i\mu_1J_1(i\mu_1)}. \tag{37}$$

The nucleation field for rotation in unison is[2] $h_n=-1$, as will be seen in the following. Therefore one is interested only in $h>-1$. This means that μ_3 is real, and[12,13] left-hand side of Eq. (37) is larger than 0.75, so that $i\mu_1>3$. According to (36d), this means that h is more negative than the nucleation field for magnetization curling given by (22).

If (37) is not fulfilled, $a_1=a_3=0$, in which case (36) gives rotation in unison. The potential for $t\geqslant 1$ is, in this case,

$$U_t=-\tfrac{1}{2}Sha_2t^{-1},$$

and the boundary condition (10c) gives

$$Sa_2=-Sha_2,\quad \text{i.e.,}\quad h=-1.$$

This value is more negative[2] than the nucleation field for magnetization buckling.

2.3. Case $\mu_3=0$.—Since k is real and $S>0$, it is seen from (16a) that μ_3 can be zero only if $h<-2$.

2.4. Equal Values of μ_n.—According to (15a) and (16a), $\mu_1=\mu_2$ or μ_3 implies that

$$\pi S^2\{2k^2+\pi S^2(h+1)\}=0,\quad \text{i.e.,}\quad h<-1.$$

In the same way one obtains from (16a) that $\mu_2=\mu_3$ implies that

$$2k^2+\pi S^2(\tfrac{1}{2}h+1)^2=0,$$

which cannot be satisfied for any real value of h.

III. HYSTERESIS CURVE

As is proved by Brown,[7,9] the equilibria after the nucleation are characterized, in general, by a set of

[14] See reference 10, p. 113, Eq. 3–63.

nonlinear partial differential equations. Actually one needs to solve these equations only for modes which reduce to the curling or to the exact buckling for small angles, since these two modes were found in Part II to be the easiest modes for nucleation. Yet even this is complicated so that simplifying assumptions are made.

A. Magnetization Curling

For the calculations of this case, the magnetization curling in the sense of reference 2 is assumed throughout the whole process. It is possible in principle (though it does not seem conceivable in this case) that the reversal is thus restricted to an unfavorable mode, so that when complete freedom is allowed, the jump will bring the spins to some other state than that calculated here. However, because of mathematical difficulties, one *has* to assume the curling throughout the process, thereby reaching the following equation[2] for the magnetization equilibria:

$$d^2\omega/dt^2+t^{-1}d\omega/dt-(\pi S^2h+t^{-2}\cos\omega)\sin\omega=0, \quad (38)$$

with the boundary conditions

$$\omega(0)=0, \quad (39a)$$

$$\omega'(1)=0. \quad (39b)$$

Here ω is the angle between the spin and the z axis, assumed to be independent of z and φ, and t is defined in (6). The boundary condition (39a) is somewhat different than that assumed previously[2] when freedom was given inside the cylinder. This freedom has been removed here, since it was not found to be necessary.[8]

Equation (38), with the boundary conditions (39), was solved by the trial and error method.[15] For this, (39a) was preserved and instead of (39b) various derivatives were tried at $t=0$, in search for a solution converging into (39b). The calculation was done for the nucleation field $\pi S^2h=-3.39$ using a fourth-order Runge Kutta method[16] on the WEIZAC—the electronic computer of this Institute. The computer was programmed to stop each solution at $t=1$ or when $\omega'\leqslant 0$. This enables the choice of small enough intervals without waste of computer time to print unwanted results. Some of the solutions are plotted in Fig. 2. It is conceivable from this figure that there is no other solution of (38) which satisfies (39) except $\omega=0$ and

$$\begin{aligned}\omega&=0 \quad \text{for} \quad t=0\\ &=\pi \quad \text{for} \quad t\neq 0.\end{aligned} \quad (40)$$

That (40) is a solution of (38) may be seen by substituting in (38)

$$\omega=\lim_{q\to\infty} 2\arctan qt.$$

[15] W. E. Milne, *Numerical Solution of Differential Equations* (John Wiley and Sons, Inc., New York, 1953), p. 102.
[16] See reference 15, p. 72.

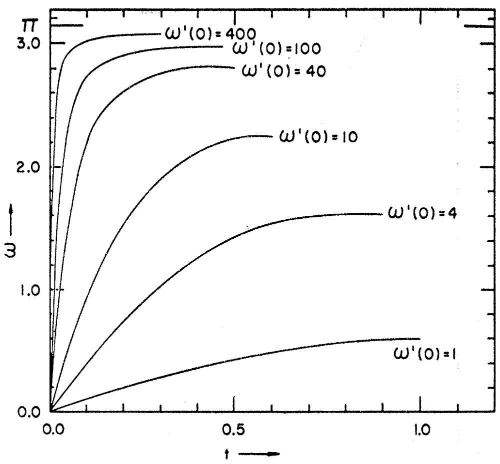

FIG. 2. Some of the solutions of Eq. (38) with the initial value $\omega(0)=0$ and $\omega'(0)$ as a parameter.

It has already been proved[2] that for $h\leqslant h_n$ the equilibrium state $\omega=0$ is unstable. This, together with the results given here, shows that at nucleation the jump is to the state (40). If one removes the restriction of curling only, the central line of spins at $t=0$ will rotate "in unison" to $\omega=\pi$ without a further change in the field. This instability of (40) may be seen from the fact that the exchange energy is finite and decreases monotonically during the rotation, while the self-magnetostatic energy is zero throughout the rotation.

It should also be noted that the solutions of (38) shown in Fig. 2 are equilibria for

$$h=h_nt_m^2,$$

with the new coordinate $t'=t/t_m$ substituted for t. Here t_m is the value of t in Fig. 2 for which $\omega'=0$. However, these equilibria are not stable since they yield negative susceptibilities.

It has thus been shown that the nucleation field in this case is identical with the coercive force, and the hysteresis curve is a symmetrical rectangular loop. This result has also been found by Brown.[8,17]

B. Magnetization Buckling

The exact calculation of the magnetization curve is much more complicated here than for the magnetization curling. Even the choice of a mode which reduces to the exact buckling for small angles was not found possible. However, since the exact buckling gave nucleation fields which were very close to those given by the approximate buckling, the latter mode was taken as the basis for the calculations. The following mode, which gives energies that are easy to deal with and nucleation fields identical to that of the approximate

[17] W. F. Brown, Jr., J. Appl. Phys **29**, 470 (1958).

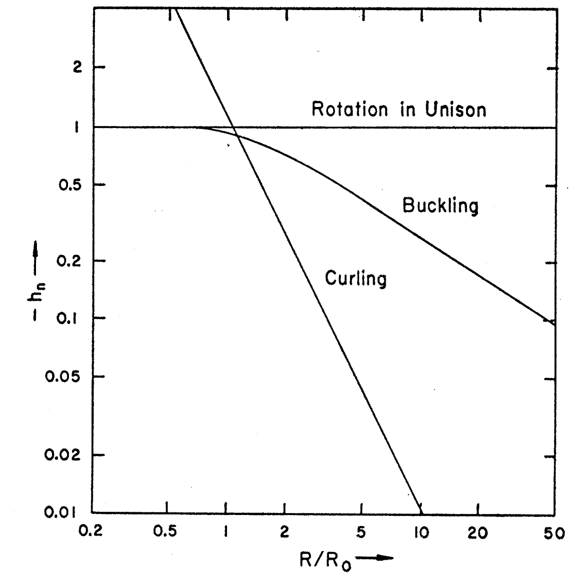

FIG. 3. The reduced nucleation field, $h_n = H_n/(2\pi I_s)$, as a function of the reduced radius of the cylinder. The actual magnetization reversal will be carried out by buckling for $R/R_0 < 1.1$ and by curling for $R/R_0 > 1.1$. The previous assumption of "rotation in unison" is plotted for the sake of comparison of the scale.

buckling, was chosen.

$$\begin{aligned} \alpha_x &= \sin\epsilon \cos kp, \\ \alpha_y &= \sin\epsilon \sin kp, \\ \alpha_z &= \cos\epsilon, \end{aligned} \tag{41}$$

where p is given by (6).

The energy per unit volume is calculated in this case analogously to the buckling case.[2] This yields

$$\bar{E} = \pi I_s^2 \left\{ \sin^2\epsilon \left[\frac{k^2}{\pi S^2} + \pi i J_1(ik) H_1^{(1)}(ik) \right] - 2h \cos\epsilon \right\}. \tag{42}$$

Here the self-magnetostatic energy is only that of the surface charges σ, since in (41) $\operatorname{div} \boldsymbol{\alpha} = 0$. These surfaces charges are

$$\sigma = \sigma_x + \sigma_y = I_s \sin\epsilon \cos kp \cos\varphi + I_s \sin\epsilon \sin kp \sin\varphi.$$

The self-magnetostatic energy is composed of the self-energy of σ_x and that of σ_y, which are given in Appendix V of reference 2, and of the interaction energy of σ_x and σ_y, which can be shown to be zero in this case.

By comparing (42) with Eq. (6.2.3) of Kittel,[18] the hysteresis curve in this case too is a rectangular loop, and the nucleation field (which is equal to the coercive force) is

$$-h = \frac{k^2}{\pi S^2} + \pi i J_1(ik) H_1^{(1)}(ik), \tag{43}$$

which is the same as given in Eq. (29) of the former paper for the nucleation of the approximate buckling.

One remark is needed here. In principle the transfer from one equilibrium to the other should be studied by means of dynamic equations of motion. It is believed however, that for the infinite cylinder there are no stable equilibria except for saturation parallel to the axis so that at nucleation, the magnetization reverses completely.

IV. CONCLUDING REMARKS

The results obtained in this paper confirm the conclusions of the former paper[2] that the hysteresis curve of an infinite cylinder with the field parallel to its axis, is a symmetrical rectangular loop so that the nucleation field is identical with the coercive force. The magnitude of the nucleation field of the exact magnetization buckling was found to be somewhat smaller than that considered before, but the deviation was about 1%, at most. In view of the results obtained by Brown,[7] the transition from exact buckling to curling (at $S \approx 1.1$) is abrupt (as is seen in Fig. 3) contrary to the former suggestions.[2] It should be especially noted that a critical size for spin rotation in unison does not exist for the infinite cylinder.

All the calculations here were carried out by neglecting the magnetocrystalline anisotropy energy. If this energy is introduced, with the easy direction coinciding with the cylinder axis, the only difference is that one should substitute $h + (K/\pi I_s^2)$ for h throughout the whole calculation of the nucleation. Here K is the anisotropy constant, assumed to be positive. The nucleation field is thus given by the same formulas with the added term. As for the hysteresis curve, it is conceivable that it will remain rectangular, at least for anisotropy energy of the type $K \cos^2\omega$. This is because the energy after nucleation will decrease faster than for the case $K = 0$ discussed before, and because $\omega = \pi$ will remain a stable equilibrium for fields past the nucleation field.

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[18] C. Kittel, Revs. Modern Phys. **21**, 575 (1949).