Reaction of Laboratory Magnetic Fields Against Their Current Coils*

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It is shown, using the virial theorem, that the forces exerted on a current-carrying wire by the magnetic field of the current in the wire are always of the same order as the stresses carried by the magnetic field. Thus it is not possible, by suitable configuration of the current-carrying wires, to construct a magnetic system wherein the currents are free of magnetic stresses, even for the so-called "force-free" fields.

WITH the present interest in producing high-density magnetic fields in the laboratory, and with the habit of fields of 10^6 gauss to explode the current coils which produce them, the question has naturally arisen as to whether it is possible to place the current-carrying wires in such a configuration that their magnetic field will not react back upon them. It is the purpose of this note to show that it is not possible to alter the net expansive forces on the current carrying wires, given a total magnetic field energy.

To prove this assertion we shall consider the wires as being composed of perfectly conducting, classical fluid, rather than rigid metal, and held in place by suitable volume forces, $f_i(x_j)$, of constraint. We let F_i be the total force per unit volume exerted on the fluid, $f_i+\partial M_{ij}/\partial x_j$, where M_{ij} is the Maxwell stress tensor, $-\delta_{ij}B^2/8\pi+B_iB_j/4\pi$. Then since the wires, being classical fluids, have no internal kinetic energies, the scalar virial equation[1] requires that in the volume V enclosing the wires we must have

$$0=\int_V dVx_iF_i \tag{1}$$

for static equilibrium.

The integration of $x_i\partial M_{ij}/\partial x_j$ by parts is elementary and has been given elsewhere.[2] The result is that (1) reduces to

$$\int_V dVB^2/8\pi=-\int_V dVx_if_i-\int_S dS_jx_iM_{ij}, \tag{2}$$

where dS_i is an element of the surface S bounding V. The left-hand side of this expression is positive definite, and is just the total field energy; we see that we must have either inward forces of constraint f_i upon the current-carrying conductors, or else we must have inward external forces, exerted by external currents. Thus forces on currents cannot be avoided.

To more readily see the order of magnitude of the forces, we note that far from the currents the field must drop off at least as fast as $1/r^3$. Thus we may choose S to be sufficiently large that the surface integral is negligible. We have remaining just

$$\int_V dVB^2/8\pi=-\int_V dVx_if_i. \tag{3}$$

If the scale of the current-carrying fluid is L, then most of the magnetic field energy lies within a volume of the order of L^3 in the vicinity of the currents. We let $\langle B^2\rangle$ be the mean square field in L^3, so that the left-hand side of (3) is $\sim L^3\langle B^2\rangle/8\pi$. The net inward force which must be exerted to hold the conductors in place is, therefore, of the order of

$$F\sim L^2\langle B^2\rangle/8\pi, \tag{4}$$

which is quite independent of the configuration of the conductors when $L^2\langle B^2\rangle$ is given.

The familiar force-free field is no exception to this condition, of course. Somewhere, either in a singular point or on a bounding surface, there must be inward forces of the order given in (4) to balance the net outward pressure $\int dVB^2/8\pi$ on the left-hand side of (2). As a specific example, consider the force-free flux tube[3] with rotational symmetry about the z axis, $B_z=B_0J_0(k\mathbf{r})$, $B_\phi=B_0J_1(k\mathbf{r})$, where $\mathbf{r}$ is distance from the z axis, $1/k$ is the scale of field in the $\mathbf{r}$ direction, and ϕ is the azimuthal angle about the z axis. The field exerts no force upon the fluid throughout its interior, but it carries stresses across any bounding surface which we might construct around the tube. The net outward force on $\mathbf{r}=R$ in the length L is

$$2\pi RLB^2(R)/8\pi=\tfrac{1}{4}RLB_0^2[J_0^2(kR)+J_1^2(kR)] \sim LB_0^2/2\pi k$$

in the limit as $R\rightarrow\infty$. Through a section across the tube there is the net compression

$$-2\pi\int_0^R d\mathbf{r}\mathbf{r}[B_z^2(\mathbf{r})-B_\phi^2(\mathbf{r})]/8\pi =-(B_0^2R/4\alpha)J_1(\alpha R)J_0(\alpha R) \sim+(B_0^2/4\pi k^2)\cos 2\alpha R.$$

Thus the total stress carried across the surface $\mathbf{r}=R$ or across the ends of the tube is of the order of the characteristic area, $1/k^2$ or L/k, multiplied by $B_0^2/8\pi$. The stress does not vanish as $R\rightarrow\infty$, and somewhere there must be an external force to act against it. The only thing that is gained in a force-free field is that the force can be spread out over a larger region and thereby rendered less intense.

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[1] E. N. Parker, Phys. Rev. **96**, 1686 (1954).

[2] E. N. Parker, Astrophys. J. Suppl. No. 27, **3**, 51 (1957).

[3] S. Lundquist, Arkiv Fysik **2**, 361 (1950).