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# Irreversible Statistical Mechanics of Incompressible Hydromagnetic Turbulence*

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The irreversible statistical mechanics of incompressible hydromagnetic turbulence driven by external forces is treated by methods which do not require that the system be close to a state of detailed balance. The equations of motion are expressed in terms of linearly independent modes formed from the wave-vector components of velocity and magnetic fields, and the nonlinear interaction is exhibited as the sum of individually conservative three-mode interactions. A fundamental statistical equation is constructed giving necessary and sufficient conditions for all members of a distribution of time-functions to satisfy the equations of motion; it involves only second-, third-, and fourth-order distribution moments. A variational criterion is proposed for specifying a distribution consistent with the fundamental equation under physically appropriate constraints. It leads to a complete formal solution of the statistical problem. This solution is not exploited. Instead, two statistical hypotheses based on the assumption of high mode density are advanced. With their aid, each three-mode interaction is treated as a small perturbation on the motion due to all the three-mode interactions and the external forces. The moments in the fundamental equation for the stationary case thereby are expressed in terms of the diagonal elements of the time-covariance matrix and distribution-averaged infinitesimal-impulse-response matrix of the system. Closed equations are obtained which fix these matrix elements in terms of the covariance matrix of the external forces. If the statistical hypotheses are sound, this provides a theory of unbounded turbulence (infinite mode density) driven by Gaussian-distributed homogeneous forces which is exact at all Reynolds numbers based on rms velocity and the macroscale determined by the driving forces. The general theory is specialized to obtain integro-differential equations determining the covariance scalars and modal impulse-response functions for stationary, isotropic hydromagnetic turbulence. In the nonmagnetic case, the asymptotic inertial-range solution yields the wave-number spectrum $E(k)=2\pi c(\epsilon v_0)^{\frac{1}{2}}k^{-\frac{3}{2}}$ and the modal time-autocorrelation function $J_1(2v_0k\tau)/(v_0k\tau)$, where $v_0$ is the rms velocity in any direction, $\epsilon$ is the mean rate of energy-cascade/unit-mass, and $c$ is a universal number fixed by the theory. This contradicts the Kolmogorov similarity hypotheses; independent arguments are advanced against the latter.

## 1. INTRODUCTION

THE turbulence problem is of considerable interest from the standpoint of both irreversible statistical mechanics and nonlinear field theory. A Gibbsian statistical mechanics of turbulence may be developed by taking the real and imaginary parts of the wave-vector components of the velocity field as phase-space coordinates analogous to the canonical variables of Hamiltonian systems.[1–3] There are crucial differences, however, between this system and those for which classical statistical mechanics has been most successful, as may be illustrated by comparison with a dilute gas of hard spheres.

In such a gas, the particles are noninteracting most of the time, and the interaction can be well described in terms of transition probabilities due to two-particle collisions which represent a small time-averaged perturbation to the Hamiltonian of the uncoupled particles. In the turbulent fluid, the motion of a typical Fourier mode is strongly dependent on simultaneous and *continuous* interaction with many other modes and the effects of interaction cannot be lumped into collision terms.

We must also note that in turbulence the mean energy of the individual Fourier modes typically depends strongly on wave number and varies markedly among modes which interact significantly. In this respect, the system is analogous to a gas of particles in which the temperature changes by its own order of magnitude within a mean free path. Methods of

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[1] J. M. Burgers, Verhandl. Konikl. Akad. Wetenschap. Amsterdam 32, 643 (1929); 36, 620 (1933); 43, 936, 1153 (1940).
[2] E. Hopf, J. Rat. Mech. Anal. 1, 87 (1952); E. Hopf and E. W. Titt, J. Rat. Mech. Anal. 2, 587 (1953).
[3] T. D. Lee, Quart. Appl. Math. 10, 69 (1952).

irreversible statistical mechanics dependent on small departure from detailed balance cannot be used to describe the transport of energy across the wave-number spectrum.

The turbulence problem also may be formulated directly in terms of the moments of the velocity distribution.[4,5] Of particular interest is the velocity space-time covariance, which determines the spectral distribution of kinetic energy in frequency and wave vector. Equations of motion for various moments readily are obtained, but as a consequence of the nonlinearity of the Navier-Stokes equation the equations of motion for second-order moments contain third-order moments, those for the third-order moments contain fourth-order moments, and so forth, leading to an infinite sequence of equations involving moments of indefinitely high order. This situation, which provides a central difficulty in turbulence theory, suggests that the Navier-Stokes equation in itself does not provide a complete definition of the problem.

It is well known that for the stationary state of a Hamiltonian system with many degrees of freedom the specification of mean energy leaves the higher distribution moments highly undetermined, since the Gibbs ensemble may be an essentially arbitrary function of the many constants of motion. An additional principle must be invoked to fix the physically appropriate distribution. If dissipation and driving forces are introduced, the ambiguity persists unless the full statistical distribution of the driving forces is specified. It seems clear by analogy that the statistics of the turbulence problem cannot be made determinate unless the equations of motion are augmented by additional conditions on the distribution. The equations of motion alone fix all the solutions but do not choose a distribution from among them.

## 2. DESCRIPTION OF THE PRESENT APPROACH

The present treatment is based on a probability measure in a space of time-functions, thereby differing from the Gibbs statistical mechanics which treats the evolution of an ensemble of instantaneous values; the latter appears not to provide a full statistical description of turbulence driven by external forces (Sec. 4.1). The Liouville equation is replaced by a fundamental statistical equation which contains (in the present problem) only second-, third-, and fourth-order moments but expresses necessary and *sufficient* conditions for all the time-functions in a distribution to satisfy the equations of motion (Sec. 4.2). This exhibits clearly the indetermination of the statistical distribution by the equations of motion alone. The variational criterion which yields the Maxwell-Boltzmann distribution in the Gibbs statistical mechanics is examined in the time-function formulation, and a generalization is proposed for dissipative systems driven by external forces (Sec. 4.3). This leads to a complete formal solution of the statistical problem and, alternately, provides a principle for reducing the infinite sequence of moment equations to a finite, complete set involving only the moments in the fundamental equation, with the assurance that the solutions belong to distributions which satisfy all the moment equations of all orders.

The variational procedure is not exploited (it involves severe difficulties); instead, two hypotheses appropriate to the case of high mode density and statistically independent modal driving forces are advanced. The first (Sec. 5.1) states that as the mode density increases without limit (approach to infinite domain) the statistical dependencies among any finite number of Fourier modes tend to zero. It is pointed out that this hypothesis does not conflict with the observed non-normality of the velocity-field distribution. The second (Sec. 5.2) states that as the same limit is approached the weak statistical dependencies among certain small groups of modes become due entirely to the most direct paths of interaction which link these modes; several arguments are given for this. The two hypotheses lead to a treatment of the dependencies (Sec. 6) in which the individual three-mode interactions which make up the total nonlinear interaction (Sec. 3) are introduced as small perturbations. The diagonal elements of the distribution-averaged infinitesimal-impulse-response matrix of the system play an essential role; they are introduced as unknowns and determined simultaneously with the diagonal elements of the time-covariance matrix (Secs. 7, 8). No assumption is made or implied concerning Reynolds number based on rms velocity and the macroscale determined by the driving forces; if the two hypotheses are sound, the theory is exact at all Reynolds numbers for unbounded homogeneous turbulence driven by Gaussian-distributed forces. The results for the stationary isotropic case (Sec. 9) contradict the Kolmogorov theory, and it is argued (Sec. 10) that the latter fails because of an implication of the fact that the Fourier coefficients of the velocity field are collective coordinates referring to the entire domain.

The structure of turbulence as pictured above is rather interesting from a general theoretical viewpoint. Each mode is strongly coupled to the system as a whole in the sense that (apart from external forces and damping) its entire motion is due to the interaction. However, very many elementary three-mode interactions contribute to the motion of a given mode, and the coupling among any few given modes is quite weak. In part this is a consequence of the particular collective coordinate description used; it is not true in the $x$ space representation of the velocity field. It would appear that a variety of many-body and nonlinear field problems could be formulated so as to permit treat-

[4] G. I. Taylor, Proc. Roy. Soc. (London) **A151**, 421 (1935); **164**, 15 (1938).

[5] G. K. Batchelor, *Theory of Homogeneous Turbulence* (Cambridge University Press, New York, 1953).

ment by methods of the type used here, in contrast to formulation in terms of coordinates such that each dynamic variable interacts principally with at most a few others at any given instant. The possibility of application to quantum-mechanical systems is of particular interest.†

### 3. REPRESENTATION OF THE FIELDS BY LINEARLY INDEPENDENT VARIABLES

We shall assume that the equations of motion of an incompressible, highly conducting, and nonrelativistic hydromagnetic fluid subjected to external body or boundary forces may be written[6]

$$\dot{u}_i(\mathbf{k})+\nu k^2u_i(\mathbf{k})=ik_lP_{ij}(\mathbf{k})\textstyle\sum_{\mathbf{k}'}[w_j(\mathbf{k}-\mathbf{k}')w_l(\mathbf{k}') -u_j(\mathbf{k}-\mathbf{k}')u_l(\mathbf{k}')]+P_{ij}(\mathbf{k})F_j(\mathbf{k}), \quad (3.1)$$

$$\dot{w}_i(\mathbf{k})+\bar{\nu}k^2w_j(\mathbf{k})=ik_j\textstyle\sum_{\mathbf{k}'}[u_i(\mathbf{k}-\mathbf{k}')w_j(\mathbf{k}') -u_j(\mathbf{k}-\mathbf{k}')w_i(\mathbf{k}')], \quad (3.2)$$

where the dots denote differentiation with respect to time. Here $u_i(\mathbf{k})$, $w_i(\mathbf{k})$, $F_i(\mathbf{k})$ are related to the velocity field $\tilde{u}_i(\mathbf{x})$, magnetic induction $\tilde{B}_i(\mathbf{x})$, and external force field $\tilde{F}_i(\mathbf{x})$ by

$$\tilde{u}_i(\mathbf{x})=\textstyle\sum_{\mathbf{k}}u_i(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}},\quad \tilde{B}_i(\mathbf{x})=(4\pi\mu\rho)^{\frac{1}{2}}\sum_{\mathbf{k}}w_i(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \tilde{F}_i(\mathbf{x})=\rho\sum_{\mathbf{k}}F_i(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (3.3)$$

and $P_{ij}(\mathbf{k})=\delta_{ij}-k^{-2}k_ik_j$. The summations range over all wave vectors allowed by the boundary conditions. The quantities $\rho$, $\nu$, $\mu$, $\sigma$, $c$, and $\bar{\nu}=(4\pi\mu\sigma)^{-1}c^2$ denote, respectively, density, kinematic viscosity, permeability, conductivity, velocity of light *in vacuo*, and "ohmic viscosity"; they all are assumed constant throughout the fluid. The fields also obey the divergence conditions

$$k_iu_i(\mathbf{k})=0,\quad k_iw_i(\mathbf{k})=0, \quad (3.4)$$

which are preserved by (3.1), (3.2), if imposed initially.

The total energy per unit mass is

$$E=\tfrac{1}{2}\textstyle\sum_{\mathbf{k}}[u_i^*(\mathbf{k})u_i(\mathbf{k})+w_i^*(\mathbf{k})w_i(\mathbf{k})]. \quad (3.5)$$

Energy is conserved by the nonlinear interaction of the Fourier modes in the sense that if $\nu$, $\bar{\nu}$, and $\mathbf{F}(\mathbf{k})$ are put equal to zero, (3.1), (3.2), and (3.4) yield $\dot{E}=0$.

In order to deal with a denumerably infinite set of dynamic variables, and still permit the description of rigorously homogeneous statistical distributions, we shall adopt the artifice that all the fields obey cyclic boundary conditions on the faces of a cube of side $L$. The limit $L\rightarrow\infty$ corresponds to the physical case of infinite domain. The allowed $\mathbf{k}$ vectors are all those whose components along the coordinate axes are integer multiples of $2\pi/L$. We shall further assume, however, that there is no uniform velocity or magnetic field, so that $\mathbf{u}(0)=\mathbf{w}(0)=0$. These conditions, and the cyclic boundary conditions, are preserved by the equations of motion, provided $\mathbf{F}(0)=0$.

The coefficients $\mathbf{u}(\mathbf{k})$ and $\mathbf{w}(\mathbf{k})$ are not linearly independent; assignment of their initial values is restricted by (3.4) and the reality requirements

$$u_i(-\mathbf{k})=u_i^*(\mathbf{k}),\quad w_i(-\mathbf{k})=w_i^*(\mathbf{k}). \quad (3.6)$$

In the statistical-mechanical treatment it is a great advantage to work with variables for which an arbitrary assignment of initial values is admissible. For this purpose we introduce those real and imaginary parts of the vector components of the $\mathbf{u}(\mathbf{k})$, $\mathbf{w}(\mathbf{k})$ which are linearly independent under (3.4), (3.6).

The vector components of $\mathbf{u}(\mathbf{k})$, $\mathbf{u}(-\mathbf{k})$, $\mathbf{w}(\mathbf{k})$, $\mathbf{w}(-\mathbf{k})$ comprise, for given $\mathbf{k}$, a total of twenty-four real and imaginary parts, of which only eight are linearly independent. We shall pick from each wave-vector pair $\mathbf{k}$, $-\mathbf{k}$ one member, say $\mathbf{k}$, and choose as these eight variables the real and imaginary parts of the components of the vectors $\mathbf{u}(\mathbf{k})$, $\mathbf{w}(\mathbf{k})$ along pairs of perpendicular axes in the plane normal to $\mathbf{k}$. For the sake of notational simplicity, let all the independent real and imaginary parts so chosen (for all pairs $\mathbf{k}$, $-\mathbf{k}$ allowed by the boundary conditions) be ordered in some single one-dimensional sequence and denoted by $q_\alpha/\sqrt{2}$, where $\alpha$ is a serial index whose values label individual variables. Then for given $\mathbf{k}$, any admissible values of $\mathbf{u}(\mathbf{k})$, $\mathbf{u}(-\mathbf{k})$, $\mathbf{w}(\mathbf{k})$, $\mathbf{w}(-\mathbf{k})$ correspond to, and are fixed completely by, some choice of values of eight of the $q_\alpha$. The $q_\alpha$ are analogous to the canonical variables of Hamiltonian systems, as we shall see later.

It follows from the linear relations between the $q$'s and the $\mathbf{u}(\mathbf{k})$, $\mathbf{w}(\mathbf{k})$ that (3.1) and (3.2) may be written in the combined form

$$\dot{q}_\alpha+\nu_\alpha q_\alpha=\sum_{\beta,\gamma}A_{\alpha\beta\gamma}q_\beta q_\gamma+f_\alpha,\quad (A_{\alpha\beta\gamma}=A_{\alpha\gamma\beta}), \quad (3.7)$$

where the coefficients $A_{\alpha\beta\gamma}$ and damping factors $\nu_\alpha$ are functions of their indices (but not of the $q$'s or of time), and the $f$'s are related to $P_{ij}(\mathbf{k})F_j(\mathbf{k})$ as the $q$'s are to $u_i(\mathbf{k})$. (We shall frequently denote the row-vectors with elements $q_\alpha$ and $f_\alpha$ by $\mathbf{q}$ and $\mathbf{f}$. No summation convention will be used for the Greek indices.) It is readily verified from (3.1), (3.2), and the definition of the $q$'s, that each $\nu_\alpha$ is positive and proportional to either $\nu$ or $\bar{\nu}$. It follows from (3.5) and the definition of the $q$'s that

$$E=\tfrac{1}{2}\textstyle\sum_\alpha q_\alpha^2. \quad (3.8)$$

It is apparent that (3.7) and (3.8) exhibit a great formal simplicity in comparison with the original Eqs. (3.1), (3.2), (3.4), (3.5), and (3.6); they provide formally identical equations for ordinary turbulence and hydromagnetic turbulence. We shall derive now

† *Note added in proof.*—In the context of the quantum-mechanical many-body problem, the present methods represent two principal departures from current approaches. First, no attempt is made to diagonalize the system even approximately; instead, a representation is sought which is sufficiently nondiagonal that the weak dependence property holds. Second, the desired solution is obtained by consideration of perturbations about the *exact* statistical state of the system, not an uncoupled state or a state in which the coupling is treated in a phenomenological fashion.

[6] S. Lundquist, Arkiv Fysik **5**, 297 (1952).

some important properties of the $A_{\alpha\beta\gamma}$, thereby expressing in the new formalism the structure of the nonlinear interaction among the Fourier modes.

We remarked before that if $\nu$, $\bar{\nu}$, and $\mathbf{F}(\mathbf{k})$ are set equal to zero, then (3.1), (3.2), and (3.4) yield $\dot{E}=0$. Therefore, setting the $\nu_\alpha$ and $f_\alpha$ equal to zero in (3.7), we have

$$\dot{E}=\sum_\alpha q_\alpha \dot{q}_\alpha = \sum_{\alpha,\beta,\gamma} A_{\alpha\beta\gamma} q_\alpha q_\beta q_\gamma = 0, \quad (\nu_\alpha,\, f_\alpha=0). \tag{3.9}$$

Since any assignment of initial values to $\mathbf{q}$ represents admissible fields, (3.9) must be an identity. Setting all the $q$'s equal to zero except some particular three, $q_\mu$, $q_\lambda$, $q_\sigma$, we deduce that

$$A_{\mu\lambda\sigma}+A_{\lambda\sigma\mu}+A_{\sigma\mu\lambda}=0. \tag{3.10}$$

If the *triangle interaction* of the modes $q_\mu$, $q_\lambda$, $q_\sigma$ is defined by the terms $2A_{\mu\lambda\sigma}q_\lambda q_\sigma$, $2A_{\lambda\sigma\mu}q_\sigma q_\mu$, $2A_{\sigma\mu\lambda}q_\mu q_\lambda$ in the respective equations of motion for $q_\mu$, $q_\lambda$, $q_\sigma$, it follows that this elementary interaction contributes zero to $\dot{E}$ in (3.9) and is individually conservative.[7]

Most of the $A_{\alpha\beta\gamma}$ defined by (3.7) vanish; it follows from (3.1), (3.2) that $A_{\alpha\beta\gamma}\neq 0$ only if $q_\alpha$, $q_\beta$, $q_\gamma$ correspond to wave vectors $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{k}''$ related by $\mathbf{k}\pm\mathbf{k}'\pm\mathbf{k}''=0$, where the $\pm$ signs are to be taken independently. We also note, from (3.4) and our assumption $\mathbf{u}(0)=\mathbf{w}(0)=0$, that only pairs of Fourier coefficients such that $\mathbf{k}'$ and $\mathbf{k}-\mathbf{k}'$ are not parallel to $\pm\mathbf{k}$ (nor, hence, to each other) can contribute to the right sides of (3.1) and (3.2). It follows that

$$A_{\alpha\beta\gamma}=0, \quad \text{unless } \alpha,\, \beta,\, \gamma \text{ are all different.} \tag{3.11}$$

We may also see from (3.1) and (3.2) that the triangle interactions connect real with imaginary parts of Fourier coefficients in only particular ways. If an even number of the variables $q_\alpha$, $q_\beta$, $q_\gamma$ represent imaginary parts of Fourier coefficients, then $A_{\alpha\beta\gamma}$, $A_{\beta\gamma\alpha}$, $A_{\gamma\alpha\beta}$ all vanish. The coefficients describing nonvanishing triangle interactions obey the symmetry relations

$$A_{\alpha_r\beta_r\gamma_i}=A_{\alpha_r\beta_i\gamma_r}=-A_{\alpha_i\beta_r\gamma_r}=A_{\alpha_i\beta_i\gamma_i}, \tag{3.12}$$

where the indices $\alpha_r$ and $\alpha_i$ correspond to $q$'s which are, respectively, the real and imaginary parts of a single independent vector-component of some given Fourier coefficient, with a similar interpretation for the other indices.

## 4. FORMULATION OF THE STATISTICAL PROBLEM

The dynamical variables $q_\alpha$ form a denumerably infinite set. We know, however, that in a physical flow sufficiently high wave numbers will be negligibly excited because of the action of viscosity. This suggests that the truncated system obtained by removing from the equations of motion all Fourier coefficients of wave number greater than an arbitrarily high limit provides a valid representation of the physical situation.[8] This corresponds to retaining in (3.7), (3.8) only the terms which involve exclusively a finite set of $N$ $q_\alpha$ or the associated $f_\alpha$, where $N$ is arbitrarily large. If follows from (3.10) that the nonlinear interaction within the truncated system is exactly conservative, as in the original system. In the rest of the present paper we shall deal exclusively with the truncated system; its advantage is that we can speak meaningfully of the dynamics and statistical mechanics of the associated conservative system in which dissipation and driving forces are taken equal to zero. This is not true of the infinite system.[9]

### 4.1 Gibbs Ensembles and Time-Function Distributions

The application of Gibbs statistical mechanics to turbulence has been discussed by Burgers,[1] Onsager,[10] Hopf,[2] and other authors. We shall present briefly some results of this approach and then point out its inadequacy for the full statistical description of turbulence maintained by external forces. We shall consider first the equilibrium statistical mechanics of the conservative system obtained by placing all $\nu_\alpha$ and $f_\alpha$ equal to zero in the truncated system. According to (3.7) and (3.11), we have

$$\textstyle\sum_\alpha \partial\dot{q}_\alpha/\partial q_\alpha=0, \quad (\nu_\alpha=0). \tag{4.1}$$

This is equivalent to the Liouville theorem for conservative Hamiltonian systems. It shows that the motion conserves measure in a phase space with the $q_\alpha$ as Cartesian coordinates, so that the time-invariant Gibbs ensembles defined by density $\rho(\mathbf{q})$ are those which satisfy

$$\sum_\alpha \dot{q}_\alpha \frac{\partial\rho}{\partial q_\alpha}=\sum_{\alpha,\beta,\gamma} A_{\alpha\beta\gamma}q_\beta q_\gamma \frac{\partial\rho}{\partial q_\alpha}=0. \tag{4.2}$$

By (3.8), (3.9), $\rho=\rho(E)$ is an obvious solution, corresponding to equipartition of energy.[11] More generally,

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[7] A rate of energy-transfer from wave number $\mathbf{k}'$ to wave number $\mathbf{k}$ frequently is defined as $Q(\mathbf{k},\mathbf{k}')=\mathrm{Im}\{\mathbf{k}\cdot\mathbf{u}(\mathbf{k}-\mathbf{k}')\mathbf{u}(\mathbf{k}')\cdot\mathbf{u}^*(\mathbf{k})\}$, since $(\partial/\partial t)[\frac{1}{2}\mathbf{u}^*(\mathbf{k})\cdot\mathbf{u}(\mathbf{k})]=\Sigma_{\mathbf{k}'}Q(\mathbf{k},\mathbf{k}')$, $(\mathbf{w},\,\nu,\,\mathbf{F}=0)$ [see reference 5]. Conservation then is expressed by $Q(\mathbf{k},\mathbf{k}')+Q(\mathbf{k}',\mathbf{k})=0$. Actually this representation of the complex of triangle interactions in terms of pair interactions is nonunique. There is a complementary term $Q(\mathbf{k},\,\mathbf{k}-\mathbf{k}')$ for each term $Q(\mathbf{k},\mathbf{k}')$ in $\Sigma_{\mathbf{k}'}Q(\mathbf{k},\mathbf{k}')$. The two terms each depend on all three amplitudes $\mathbf{u}(\mathbf{k})$, $\mathbf{u}(\mathbf{k}')$, $\mathbf{u}(\mathbf{k}-\mathbf{k}')$ (or their complex conjugates) and it is capricious to call $Q(\mathbf{k},\mathbf{k}')$ the rate of transfer from $\mathbf{k}'$ and $Q(\mathbf{k},\,\mathbf{k}-\mathbf{k}')$ that from $\mathbf{k}-\mathbf{k}'$. This is exhibited by noting that $Q(\mathbf{k},\mathbf{k}')=Q(\mathbf{k},\mathbf{k}')+K(k,k',\,|\mathbf{k}-\mathbf{k}'|)$, where $K$ is any function antisymmetric in each pair of arguments, may validly replace $Q(\mathbf{k},\mathbf{k}')$ since $\bar{Q}(\mathbf{k},\mathbf{k}')+\bar{Q}(\mathbf{k}',\mathbf{k})=0$ and $\bar{Q}(\mathbf{k},\mathbf{k}')+\bar{Q}(\mathbf{k},\,\mathbf{k}-\mathbf{k}')=Q(\mathbf{k},\mathbf{k}')+Q(\mathbf{k},\,\mathbf{k}-\mathbf{k}')$. It may be noted that (3.10) is a stronger statement than $Q(\mathbf{k},\mathbf{k}')+Q(\mathbf{k}',\mathbf{k})=0$, since it does not involve averaging over the vector components of $\mathbf{u}(\mathbf{k})$.

[8] We may regard this as the introduction of infinite damping (infinite resistance) for the degrees of freedom removed.

[9] The inviscid Navier-Stokes equation leads to solutions which develop discontinuities.

[10] L. Onsager, Suppl. Nuovo cimento 6, 279 (1949).

[11] This artificial equilibrium case does *not* describe turbulence at infinite Reynolds number.

the solutions are arbitrary functions of functions $\chi_i(\mathbf{q})$ such that the relations $\chi_i(\mathbf{q})=\text{const}$ are the integrals of the simultaneous system

$$dq_\alpha/\sum_{\beta,\gamma} A_{\alpha\beta\gamma}q_\beta q_\gamma = dq_\lambda/\sum_{\beta,\gamma} A_{\lambda\beta\gamma}q_\beta q_\gamma. \quad (4.3)$$

According to the theory of such systems, there should exist $N-1$ independent $\chi_i$. This implies that there are $N-1$ constants of motion, with a corresponding restriction on the ergodicity of the system.

For $\mathbf{f}=0$, whether or not the dissipation factors $\nu_\alpha$ vanish, the Gibbs density $\rho(\mathbf{q})$, prescribed at some instant, determines the complete structure of the distribution of solutions. Averages of any functions of simultaneous values of the $q$'s may be obtained directly by integration over $\rho$, and $\rho$ at later times is determined by Liouville's equation. Nonsimultaneous averages, such as $\langle q_\alpha(t)q_\beta(t')\rangle$, also are fixed by $\rho$, but their evaluation is not so direct. For the example given, $q_\beta(t')$ could be expanded in a Taylor series about time $t$, the various derivatives in the series expressed in terms of $\mathbf{q}(t)$ by repeated use of (3.7) and its time-derivatives, and the result integrated over $\rho$. Since nonsimultaneous averages are of great interest in turbulence theory, this lack of symmetry of the Gibbs method may be considered a disadvantage.

When external forces act on the system, the situation is more serious; averages involving nonsimultaneous values of the $q$'s cannot be evaluated from knowledge of $\rho$. To see this, we note that when $\mathbf{f}\neq 0$, the use of (3.7) and its derivatives to express higher derivatives of $\mathbf{q}$ in terms of lower introduces the time-derivatives of $\mathbf{f}$. Thus, the evaluation of $\langle q_\alpha(t)q_\beta(t')\rangle$ by the Taylor series method requires knowledge not only of the distribution of $\mathbf{q}$ at some instant, but also of the full joint-distribution, at that instant, of $\mathbf{q}$, $\mathbf{f}$, *and all the time-derivatives of* $\mathbf{f}$. Alternately, one would have to know the full joint-distribution, at an instant, of $\mathbf{q}$ and all the time-derivatives of $\mathbf{q}$. Therefore, even if it were possible to solve the Liouville equation completely in the case $\mathbf{f}\neq 0$, and obtain $\rho$ as a function of time, this would not provide a full solution of the statistical problem.

The complete statistical description of the dissipative system subjected to external forces requires the determination of a probability measure $\Psi[\mathbf{q}(\ ), \mathbf{f}(\ )]$ in the function space of all time-function pairs $\mathbf{q}(t)$, $\mathbf{f}(t)$ which satisfy (3.7). Nonsimultaneous as well as simultaneous averages of any functions of $\mathbf{q}$ and $\mathbf{f}$ are fixed thereby. In view of the difficulties just discussed, we shall attempt to develop a statistical-mechanical treatment directly in terms of time-function distributions rather than in terms of instantaneous-value distributions which evolve in time.

### 4.2 Fundamental Statistical Equation

Since the measure, or distribution functional, $\Psi$ contains the entire structure of the distribution for all times, we cannot hope to find for it an equation of motion like the Liouville equation. However, the functional equation

$$\langle \textstyle\sum_\alpha L_\alpha(t)L_\alpha(t')\rangle_\Psi = 0, \quad (4.4)$$

where

$$L_\alpha(t)\equiv \dot{q}_\alpha(t)+\nu_\alpha q_\alpha(t)-\sum_{\beta,\gamma} A_{\alpha\beta\gamma}q_\beta(t)q_\gamma(t)-f_\alpha(t) \quad (4.5)$$

and $\langle\ \rangle_\Psi$ indicates an average taken with measure $\Psi$, provides both a necessary and a sufficient condition that all the function-pairs $\mathbf{q}(\ )$, $\mathbf{f}(\ )$ in the distribution (except possibly for a set of zero measure) satisfy (3.7). The necessity of (4.4) is obvious, and it is sufficient if satisfied for $t=t'$. This follows from the fact that the left side is a sum with non-negative measure ($\langle\ \rangle_\Psi$) over a sum ($\sum_\alpha$) of squares of real quantities,[12] so that it can vanish only if $L_\alpha(t)=0$ for all $\alpha$ and for all members of the distribution (except for a set of zero measure). The integral of (4.4) (with $t=t'$) over an arbitrary time interval is the condition that the members of the distribution satisfy (3.7) in a mean-square sense within the integration interval. Equation (4.4) is the fundamental statistical equation of the present treatment.

Since $L_\alpha(t)$ contains only linear and bilinear terms, (4.4) contains only second-, third-, and fourth-order moments of the joint-distribution of $\mathbf{q}$ and $\mathbf{f}$. *Therefore, we need evaluate only these moments to determine whether a given distribution satisfies the moment equations of all orders obtained by multiplying* (3.7) *with arbitrary functions of* $\mathbf{q}$ *and* $\mathbf{f}$ *and averaging.* It is important to note, however, that not every assignment of values to second-, third-, and fourth-order moments which satisfies (4.4) corresponds to a possible distribution. The non-negativity of $\Psi$ implies that the distribution average of every functional of $\mathbf{q}$ and $\mathbf{f}$ which is non-negative everywhere in the function space must itself be non-negative.[13]

Despite the realizability restrictions, it is clear that the specification of second-, third-, and fourth-order moments which satisfy (4.4) must leave the higher structure of the distribution largely undetermined. Since (4.4) represents the *entire* constraint imposed by the equations of motion, this indicates the necessity of supplementing them with additional conditions.

### 4.3 Variational Criterion for Specifying the Distribution

In the Gibbs statistical mechanics of conservative systems satisfying Liouville's theorem, the most probable distribution (m.p.d.) is that which minimizes $\int\rho\ln\rho d\sigma$ where the integration is over all phase space subject to the condition $\int\rho d\sigma=1$ and to specifi-

[12] Since $\Psi$ is a probability measure it is everywhere non-negative.

[13] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, New York, 1943).

cation of mean values of appropriate constants of motion. This distribution is the one which can be realized in a maximum number of ways by arrangement of system-points in accordance with the constraints; it is time-invariant. When only the mean-energy is specified, $\rho$ is the Maxwell-Boltzmann (M-B) distribution.[14] We shall now examine this variational criterion in the time-function formulation and propose a generalization for the dissipative system driven by external forces.

Let $\mathbf{q}(t)$ and $\mathbf{f}(t)$ be expanded in Fourier series with coefficients $\mathfrak{q}(\omega)$, $\mathfrak{f}(\omega)$ within an arbitrarily large interval $T$. If we adopt the artifice of removing from the equations all Fourier coefficients such that $|\omega|>\Omega$, an arbitrarily high frequency, the distribution may be described by a density-function $\Phi[\mathfrak{q}(\ ),\ \mathfrak{f}(\ )]$ in the space of finite dimensionality with the independent real and imaginary parts of the $\mathfrak{q}_\alpha(\omega)$ and $\mathfrak{f}_\alpha(\omega)$ as Cartesian coordinates. After computing desired moments as averages over the finite dimensional space, we may take the limits $\Omega\to\infty$ and then $T\to\infty$. In the present representation, (4.4) may be replaced by the equation

$$\int\sum_{\omega,\alpha}\mathfrak{L}_\alpha^*(\omega)\mathfrak{L}_\alpha(\omega)\Phi d\Sigma=0, \tag{4.6}$$

where $i\omega\,\mathfrak{L}_\alpha(\omega)$ is the Fourier coefficient of $L_\alpha(t)$ and the integration is over the entire space. All the $\mathfrak{q}_\alpha(\omega)$ enter (4.6) with a kind of equal *a priori* weight.

For the conservative system all the $f$'s and $\nu$'s vanish, and the space collapses to that of the $\mathfrak{q}_\alpha(\omega)$ alone. Let us perform a rotation in this space, transforming to new Cartesian coordinates which are the values $q_\alpha(t_n)$ of the time-dependent variables at a discrete set of instants $t_n$. A system-point in this space has coordinates $q_\alpha(t_1)$, $q_\alpha(t_2)$, $\cdots$ which are related by the equations of motion. Since these are first-order equations, all the coordinates are completely fixed by the location of the point in the subspace corresponding to any instant $t_n$. The m.p.d. above is the (time-invariant) instantaneous distribution realizable in a maximum number of ways by arrangement of system-points in any of the subspaces, and it therefore should correspond to the distribution in the full space of the $\mathfrak{q}_\alpha(\omega)$ (product space of all the subspaces) which is realizable in a maximum number of ways subject to the constraints represented by the equations of motion. Since (4.6) fully expresses these constraints, this suggests the variational criterion that $I=\int\Phi\ \ln\Phi d\Sigma$ be minimized subject to (4.6), the condition $\int\Phi d\Sigma=1$, and appropriate integral constraints such as prescription of mean-energy-per-system.

An immediate complication is that the equations of motion confine the system-points to hypersurfaces in the space of the $\mathfrak{q}_\alpha(\omega)$ so that the $\Phi$ thus specified is singular and $I$ diverges. This may be handled as follows. We may find the (nonsingular) $\Phi$ which minimizes $I$ with the constraint

$$\int\sum_{\omega,\alpha}\mathfrak{L}_\alpha^*(\omega)\mathfrak{L}_\alpha(\omega)\Phi d\Sigma=\kappa^2 \tag{4.6$'$}$$

replacing (4.6). Then we may take the limit $\kappa\to 0$. When the additional constraint is fixed mean-energy, this procedure leads to the formal solution

$$\Phi=\lim_{\lambda\to 0}N(\lambda,\mu)\exp(-\Lambda/\lambda-A/\mu), \tag{4.7}$$

where $N$ is a normalization factor, $\Lambda=\sum_{\omega,\alpha}\mathfrak{L}_\alpha^*(\omega)\,\mathfrak{L}_\alpha(\omega)$, $A=\sum_{\omega,\alpha}\mathfrak{q}_\alpha^*(\omega)\mathfrak{q}_\alpha(\omega)$, and $1/\lambda$, $1/\mu$ are real, positive Lagrange multipliers. The limit $\lambda\to 0$ corresponds to $\kappa\to 0$, as is clear from the fact that $\Lambda$ is a sum of squared moduli of quantities which vanish when the constraints are satisfied for $\kappa=0$. Of course, we may not assert from the crude arguments given that (4.7) actually yields the M-B distribution.[15]

For the dissipative system driven by external forces, we now propose the criterion that $I$ be minimized as above, where $\Phi$ now is defined over the full product space of the $\mathfrak{q}_\alpha(\omega)$ and $\mathfrak{f}_\alpha(\omega)$. In this case the appropriate integral constraints in addition to (4.6′) and normalization are the specification not of mean energy but rather of those moments of $\mathbf{f}$ which express the known information about the distribution of the driving forces. The distribution chosen by our criterion is one in which the $\mathfrak{q}_\alpha(\omega)$ and $\mathfrak{f}_\alpha(\omega)$ are as statistically independent as the equations of motion and other constraints permit.

The relevance of the M-B distribution to the state of a conservative physical system has not been fully established in the Gibbs statistical mechanics. Our variational criterion for a dissipative, driven system would appear to embody a large degree of additional arbitrariness, since the absolute measure indicated by the Liouville theorem for the associated conservative system is of questionable relevance and we have presented no basis at all for establishing a probability measure in the space of the $\mathfrak{f}_\alpha(\omega)$. However, if we consider the system to have been excited from rest in the distant past, the entire distribution should be determined by the distribution of $\mathbf{f}$ alone. Our criterion yields a distribution whose members obey (3.7) and which is consistent with any degree of specification of the moments of the $\mathbf{f}$ distribution, expressed as integral constraints on the variation. Its arbitrariness then consists only in fixing the higher moments of the

[14] R. C. Tolman, *Principles of Statistical Mechanics* (Oxford University Press, New York, 1938).

[15] A possible indication of the appropriateness of our procedure is afforded by application to a linear harmonic oscillator with equations of motion $\dot{q}=ap$, $\dot{p}=-aq$. Equation (4.7) gives a distribution of the $\mathfrak{q}(\omega)$ and $\mathfrak{p}(\omega)$ which yields the M-B Gibbs ensemble and whose members obey the equations of motion $[i\mathfrak{q}(\pm a)=\pm\mathfrak{p}(\pm a)$; $\mathfrak{q}(\omega)$, $\mathfrak{p}(\omega)$ vanish unless $|\omega|=a]$.

**f** distribution which are not explicitly prescribed. In an actual physical situation, the **f** distribution will be affected importantly by the reaction of the system on its environment, and its *a priori* prescription is unrealistic in any event. The arbitrariness associated with our variational criterion does not seem significant in view of this irreducible artificiality associated with the entire procedure of treating the external forces as parameters rather than as the interaction with an explicitly defined environment.[16]

The variational criterion may be used to write down a formal solution for the dissipative, driven case, corresponding to (4.7) for the conservative case. Alternately, we may choose not to apply constraint (4.6′) directly, but instead find the $\Phi$ which minimizes $I$ with given values of all the moments which appear in (4.4), thereby fixing all the other moments in terms of these. Then we should be able to form from (3.7) a complete set of equations, of which (4.4) is one, to determine the moments in (4.4). Thus the variational method provides a principle for reducing the infinite sequence of moment equations derivable from (3.7) to a finite, complete set with the assurance, since (4.4) is satisfied, that the solutions represent distributions which satisfy all the possible moment equations of all orders. Severe practical difficulties appear to arise both in carrying out this procedure and in evaluating moments from the formal expression for $\Phi$ given directly by the variational criterion. Thus these methods may be only of formal interest. In the following sections we shall develop a simpler method for determining the moments appearing in (4.4) based on two statistical hypotheses. Once these moments are determined, the variational criterion could be used, in principle at least, to find the higher moments, but we shall not attempt this in the present paper.[17]

### 4.4 Stationarity and Homogeneity

Equation (3.7) is invariant under time-displacement and therefore admits distributions whose moments depend only on time differences. For this stationary case we define the moments

$$\begin{aligned} R_{\alpha\beta}(\tau)&=\langle q_\alpha(t+\tau)q_\beta(t)\rangle,\\ S_{\beta\gamma\alpha}(\tau)&=\langle q_\beta(t+\tau)q_\gamma(t+\tau)q_\alpha(t)\rangle,\\ T_{\mu\lambda\beta\gamma}(\tau)&=\langle q_\mu(t+\tau)q_\lambda(t+\tau)q_\beta(t)q_\gamma(t)\rangle,\\ F_{\alpha\beta}(\tau)&=\langle f_\alpha(t+\tau)f_\beta(t)\rangle,\\ G_{\alpha\beta}(\tau)&=\langle f_\alpha(t+\tau)q_\beta(t)\rangle,\\ H_{\alpha\beta\gamma}(\tau)&=\langle f_\alpha(t+\tau)q_\beta(t)q_\gamma(t)\rangle, \end{aligned} \tag{4.8}$$

where the subscript $\Psi$ on the averages has been dropped for compactness. We shall denote these moment matrices by **R**, **S**, **T**, **F**, **G**, **H**. It follows from stationarity that $R_{\alpha\beta}(\tau)=R_{\beta\alpha}(-\tau)$, with similar properties for **F**. We also note $S_{\beta\gamma\alpha}(\tau)=S_{\gamma\beta\alpha}(\tau)$. Equation (4.4) in the stationary case may be written

$$\sum_\alpha \mathrm{Sym}\Big\{-\frac{\partial^2 R_{\alpha\alpha}(\tau)}{\partial\tau^2}+\nu_\alpha{}^2R_{\alpha\alpha}(\tau)+2\sum_{\beta,\gamma}A_{\alpha\beta\gamma}\times\Big[\frac{\partial S_{\beta\gamma\alpha}(\tau)}{\partial\tau}-\nu_\alpha S_{\beta\gamma\alpha}(\tau)+H_{\alpha\beta\gamma}(\tau)\Big]+\sum_{\beta,\gamma,\mu,\lambda}A_{\alpha\beta\gamma}A_{\alpha\mu\lambda}T_{\beta\gamma\mu\lambda}(\tau)+2\frac{\partial G_{\alpha\alpha}(\tau)}{\partial\tau}-2\nu_\alpha G_{\alpha\alpha}(\tau)+F_{\alpha\alpha}(\tau)\Big\}=0, \tag{4.9}$$

where $\mathrm{Sym}\{g(\tau)\}\equiv\frac{1}{2}[g(\tau)+g(-\tau)]$.

Equations (3.1), (3.2), (3.4) correspond to $x$-space equations which are invariant under spatial displacement, inversion, and change of sign of **w**.[18] Therefore they admit spatially homogeneous, inversion-invariant distributions in which **u** and **w** are uncorrelated. In such distributions, Fourier coefficients associated with different wave-vector pairs $\pm\mathbf{k}$, $\pm\mathbf{k}'$ are uncorrelated, and the real parts of the coefficients are uncorrelated with the imaginary parts. If we take the $q$'s as components along suitable principal axes in the normal planes to the wave vectors (the choice may have to be different for **u** and for **w**), those representing the vector components of a given Fourier coefficient will be uncorrelated. With all the conditions stated, the covariance matrix of the $q$'s is diagonal:

$$R_{\alpha\beta}(\tau)=\delta_{\alpha\beta}R_{\alpha\alpha}(\tau). \tag{4.10}$$

The homogeneously distributed driving forces implied by the conditions above are rather unrealistic physically. (They may be visualized as arising from a suitably random volume distribution of stirring devices.) Such forces clearly are required, however, in a consistent idealization of turbulence as a rigorously homogeneous and stationary phenomenon.

## 5. TWO STATISTICAL HYPOTHESES

### 5.1 Weak-Dependence Hypothesis

Let us take homogeneous turbulence in which the energy is spread over many individual modes $q_\alpha$ and

[16] A more realistic problem is turbulence supported by a steady shear flow so that there are no external forces as such (the parameters of the steady flow enter the equations of motion in terms representing nonconservative couplings between pairs of modes). In this case it would seem that the most probable distribution should be one which is stable under random infinitesimal perturbations, and it cannot be asserted without further investigation that the variationally determined $\Phi$ has this property. However, it is to be expected by analogy to equilibrium statistical mechanics that in a system with very many degrees of freedom physically important averages may not be sensitive to the exact form of distribution; thus, the arbitrariness of our criterion may be of only academic importance in this case also.

[17] The difficulties involved are, again, severe. It appears more practical to find the higher moments by direct extension of the methods of Secs. 5–8.

[18] S. Chandrasekhar, Proc. Roy. Soc. (London) **A233**, 322 (1955).

in which the external forces $f_\alpha$ for different $\alpha$'s are statistically independent. It is clear from (3.1) and (3.2) that the various excited modes are paired differently in the right sides of the equations of motion for different modes. Any particular few modes appear in only several of the many bilinear terms contributing to the motion of a given mode. It then is difficult to see how the nonlinear interaction could give rise to other than quite weak statistical dependencies among any small group of modes.

Now suppose we increase without limit the dimension $L$ of the "cyclic box" confining the turbulence and adjust the driving forces to yield some given limiting energy per unit fluid volume in any given wave-vector range. The number of modes over which the energy is spread will increase without limit because of the increase of mode density,[19] and we should expect, on the basis of the observation above, that the statistical dependencies among any finite number of modes should decrease without limit as we approach the case of infinite domain. Let us call those cross-moments which necessarily vanish in a rigorously independent joint-distribution *skew moments*. Then for turbulence excited as described, we hypothesize that as $L\rightarrow\infty$ the normalized values of the skew moments which can be formed from a finite number of $q$'s approach the limit zero and the values of nonskew moments so formed approach as limits values corresponding to an exactly independent joint-distribution of the $q$'s.

The weak-dependence hypothesis just stated does not conflict with the well-established experimental fact[5] that the two-point velocity distribution in turbulent flow is strongly non-normal, although at first sight the central-limit theorem might seem to indicate a contradiction. Consider a measure of non-normality such as $\langle(\partial\tilde{u}_1/\partial x_1)^4\rangle/\langle(\partial\tilde{u}_1/\partial x_1)^2\rangle^2$. If $\langle(\partial\tilde{u}_1/\partial x_1)^4\rangle$ is expanded as a sum of moments of the Fourier modes, it readily is seen that the skew moments included in the summation outnumber the nonskew by a factor the order of the total number of modes excited. As $L\rightarrow\infty$ in the fashion described, this number increases as $L^3$, and the fact that the normalized skew moments tend individually to zero by no means indicates that their total contribution is negligible in the limit. *Thus the weak-dependence hypothesis does not imply the neglect of cross-dependencies in the evaluation of extended sums over Fourier modes.*[20] Crudely speaking, although individual cross-dependencies among a large group of variables may be very small, the effective over-all dependency may be substantial because the number of possible cross-dependencies is very large compared to the number of variables.

It is important to remember that the Fourier coefficients are *collective* coordinates which describe the motion throughout the entire fluid. Observation indicates that the small-scale turbulent motion tends to concentrate into vortex sheets and filaments. The Fourier components of a single such disturbance necessarily must be strongly correlated in phase, but in turbulence filling a large domain there are many such disturbances and their Fourier components must interfere, giving rise to rapid and complicated oscillations of phase with $\mathbf{k}$. Again, there is no conflict with the principle of weak dependence.

### 5.2 Direct-Interaction Hypothesis

Now we shall formulate a related hypothesis concerning the manner in which the presumed weak cross-dependencies are determined by the structure of the nonlinear interaction. The two hypotheses together provide the foundation for a perturbation-theoretic evaluation of the skew moments which appear in (4.4).

Let $q_\alpha$, $q_\beta$, $q_\gamma$ be modes such that the coefficients $A_{\alpha\beta\gamma}$, $A_{\beta\gamma\alpha}$, $A_{\gamma\alpha\beta}$ of their direct triangle interaction, which we shall denote by $(\alpha,\beta,\gamma)$, do not all vanish. Because of the three bilinear terms of which $2A_{\alpha\beta\gamma}q_\beta q_\gamma$ in expression (3.7) for $\dot{q}_\alpha+\nu_\alpha q_\alpha$ is one, $(\alpha,\beta,\gamma)$ can be expected to induce a contribution to the triple moment $\langle q_\alpha q_\beta' q_\gamma''\rangle$, where $\mathbf{q}=\mathbf{q}(t)$, $\mathbf{q}'=\mathbf{q}(t')$, $\cdots$. There will also be a contribution to this moment due to indirect interaction of $q_\alpha$, $q_\beta$, $q_\gamma$ through their couplings with the rest of the dynamical system. We hypothesize that as we approach the limit $L\rightarrow\infty$, for turbulence excited as described previously, the triple moment becomes determined by the contribution of the direct interaction in the sense that if $(\alpha,\beta,\gamma)$ were removed and the other triangle interactions left unaltered, the value of $\langle q_\alpha q_\beta' q_\gamma''\rangle$ would be negligible compared to its value with $(\alpha,\beta,\gamma)$ present.

For the present we justify the direct-interaction hypothesis on the intuitive ground that turbulence is a mixing process which degrades information so that the indirect interaction of three modes through the turbulent motion as a whole should not convey phase information among them in the limit where the motion consists of the excitation of an infinitely large number of weakly dependent degrees of freedom. It is possible to give some further justification of an *a posteriori* nature by estimating, on the basis of the perturbation techniques to be presented in the following section, the contributions to $\langle q_\alpha q_\beta' q_\gamma''\rangle$ from indirect interaction of $q_\alpha$, $q_\beta$, $q_\gamma$ through assumed perturbations in classes of triangle interaction chains involving groups of other modes. [An example of such a chain is $(\alpha,\mu,\lambda)$ $(\beta,\lambda,\sigma)$ $(\gamma,\sigma,\mu)$.] The results which have been obtained in this way support the hypothesis.[21] A rigorous and complete investigation of validity seems out of the question at present; the difficulties involved resemble

[19] Let us adjust the labelling of the $q$'s and $f$'s during this process so that given values of $\alpha$, $\beta$, etc. continue to correspond to given wave-vector neighborhoods.

[20] R. H. Kraichnan, Phys. Rev. **107**, 1485 (1957).

[21] Property (3.12) is important to the analysis. It results in cancellation of indirect contributions by symmetry for the homogeneous case.

somewhat those connected with perturbation expansions in quantum electrodynamics, with severe complications because of the essential nonlinearity of the fields.

The plausibility of our hypothesis perhaps is enhanced by the following argument. Let us consider the distribution determined by the variational criterion when the constraints, in addition to (4.6) and normalization, are the specification of the (diagonal) covariance matrix of **f**. We have then a distribution in which essentially the variables are as statistically independent as the constraints permit; thus, in the case taken it presumably corresponds closely to the weakly dependent state. Now it is clear from (4.4) and the relation between (4.4) and (4.6) that $\langle q_\alpha q_\beta' q_\gamma''\rangle$ only enters into the constraints multiplied by the coefficients $A_{\alpha\beta\gamma}$, $A_{\beta\gamma\alpha}$, $A_{\gamma\alpha\beta}$ of the direct interaction. If this interaction were switched off, we might surmise that $\langle q_\alpha q_\beta' q_\gamma''\rangle$ would be negligible in a maximally independent distribution since it could not contribute toward satisfying the constraints. The situation is complicated by the necessity of satisfying realizability inequalities, but it seems reasonable, in view of the presumed weak-dependence property, that they would not invalidate this conclusion.

We shall now extend the direct-interaction hypothesis to the other skew moments which enter (4.4). We assume that as we approach the limit $L\rightarrow\infty$ the value of $\langle q_\beta q_\gamma' f_\alpha''\rangle$, where $\alpha$, $\beta$, $\gamma$ are all different, becomes determined, in the sense stated previously, by the contribution from the direct-interaction path consisting of $(\alpha,\beta,\gamma)$ and the coupling of $f_\alpha$ to $q_\alpha$. In the case of the skew moments $\langle q_\beta q_\gamma' q_\mu'' q_\lambda'''\rangle$ which appear in (4.4), the selection rules determining which triangle interactions are nonvanishing establish that the most direct paths of interaction which link all four modes are chains involving an external mode such as $(\mu,\lambda,\alpha)$ $(\alpha,\beta,\gamma)$. The contribution of this path represents a distribution-averaged phase relation between the quantities $q_\beta q_\gamma'$ and $q_\mu'' q_\lambda'''$ induced by their couplings to the mode $q_\alpha$. The selection rules permit several such chains for each choice of $\beta$, $\gamma$, $\mu$, $\lambda$. We assume that as we approach the limit $L\rightarrow\infty$ the values of the fourth-order skew moments appearing in (4.4) become determined, in the sense stated previously, by the contributions from these paths.

In connection with the plausibility argument just given, we may note that $\langle q_\beta q_\gamma' f_\alpha''\rangle$ only appears in (4.4) multiplied by the coefficients of $(\alpha,\beta,\gamma)$, while the fourth-order skew moments only appear multiplied bilinearly with the coefficients of the pairs of triangle interactions which constitute the chains we have designated as the direct interaction.

## 6. EVALUATION OF SKEW MOMENTS

In the limit $L\rightarrow\infty$, an excited mode interacts with an infinitely large number of other modes; thus, the triangle interactions which comprise the direct interaction in each of the cases just discussed make an infinitesimal contribution to the motion of the modes involved. This suggests that the skew moments appearing in (4.4) be evaluated by introducing the direct interactions as perturbations on the total motion. In the present paper we shall carry out this program for the stationary case with Gaussian driving force distribution. It is important to note that the perturbations to be considered are about the *exact* state of the system, not a state in which all the modes are uncoupled.

Let us introduce an arbitrary perturbation term $\xi_\alpha(t)$ on the right side of (3.7) for $t\geq t_0$. If $\boldsymbol{\xi}$ is small enough, we may assume that the perturbation in $\mathbf{q}(t)$ can be expanded in powers of $\boldsymbol{\xi}$ in the form

$$\delta q_\mu(t)=\sum_\lambda\int_{t_0}^t \zeta_{\mu\lambda}(t,t')\xi_\lambda(t')dt' +\sum_{\lambda,\sigma}\int_{t_0}^t\int_{t_0}^t \eta_{\mu\lambda\sigma}(t,t',t'')\xi_\lambda(t')\xi_\sigma(t'')dt'dt''+\cdots, \quad (6.1)$$

where $\zeta_{\mu\lambda}(t,t')$, $\eta_{\mu\lambda\sigma}(t,t',t'')$, $\cdots$ are certain implicit functionals of the unperturbed $\mathbf{q}$ and $\mathbf{f}$ and satisfy

$$\zeta_{\mu\lambda}(t,t')=0,\quad (t<t'),\quad \eta_{\mu\lambda\sigma}(t,t',t'')=0,\quad (t<t' \text{ or } t''),\cdots.$$

For a linear system, $\eta_{\mu\lambda\sigma},\cdots$ would vanish, and $\zeta_{\mu\lambda}(t,t')$ (which would be independent of the unperturbed variable values) would be the impulse-response matrix. In the nonlinear case at hand, $\zeta_{\mu\lambda}(t,t')$ gives the response to an infinitesimal impulse. Since (3.7) is a first-order equation, we have $\zeta_{\mu\lambda}(t,t)=\delta_{\mu\lambda}$. For a typical member system of a weakly dependent distribution, the off-diagonal elements of $\zeta_{\mu\lambda}(t,t')$ for $t>t'$ individually are infinitesimally small compared to the diagonal elements in the limit $L\rightarrow\infty$, since a given off-diagonal element represents the response of one mode to a perturbation in the equation of motion of another to which it is only weakly coupled.

Now let us imagine that the triangle interaction $(\alpha,\beta,\gamma)$ is switched on at $t=t_0$. According to our statistical hypotheses, when $L$ is large $\langle q_\alpha(t-\tau)q_\beta(t)q_\gamma(t)\rangle$ should be given to first order by

$$\langle q_\alpha(t-\tau)q_\beta(t)q_\gamma(t)\rangle=\langle \delta q_\alpha(t-\tau)q_\beta(t)q_\gamma(t)\rangle +\langle q_\alpha(t-\tau)\delta q_\beta(t)q_\gamma(t)\rangle+\langle q_\alpha(t-\tau)q_\beta(t)\delta q_\gamma(t)\rangle, \quad (6.2)$$

where

$$\delta q_\gamma(t)=2A_{\gamma\alpha\beta}\int_{t_0}^t \zeta_{\gamma\gamma}(t,t')q_\alpha(t')q_\beta(t')dt', \quad (6.3)$$

with corresponding expressions for $\delta q_\alpha(t-\tau)$, $\delta q_\beta(t)$ involving $2A_{\alpha\beta\gamma}q_\beta(t')q_\gamma(t')$, $2A_{\beta\gamma\alpha}q_\gamma(t')q_\alpha(t')$ respectively. The weak dependence hypothesis rather clearly implies that $\zeta_{\gamma\gamma}$ is statistically independent of $q_\alpha$ and $q_\beta$ in the limit $L\rightarrow\infty$, and for a stationary unperturbed distribution $\langle\zeta_{\gamma\gamma}(t,t')\rangle$ depends only on $t-t'$. Then, to first order,

$$\langle q_\alpha(t-\tau)q_\beta(t)\delta q_\gamma(t)\rangle$$
$$=2A_{\gamma\alpha\beta}\int_{t_0}^{t}\langle\zeta_{\gamma\gamma}(t,t')\rangle\times\langle q_\alpha(t-\tau)q_\beta(t)q_\alpha(t')q_\beta(t')\rangle dt'$$
$$=2A_{\gamma\alpha\beta}\int_0^{t-t_0}g_{\gamma\gamma}(s)R_{\alpha\alpha}(s-\tau)R_{\beta\beta}(s)ds, \quad (6.4)$$

where $s=t-t'$,

$$g_{\mu\lambda}(s)=\langle\zeta_{\mu\lambda}(t,\ t-s)\rangle, \quad (6.5)$$

and we have used (3.11) in addition to the weak-dependence hypothesis to obtain the third member. The matrix $g_{\mu\lambda}(\tau)$ gives the average response to an infinitesimal impulsive perturbing force. We shall call it simply the impulse-response matrix, and denote it by g.

Working out the expressions corresponding to (6.4) for the other two terms on the right side of (6.2), and letting $t_0\rightarrow-\infty$ to describe the new stationary state which is reached sufficiently long after switching on the perturbation, we obtain

$$S_{\beta\gamma\alpha}(\tau)=2A_{\alpha\beta\gamma}\int_\tau^\infty g_{\alpha\alpha}(s-\tau)R_{\beta\beta}(s)R_{\gamma\gamma}(s)ds$$
$$+2A_{\beta\gamma\alpha}\int_0^\infty R_{\alpha\alpha}(s-\tau)g_{\beta\beta}(s)R_{\gamma\gamma}(s)ds$$
$$+2A_{\gamma\alpha\beta}\int_0^\infty R_{\alpha\alpha}(s-\tau)R_{\beta\beta}(s)g_{\gamma\gamma}(s)ds. \quad (6.6)$$

If our statistical hypotheses are well-founded, the first-order perturbation theory, and consequently (6.6), should be exact for the infinite domain, $L\rightarrow\infty$.

When the distribution of **f** is Gaussian, as assumed, we may write $f_\alpha=\Sigma f_{\alpha n}$, where the $f_{\alpha n}$ are statistically independent infinitesimals. Treating any single $f_{\alpha n}$ as a perturbation, we obtain in direct analogy to the preceding analysis,

$$\langle f_{\alpha n}(t+\tau)q_\alpha(t)\rangle=\int_0^\infty g_{\alpha\alpha}(s)F_{\alpha n\alpha n}(s+\tau)ds \quad (6.7)$$

in the new stationary state, where

$$F_{\alpha n\alpha n}(s)=\langle f_{\alpha n}(t+s)f_{\alpha n}(t)\rangle.$$

Summing over all $n$, we have

$$G_{\alpha\alpha}(\tau)=\int_0^\infty g_{\alpha\alpha}(s)F_{\alpha\alpha}(s+\tau)ds, \quad (6.8)$$

since $F_{\alpha\alpha}(\tau)=\Sigma F_{\alpha n\alpha n}(\tau)$.

Again introducing the triangle interaction $(\alpha,\beta,\gamma)$ as a perturbation, the direct-interaction hypothesis implies that

$$\langle f_\alpha(t+\tau)q_\beta(t)q_\gamma(t)\rangle=\langle f_\alpha(t+\tau)\delta q_\beta(t)q_\gamma(t)\rangle+\langle f_\alpha(t+\tau)q_\beta(t)\delta q_\gamma(t)\rangle, \quad (6.9)$$

where $\delta q_\gamma(t)$ is given by (6.3) with a corresponding expression for $\delta q_\beta(t)$. Asserting weak dependence, we find

$$\langle f_\alpha(t+\tau)q_\beta(t)\delta q_\gamma(t)\rangle$$
$$=2A_{\gamma\alpha\beta}\int_0^{t-t_0}g_{\gamma\gamma}(s)R_{\beta\beta}(s)\langle f_\alpha(t+\tau)q_\alpha(t-s)\rangle ds. \quad (6.10)$$

Noting $\langle f_\alpha(t+\tau)q_\alpha(t-s)\rangle=G_{\alpha\alpha}(s+\tau)$, working out the expression corresponding to (6.10) for the other term on the right side of (6.9), and letting $t_0\rightarrow-\infty$, we obtain

$$H_{\alpha\beta\gamma}(\tau)=2\int_0^\infty ds\int_0^\infty dr g_{\alpha\alpha}(r)[A_{\beta\gamma\alpha}g_{\beta\beta}(s)R_{\gamma\gamma}(s)+A_{\gamma\alpha\beta}g_{\gamma\gamma}(s)R_{\beta\beta}(s)]F_{\alpha\alpha}(r+s+\tau). \quad (6.11)$$

The skew moments $T_{\beta\gamma\mu\lambda}(\tau)$ in (4.9) may be evaluated by the foregoing methods if, in accordance with the direct-interaction hypothesis, triangle interaction pairs such as $(\mu,\lambda,\alpha)$, $(\alpha,\beta,\gamma)$ are introduced as perturbations. For all of the skew $T_{\beta\gamma\mu\lambda}(\tau)$ in (4.9) except a fraction which is vanishingly small in the limit $L\rightarrow\infty$,[22] $\beta$, $\gamma$, $\mu$, and $\lambda$ all are different, and there are no nonvanishing pairs of the form $(\mu,\beta,\alpha)$, $(\alpha,\lambda,\gamma)$ or $(\mu,\gamma,\alpha)$, $(\alpha,\lambda,\beta)$. In this case, for $L\rightarrow\infty$ one finds

$$T_{\beta\gamma\mu\lambda}(\tau)=4\sum_\alpha A_{\beta\gamma\alpha}A_{\mu\lambda\alpha}\int_0^\infty ds\int_\tau^\infty dr g_{\beta\beta}(s)R_{\gamma\gamma}(s)R_{\alpha\alpha}(r-s)g_{\mu\mu}(r-\tau)R_{\lambda\lambda}(r-\tau)$$

+3 similar terms obtained by permuting $\beta$ with $\gamma$ and/or $\mu$ with $\lambda$

$$+4\sum_\alpha A_{\beta\gamma\alpha}A_{\alpha\mu\lambda}\int_0^\infty ds\int_s^\infty dr g_{\beta\beta}(s)R_{\gamma\gamma}(s)g_{\alpha\alpha}(r-s)R_{\mu\mu}(r-\tau)R_{\lambda\lambda}(r-\tau)$$

+a similar term obtained by permuting $\beta$ with $\gamma$

$$+4\sum_\alpha A_{\mu\lambda\alpha}A_{\alpha\beta\gamma}\int_\tau^\infty ds\int_s^\infty dr R_{\beta\beta}(r)R_{\gamma\gamma}(r)g_{\alpha\alpha}(r-s)g_{\mu\mu}(s-\tau)R_{\lambda\lambda}(s-\tau)$$

+a similar term obtained by permuting $\mu$ with $\lambda$, ($\beta$, $\gamma$, $\mu$, $\lambda$ all different). (6.12)

[22] This fraction may be ignored in the limit $L\rightarrow\infty$ because the skew $T_{\beta\gamma\mu\lambda}(\tau)$ appear in observable expressions only in extended sums over modes.

Finally, for $L\to\infty$ the weak-dependence hypothesis implies

$$T_{\beta\gamma\beta\gamma}(\tau)=R_{\beta\beta}(\tau)R_{\gamma\gamma}(\tau),\quad(\beta\neq\gamma).\tag{6.13}$$

By (3.11), this case includes all the nonskew $T_{\beta\gamma\mu\lambda}(\tau)$ in (4.9).

We now have expressed all the moments in (4.9) in terms of the diagonal elements of g, $\mathbf{R}$, and $\mathbf{F}$.[23] If the theory is well-founded, these expressions all should be exactly valid in the limit $L\to\infty$.

## 7. NONDISSIPATIVE EQUILIBRIUM

It is shown in the Appendix that when all $\nu$'s and $f$'s vanish and the instantaneous $\mathbf{q}$ distribution is Maxwell-Boltzmann (M-B),[11]

$$g_{\mu\lambda}(\tau)=R_{\mu\lambda}(\tau)/R,\quad(\tau\geq 0),\tag{7.1}$$

where $R=R_{\alpha\alpha}(0)$ (all $\alpha$) is twice the mean energy per mode. In this case we find, using (3.10), that (6.6) may be written

$$S_{\beta\gamma\alpha}(\tau)=-2\frac{A_{\alpha\beta\gamma}}{R}\int_0^\tau R_{\alpha\alpha}(\tau-s)R_{\beta\beta}(s)R_{\gamma\gamma}(s)ds,\tag{7.2}$$

which shows that $S_{\beta\gamma\alpha}(\tau)$ is an odd function of $\tau$ and goes to zero as $\tau\to\pm\infty$, provided $\mathbf{R}(\tau)$ falls off sufficiently fast for large argument.

If we multiply (3.7) by $q_\alpha'$ and average over the distribution, we find

$$\dot{R}_{\alpha\alpha}(\tau)=\sum_{\beta,\gamma}A_{\alpha\beta\gamma}S_{\beta\gamma\alpha}(\tau),\quad(\nu_\alpha,\,f_\alpha=0),\tag{7.3}$$

where the dot now indicates differentiation with respect to $\tau$, so that, by (7.2),

$$\dot{R}_{\alpha\alpha}(\tau)+2\sum_{\beta,\gamma}\frac{(A_{\alpha\beta\gamma})^2}{R}\int_0^\tau R_{\alpha\alpha}(\tau-s)R_{\beta\beta}(s)\times R_{\gamma\gamma}(s)ds=0.\tag{7.4}$$

Finally, we may express (7.4), with the boundary condition $g_{\alpha\alpha}(+0)=1$, as the "universal" integral equation

$$g_{\alpha\alpha}'(x)=1-2\sum_{\beta,\gamma}(A_{\alpha\beta\gamma})^2\int_0^x dy\int_0^y dz g_{\alpha\alpha}'(y-z)\times g_{\beta\beta}'(z)g_{\gamma\gamma}'(z),\quad(x\geq 0),\tag{7.5}$$

where $g(\tau)=g'(\tau R^{\frac{1}{2}})$. Equations (7.5) may be solved through step-by-step integration, starting at $x=0$, thereby determining all the diagonal elements of $g(\tau)$ and $\mathbf{R}(\tau)$.[24] The M-B distribution is a function only of $E$ and necessarily has the homogeneity and other independence properties expressed by (4.10). By (7.1), the off-diagonal elements of both $g(\tau)$ and $\mathbf{R}(\tau)$ then all must vanish.

It easily is verified that the moments we have determined satisfy (4.9) for $\tau=0$ and therefore represent an exact solution of the statistical problem, provided they belong to some everywhere non-negative distribution. By using (3.10) and (7.1), it may be shown that the skew $T_{\beta\gamma\mu\lambda}(0)$ given by (6.12) vanish. Then by (3.11) and (6.13), we have

$$\sum_{\beta,\gamma,\mu,\lambda}A_{\alpha\beta\gamma}A_{\alpha\mu\lambda}T_{\beta\gamma\mu\lambda}(0)=2\sum_{\beta,\gamma}(A_{\alpha\beta\gamma})^2R^2,\tag{7.6}$$

in the limit $L\to\infty$. Both these results may be anticipated from the fact that the M-B distribution is a function only of $E=\frac{1}{2}\sum_\alpha q_\alpha^2$. Differentiating (7.2) (noting $\dot{R}_{\alpha\alpha}(0)=0$), we find

$$\sum_{\beta,\gamma}A_{\alpha\beta\gamma}\dot{S}_{\beta\gamma\alpha}(0)=-2\sum_{\beta,\gamma}(A_{\alpha\beta\gamma})^2R^2.\tag{7.7}$$

The sum of (7.6) and (7.7) less the derivative of (7.3) at $\tau=0$ then yields (4.9) for $\tau=0$, if all the $\nu$'s and $f$'s vanish.

The conservative case we have just treated would not appear to be of any direct physical interest, but it will be instructive to compare the equations obtained with those for the dissipative system driven by stationary external forces.

## 8. DISSIPATIVE STATIONARY STATE

Stationary turbulence subjected to viscosity and driving forces represents a strong departure from equipartition and detailed balance in which the energy of the system fluctuates. In this case, (7.1) may not be asserted, but we can deduce a set of integro-differential equations which determine the $g_{\alpha\alpha}(\tau)$ and $R_{\alpha\alpha}(\tau)$ simultaneously in terms of the $F_{\alpha\alpha}(\tau)$. Multiplying (3.7) by $f_\alpha'$ and averaging, we obtain

$$-\dot{G}_{\alpha\alpha}(\tau)+\nu_\alpha G_{\alpha\alpha}(\tau)=\sum_{\beta,\gamma}A_{\alpha\beta\gamma}H_{\alpha\beta\gamma}(\tau)+F_{\alpha\alpha}(\tau).\tag{8.1}$$

Now let us introduce an arbitrary infinitesimal perturbation $\delta f_\alpha$, statistically independent of $\mathbf{f}$, with autocovariance $\delta F_{\alpha\alpha}(\tau)$. The perturbations in the terms of (8.1) may be obtained from (6.8), (6.11) by replacing $F_{\alpha\alpha}$ with $\delta F_{\alpha\alpha}$. With some manipulation, and the use of partial integration for $\delta\dot{G}_{\alpha\alpha}(\tau)$ [noting that $g_{\alpha\alpha}(+0)=1$ and taking $\delta F_{\alpha\alpha}(\infty)=0$], we find[25]

$$\begin{aligned}\delta G_{\alpha\alpha}(\tau)&=\int_0^\infty g_{\alpha\alpha}(r)\delta F_{\alpha\alpha}(r+\tau)dr,\\-\delta\dot{G}_{\alpha\alpha}(\tau)&=\int_0^\infty \dot{g}_{\alpha\alpha}(r)\delta F_{\alpha\alpha}(r+\tau)dr+\delta F_{\alpha\alpha}(\tau),\\\delta H_{\alpha\beta\gamma}(\tau)&=2\int_0^\infty\int_0^\infty g_{\alpha\alpha}(r-s)[A_{\beta\gamma\alpha}g_{\beta\beta}(s)R_{\gamma\gamma}(s)\\&\quad+A_{\gamma\alpha\beta}g_{\gamma\gamma}(s)R_{\beta\beta}(s)]\eta(r-s)\delta F_{\alpha\alpha}(r+\tau)drds,\end{aligned}\tag{8.2}$$

[23] As $L\to\infty$ in the manner described in Sec. 5, $R_{\alpha\alpha}(\tau)$, $F_{\alpha\alpha}(\tau)\propto 1/M$, where $M=(L/2\pi)^3$. Then our expressions for the skew moments give $S_{\beta\gamma\alpha}(\tau)/[R_{\beta\beta}(0)R_{\gamma\gamma}(0)R_{\alpha\alpha}(0)]^{\frac{1}{2}}\propto 1/\sqrt{M}$, $H_{\alpha\beta\gamma}(\tau)/[R_{\beta\beta}(0)R_{\gamma\gamma}(0)F_{\alpha\alpha}(0)]^{\frac{1}{2}}\propto 1/\sqrt{M}$, $T_{\beta\gamma\mu\lambda}(\tau)/[R_{\beta\beta}(0)R_{\gamma\gamma}(0)R_{\mu\mu}(0)\times R_{\lambda\lambda}(0)]^{\frac{1}{2}}\propto 1/M$, which supports the self-consistency of the weak-dependence hypothesis.

[24] If the present method is applied to a linear dynamic system, the equations analogous to (7.5) become a system of simultaneous bilinear algebraic equations under Laplace transformation.

[25] Equation (8.2) may be derived directly by the perturbation theory by using only the independence of $\mathbf{f}$ and $\delta\mathbf{f}$, without assuming that the distribution of $\mathbf{f}$ is Gaussian.

where $\eta(x)=\frac{1}{2}(1+x/|x|)$. Equation (8.1) can be satisfied by the perturbed values of $F_{\alpha\alpha}$, $G_{\alpha\alpha}$, $H_{\alpha\beta\gamma}$ only if the sums, on right and left sides, of the coefficients of the arbitrary auto-covariance $\delta F_{\alpha\alpha}(r+\tau)$ are equal. We thereby find (making a change of variable),

$$\dot{g}_{\alpha\alpha}(\tau)+\nu_\alpha g_{\alpha\alpha}(\tau)$$
$$=2\sum_{\beta,\gamma}A_{\alpha\beta\gamma}\int_0^\tau g_{\alpha\alpha}(\tau-s)[A_{\beta\gamma\alpha}g_{\beta\beta}(s)R_{\gamma\gamma}(s)+A_{\gamma\alpha\beta}g_{\gamma\gamma}(s)R_{\beta\beta}(s)]ds,\quad(\tau\geq 0).\quad(8.3)$$

Now, multiplying (3.7) by $q_\alpha'$ and averaging, we obtain

$$\dot{R}_{\alpha\alpha}(\tau)+\nu_\alpha R_{\alpha\alpha}(\tau)=\sum_{\beta,\gamma}A_{\alpha\beta\gamma}S_{\beta\gamma\alpha}(\tau)+G_{\alpha\alpha}(\tau).\quad(8.4)$$

Equations (8.3), (8.4), (6.6), and (6.8) form a complete simultaneous system which should uniquely determine both the $g_{\alpha\alpha}(\tau)$ and the $R_{\alpha\alpha}(\tau)$ in terms of given $F_{\alpha\alpha}(\tau)$, if the required conditions $\dot{R}_{\alpha\alpha}(0)=0$, $g_{\alpha\alpha}(+0)=1$ are imposed.[26] If the distribution is known to be homogeneous and to satisfy (4.10), this then provides a determination of the full matrix $\mathbf{R}(\tau)$.[27] To verify the consistency of the present results with those of Sec. 7, we may set $\nu_\alpha$, $F_{\alpha\alpha}=0$ and, using (7.1) and (3.10), find without difficulty that (8.3) and (8.4) reduce to (7.4). In general, however, (8.3) and (8.4) constitute a profoundly more difficult system than (7.4). Although (8.3) could be solved for the $g_{\alpha\alpha}(\tau)$ through step-by-step integration from $\tau=0$ if the $R_{\alpha\alpha}(\tau)$ were known, the right side of (8.4) for given $\tau$ depends on g and $\mathbf{R}$ for all argument values. The difference in difficulty is not surprising when we consider that the nondissipative case could have been handled in elementary fashion by expanding $R_{\alpha\alpha}(\tau)$ as a Taylor series about $\tau=0$, expressing the derivatives in terms of instantaneous averages by repeated use of (3.7), and evaluating these averages in the M-B distribution.

Although there may be no generally valid relation between g and $\mathbf{R}$ comparable in simplicity to (7.1), we may conjecture that in many cases a useful approximation will be

$$g_{\alpha\alpha}(\tau)=e^{-\nu_\alpha\tau}R_{\alpha\alpha}(\tau)/R_{\alpha\alpha}(0),\quad(\tau\geq 0).\quad(8.5)$$

For the conservative case treated, (8.5) reduces to (7.1), and (since $\dot{R}_{\alpha\alpha}(0)=0$) it yields $\dot{g}_{\alpha\alpha}(+0)+\nu_\alpha\cdot g_{\alpha\alpha}(+0)=0$, as required by (8.3) in the dissipative case. For $\nu_\alpha$ very large, (8.5) implies $g_{\alpha\alpha}(\tau)\sim\exp(-\nu_\alpha\tau)$, $(\tau\geq 0)$, the correct result for the impulse response of a system controlled by linear damping. For $\nu_\alpha$ small enough that the factor $\exp(-\nu_\alpha\tau)$ may be ignored except for very large $\tau$, (8.5) seems qualitatively plausible on the intuitive ground that the time characterizing the relaxation of an initial perturbation of a mode through interaction with the rest of the system should approximate the characteristic correlation time of that mode.

The equation

$$\nu_\alpha R_{\alpha\alpha}(0)=\sum_{\beta,\gamma}A_{\alpha\beta\gamma}S_{\beta\gamma\alpha}(0)+G_{\alpha\alpha}(0)\quad(8.6)$$

expresses the energy balance in the stationary state. The left side is the mean rate of dissipation (per unit mass) for the mode $q_\alpha$, $G_{\alpha\alpha}(0)$ is the mean rate of input by the driving force $f_\alpha$, and $2A_{\alpha\beta\gamma}S_{\beta\gamma\alpha}(0)$ is the mean rate of input from the triangle interaction $(\alpha,\beta,\gamma)$.[28] Equation (6.6) gives $S_{\beta\gamma\alpha}(0)=S_{\gamma\alpha\beta}(0)=S_{\alpha\beta\gamma}(0)$ [as required from the definition (4.8)], so that by (3.10)

$$A_{\alpha\beta\gamma}S_{\beta\gamma\alpha}(0)+A_{\beta\gamma\alpha}S_{\gamma\alpha\beta}(0)+A_{\gamma\alpha\beta}S_{\alpha\beta\gamma}(0)=0,\quad(8.7)$$

which shows that the present theory embodies rigorously the detailed conservation properties of the individual triangle interactions discussed in Sec. 3. By (7.2), $S_{\gamma\beta\alpha}(0)=0$ for the equilibrium distribution treated in Sec. 7, which shows that every triad of modes is in detailed balance. In the dissipative stationary case, it can be seen from (6.6) and (3.10) that if (8.5) is approximately true the individual triangle interactions typically transfer energy from strongly to weakly excited modes, as should be expected.[29] This will be illustrated in the next section.

## 9. ISOTROPIC TURBULENCE

### 9.1 Nonmagnetic Turbulence

In the case of isotropic turbulence, the theory developed in Secs. 6 and 8 can be expressed conveniently in terms of the characteristic scalars of the moments involved, without separating explicitly the several $q$ variables belonging to each $\mathbf{k}$. Since only the divergenceless part of $\mathbf{F}(\mathbf{k},t)$ appears in (3.1), we may take it divergence-free without loss of generality. Then we may define the real scalar functions $U$, $F$, and $G$ by

$$\begin{aligned}\tfrac{1}{2}P_{ij}(\mathbf{k})U(k,\tau)&=(L/2\pi)^3\langle u_i(\mathbf{k},t+\tau)u_j^*(\mathbf{k},t)\rangle,\\ \tfrac{1}{2}P_{ij}(\mathbf{k})F(k,\tau)&=(L/2\pi)^3\langle F_i(\mathbf{k},t+\tau)F_j^*(\mathbf{k},t)\rangle,\\ \tfrac{1}{2}P_{ij}(\mathbf{k})G(k,\tau)&=(L/2\pi)^3\mathrm{Re}\{\langle F_i(\mathbf{k},t+\tau)u_j^*(\mathbf{k},t)\rangle\},\end{aligned}\quad(9.1)$$

where Re{ } denotes real part. $U$ and $F$ are even functions of $\tau$. In the limit $L\rightarrow\infty$ (which is required for rigorous isotropy),

$$U(k,\tau)=(2\pi)^{-3}\int\tilde{U}(r,\tau)e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r,\quad(9.2)$$

[26] The condition $\dot{R}_{\alpha\alpha}(0)=0$ is necessary for stationarity; by (8.4), it is equivalent to the integral condition (8.6).

[27] It is easily seen that for a distribution satisfying (4.10) the off-diagonal elements of $g_{\alpha\beta}(\tau)$ must vanish also; otherwise, homogeneously distributed perturbing forces would induce correlations between different wave vectors, which would be inconsistent with the symmetry of the equations of motion.

[28] We have used $A_{\alpha\gamma\beta}=A_{\alpha\beta\gamma}$, $S_{\gamma\beta\alpha}(\tau)=S_{\beta\gamma\alpha}(\tau)$.

[29] This may not be true, however, for triangle interactions linking modes for which $\nu_\alpha$ and $G_{\alpha\alpha}(0)$ are very large, in which case the behavior may deviate substantially from what would be anticipated on the basis of near-equilibrium theory.

where $\tilde{U}(r,\tau)=\langle\tilde{u}_i(\mathbf{x}+\mathbf{r},\,t+\tau)\tilde{u}_i(\mathbf{x},t)\rangle$, with corresponding expressions for $F(k,\tau)$ and $G(k,\tau)$. From (3.1) (with $\mathbf{w}=0$), we may obtain

$$\dot{U}(k,\tau)+\nu k^2U(k,\tau)=S(k,\tau)+G(k,\tau), \qquad (9.3)$$

and

$$-\dot{G}(k,\tau)+\nu k^2G(k,\tau)=H(k,\tau)+F(k,\tau), \qquad (9.4)$$

where

$$S(k,\tau)=-(L/2\pi)^3\,\mathrm{Im}\{k_m\textstyle\sum_{\mathbf{k}'}\langle u_i^*(\mathbf{k}',t)\times u_m^*(\mathbf{k}-\mathbf{k}',\,t)u_i(\mathbf{k},\,t-\tau)\rangle\} \qquad (9.5)$$

and

$$H(k,\tau)=-(L/2\pi)^3\,\mathrm{Im}\{k_m\textstyle\sum_{\mathbf{k}'}\langle u_i^*(\mathbf{k}',\,t)\times u_m^*(\mathbf{k}-\mathbf{k}',\,t)F_i(\mathbf{k},\,t+\tau)\rangle\}. \qquad (9.6)$$

Im{ } denotes imaginary part. These relations express (8.4) and (8.1).

We may evaluate the summand in (9.5) with the methods of Sec. 6 by introducing as perturbations the several triangle interactions which link the modes corresponding to wave vectors $\pm\mathbf{k}$, $\pm\mathbf{k}'$, $\pm(\mathbf{k}-\mathbf{k}')$.[30] Then, we obtain expressions [equivalent to (6.2)] involving $\delta u_i^*(\mathbf{k}',t)$, $\delta u_m^*(\mathbf{k}-\mathbf{k}',\,t)$, and $\delta u_i(\mathbf{k},\,t-\tau)$. In correspondence to the second member of (6.4) we find a contribution $\langle u_i^*(\mathbf{k}',t)u_m^*(\mathbf{k}-\mathbf{k}',\,t)\bar{\delta}u_i(\mathbf{k},\,t-\tau)\rangle$, where

$$\bar{\delta}u_i(\mathbf{k},\,t-\tau)=\int_{t_0}^{t-\tau}g(k,t')\{-ik_lP_{ij}(\mathbf{k})[u_j(\mathbf{k}-\mathbf{k}',\,t')u_l(\mathbf{k}',t')+u_j(\mathbf{k}',t')u_l(\mathbf{k}-\mathbf{k}',\,t')]\}dt'. \qquad (9.7)$$

Here $g(k,t')$ is the diagonal element of $\mathsf{g}(t')$ corresponding to a mode of wave number $k$,[31] and { } represents the contribution to the right side of (3.1) from the triangle interactions involved. The indicated average may be evaluated by (9.1) upon asserting weak dependence. Summing all the perturbation expressions, letting $t_0\rightarrow-\infty$, and replacing $\sum_{\mathbf{k}'}$ by $(L/2\pi)^3\int d^3k'$ in the limit $L\rightarrow\infty$, the final result for $S(k,\tau)$ may be written

$$S(k,\tau)=\pi k\int_0^\infty k'dk'\int_{|k-k'|}^{k+k'}k''dk''\int_0^\infty ds\times[a(k,k',k'')g(k,s)U(k',\,s+\tau)U(k'',\,s+\tau)-b(k,k',k'')g(k',s)U(k,\,s-\tau)U(k'',s)], \qquad (9.8)$$

where

$$a(k,k',k'')=(1-xyz-2y^2z^2),$$
$$b(k,k',k'')=2k^{-1}k'(-z+xy+x^2z+y^2z+2z^3+2xyz^2), \qquad (9.9)$$

$x$, $y$, $z$ being the cosines of the interior angles opposite $k$, $k'$, $k''$, respectively, in a triangle with these quantities as sides.

The factors $a$ and $b$ arise from products of the projection operators $P_{ij}$. They satisfy the identities

$$a(k,k',k'')=a(k,k'',k')\geq 0,$$
$$k^2b(k,k',k'')=k'^2b(k',k,k''), \qquad (9.10)$$
$$b(k,k',k'')+b(k,k'',k')=2a(k,k',k''),$$

and express the structure of the triangle interactions among the Fourier modes of wave numbers $k$, $k'$, and $k''$. The domain of integration in (9.8) includes and is limited to all pairs of values of $k'$ and $k''$ which can form a triangle with $k$; the limits actually are symmetric in $k'$ and $k''$ since if $|k-k'|\leq k''\leq k+k'$, then $|k-k''|\leq k'\leq k+k''$.

The quantity $E(k)dk=2\pi k^2U(k,0)dk$ is the mean energy/unit-mass lying between $k$ and $k+dk$. For $\tau=0$, (9.3) expresses (8.6) for the isotropic case, and it shows that $S(k)dk=4\pi k^2S(k,0)dk$ is the mean rate of net transfer of energy/unit-mass by the nonlinear interaction to modes lying between $k$ and $k+dk$. The identities (9.10) express the conservation properties of the nonlinear interaction. It follows readily from them that $\int_0^\infty S(k)dk=0$ [provided $E(k)$ vanishes rapidly enough as $k\rightarrow\infty$] and that the contributions to $S(k)dk$, $S(k')dk'$, $S(k'')dk''$ from interactions among modes in the intervals $dk$, $dk'$, $dk''$ add to zero in accordance with the detailed conservation properties of the triangle interactions.

Following the procedure which led to (6.8), we find

$$G(k,\tau)=\int_0^\infty ds\,g(k,s)F(k,\,s+\tau). \qquad (9.11)$$

Finally, following the procedure which led from (8.1) to (8.3), we may obtain from (9.4) the relation[32]

$$\dot{g}(k,\tau)+\nu k^2g(k,\tau)=-\pi k\int_0^\infty k'dk'\int_{|k-k'|}^{k+k'}k''dk''b(k,k',k'')\times\int_0^\tau g(k,\,\tau-s)g(k',s)U(k'',s)ds,\quad(\tau\geq 0). \qquad (9.12)$$

Equations (9.3), (9.8), (9.11), and (9.12) form a complete set which should determine uniquely both $g(k,\tau)$ and $U(k,\tau)$ in terms of given $F(k,\tau)$, if the required conditions $\dot{U}(k,0)=0$, $g(k,+0)=1$ are imposed. If the statistical hypotheses underlying the present theory are well-founded, these equations should provide an exact description of stationary, isotropic turbulence in an infinite domain, maintained by normally distributed driving forces. The skew fourth-

[30] As a consequence of weak dependence, the results are indentical with those obtained by introducing the triangle interactions individually, as strictly required by the direct-interaction hypothesis.

[31] As a consequence of isotropy, the distribution-averaged impulse-response tensor has the form $P_{ij}(k)g(k,\tau)$.

[32] Equation (9.12) could also be deduced from inspection of (9.8) by noting the relation between (8.3) and (6.6).

order moments appearing in (4.9) for the isotropic case can be expressed in terms of $g(k,\tau)$ and $U(k,\tau)$ by the methods used above, and then the variational principle of Sec. 4.3 could be invoked to determine all the higher distribution moments, thus providing, in principle, a complete solution.

The integrodifferential equations for $U(k,\tau)$ and $g(k,\tau)$ are of a type which does not appear to have been studied, and their general solution presents formidable difficulties. However, it is not difficult to find an asymptotic solution which describes the inertial and dissipation ranges of turbulence at very high Reynolds number based on macroscale and macrovelocity. This problem will be treated in detail in a future paper,[33] but we shall outline the solution for the inertial range here because it displays important features of the theory.

Let us anticipate that at sufficiently high Reynolds number there exists a range of wave numbers $k$ such that (1) all but a negligible fraction of the total energy lies below a wave number $k_e \ll k$, (2) driving forces and viscosity effects within the range are negligible, (3) mean energy transfer proceeds by a cascade process from lower to successively higher wave numbers within the range, with negligible direct mean transfer from below to above the range, and (4) the times characterizing modes in the range are very much smaller than those for modes below $k_e$. In the limit where these approximations are presumed exact, let us seek a solution within the range of the form

$$U(k,\tau)=U(k,0)g(k,|\tau|). \tag{9.13}$$

Then it may be found that (9.3) and (9.12) each reduce to

$$\dot{g}(k,\tau)=-\tfrac{2}{3}k^2E\int_0^\tau g(k,\tau-s)g(k,s)ds, \quad (\tau\geq 0), \tag{9.14}$$

where $E$ is the mean energy/unit-mass. The solution which satisfies $g(k,+0)=1$ is

$$g(k,\tau)=J_1(2v_0k\tau)/(v_0k\tau), \quad (\tau\geq 0), \tag{9.15}$$

where $v_0=\sqrt{(\tfrac{2}{3}E)}$ is the macrovelocity (rms velocity in any direction).

Under the assumptions made, the energy-balance equation within the range has the asymptotic form

$$S(k,0)=\pi k\int_0^\infty k'dk'\int_{|k-k'|}^{k+k'}k''dk'' \times[a(k,k',k'')U(k',0)-b(k,k',k'')U(k,0)] \times U(k'',0)\int_0^\infty g(k,s)g(k',s)g(k'',s)ds=0, \tag{9.16}$$

for $U(k,\tau)$ of the form (9.13). Equation (9.16) expresses conservation in the energy-cascade process and is a consistency condition for the reduction of (9.3) to (9.14). The solution consistent with (9.15) is

$$U(k,0)=c(\epsilon v_0)^{\frac{1}{2}}k^{-7/2}, \tag{9.17}$$

where the as yet undetermined constant of proportionality has been expressed in terms of $v_0$, the rate of mean energy-transfer/unit-mass by the cascade process $\epsilon$, and a numerical constant $c$.[34]

The $U(k,\tau)$ given by (9.13), (9.15) has a non-negative Fourier transform with respect to $\tau$, as it must to be a realizable covariance scalar. A consequence is that the $s$ integral in (9.16) is positive for any $k$, $k'$, and $k''$. Then it is easy to see from the first and third of identities (9.10) that the terms in (9.16) involving $a$ and $b$ respectively represent positive and (typically) negative contributions to $S(k,0)$ from the individual triangle interactions. Since $U(k,0)$ is a decreasing function of $k$, this suggests that interactions among wave numbers $k$, $k'$, $k''$ give a net contribution to $S(k,0)$ which is positive if $k', k''<k$ and negative if $k', k''>k$.

If $k_1$ is any wave number in the range, the quantity $\epsilon$ defined above equals the net energy-flow/unit-mass *to* all modes $k>k_1$ due to all triangle interactions with modes $k', k''<k_1$, plus the net flow *from* all modes $k<k_1$ due to all triangle interactions with modes $k', k''>k_1$. (These two nonoverlapping classes exhaust all the triangle interactions which transport energy across wave number $k_1$.) By using this fact, an expression for $\epsilon$ in terms of $g(k,\tau)$ and $U(k,\tau)$ is readily constructed, and the constant $c$ then may be evaluated in terms of a definite multiple integral. This completes the inertial-range solution. A calculation of the direct transfer of energy from all modes $<k_1$ to all modes $>k_2\gg k_1$ gives a result $\propto\epsilon(k_1/k_2)^{\frac{3}{2}}$ if $k_1$, $k_2$ are both in the range. This confirms the initial assumption of an essentially local energy-cascade process and provides *a posteriori* justification for using the inertial-range form of solution over the entire $k'$, $k''$ domain in (9.16).

The energy spectrum corresponding to (9.17) is

$$E(k)=2\pi c(\epsilon v_0)^{\frac{1}{2}}k^{-\frac{3}{2}}. \tag{9.18}$$

This expression contains the macrovelocity $v_0$ in contradiction of Kolmogorov's hypothesis[35,5] that the inertial-range spectrum should depend only on $\epsilon$. The reason why Kolmogorov's similarity hypotheses are not supported by the present theory, despite its prediction of the energy-cascade process central to his reasoning, is that according to (9.15) the impulse response and autocorrelation functions for modes in the inertial range are determined by $v_0$. Thus, although

[33] R. H. Kraichnan, Report MH-9, Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, 1958 (unpublished).

[34] Identities (9.10) show that (9.17) solves $S(k,0)=0$, where $S(k,0)$ is given by (9.8), if $g(k,\tau)$ and $U(k,\tau)/U(k,0)$ are any functions of the single argument $v_0k\tau$ with certain integrability properties.

[35] A. N. Kolmogorov, Doklady Akad. Nauk S.S.S.R. **30**, 301 (1941); **32**, 16 (1941).

asymptotically there is no direct energy-transfer from low wave numbers to modes in the inertial range, the rates of transfer by triangle interactions *within* the inertial range are dependent on $v_0$. The physical interpretation of this phenomenon and the breakdown in the Kolmogorov theory implied thereby are discussed in Sec. 10.

### 9.2 Magnetic Turbulence

The magnetic case may be formulated in close analogy to the foregoing. We shall give only the results. Let us define $W(k,\tau)$ by

$$\tfrac{1}{2}P_{ij}(\mathbf{k})W(k,\tau)=(L/2\pi)^3\langle w_i(\mathbf{k},\, t+\tau)w_j^*(\mathbf{k},t)\rangle, \quad (9.19)$$

and let $g_m(k,\tau)$ be the impulse response for magnetic modes. Then,

$$\dot{W}(k,\tau)+\bar{\nu}k^2W(k,\tau)=Q(k,\tau), \quad (9.20)$$

$$\dot{U}(k,\tau)+\nu k^2U(k,\tau)=S(k,\tau)+R(k,\tau)+G(k,\tau), \quad (9.21)$$

$$\dot{g}_m(k,\tau)+\bar{\nu}k^2g_m(k,\tau)=m(k,\tau), \quad (\tau\geq 0), \quad (9.22)$$

$$\dot{g}(k,\tau)+\nu k^2g(k,\tau)=n(k,\tau)+l(k,\tau), \quad (\tau\geq 0), \quad (9.23)$$

where

$$\begin{aligned}Q(k,\tau)=\pi k\int_0^\infty k'dk'\int_{|k-k'|}^{k+k'}k''dk''\int_0^\infty ds\\ \times[d(k,k',k'')g_m(k,\,s)W(k',\,s+\tau)U(k'',\,s+\tau)\\ -h(k,k',k'')W(k,\,s-\tau)g_m(k',s)U(k'',s)\\ -j(k,k',k'')W(k,\,s-\tau)g(k',s)W(k'',s)], \quad (9.24)\end{aligned}$$

$$\begin{aligned}R(k,\tau)=\pi k\int_0^\infty k'dk'\int_{|k-k'|}^{k+k'}k''dk''\int_0^\infty ds\\ \times[a(k,k',k'')g(k,s)W(k',\,s+\tau)W(k'',\,s+\tau)\\ -c(k,k',k'')U(k,\,s-\tau)g_m(k',s)W(k'',s)], \quad (9.25)\end{aligned}$$

$$\begin{aligned}m(k,\tau)=-\pi k\int_0^\infty k'dk'\int_{|k-k'|}^{k+k'}k''dk''\int_0^\tau g_m(k,\,\tau-s)\\ \times[h(k,k',k'')g_m(k',s)U(k'',s)\\ +j(k,k',k'')g(k',s)W(k'',s)]ds, \quad (9.26)\end{aligned}$$

$$\begin{aligned}l(k,\tau)=-\pi k\int_0^\infty k'dk'\int_{|k-k'|}^{k+k'}k''dk''c(k,k',k'')\\ \times\int_0^\tau g(k,\,\tau-s)g_m(k',s)W(k'',s)ds, \quad (9.27)\end{aligned}$$

and $n(k,\tau)$ is the right side of (9.12). The new angular factors are

$$c(k,k',k'')=2k^{-1}k'z(1-y^2), \quad (9.28)$$

$$d(k,k',k'')=2(1+xyz), \quad (9.29)$$

$$h(k,k',k'')=2k^{-1}k'(1+xy+y^2z-z^2), \quad (9.30)$$

$$j(k,k',k'')=2k^{-1}k'(z-xy-x^2z-2xyz^2). \quad (9.31)$$

$S(k,\tau)$ and $G(k,\tau)$ are given by the expressions previously derived.

## 10. DISCUSSION

The function $g_{\alpha\alpha}(\tau)$ describes the relaxation of an initial impulsive disturbance by the spreading of its energy (apart from viscous dissipation) over many modes; thus it describes the effect on the mode $q_\alpha$ of the mixing action of the motion as a whole. From this it can be seen that the theory developed in Secs. 5–9 may be given the following qualitative characterization: the direct interaction among small groups of modes tends to establish mutual phase relations which the mixing action of the total motion tends to destroy. The exchange of energy among given modes is dependent on these phase relations and thereby is affected by the general mixing action, as is displayed clearly in the asymptotic solution for the inertial range of isotropic turbulence outlined in Sec. 9.1.

The present theory is quite inconsistent with the widely accepted hypothesis of Kolmogorov[35,5] which holds that at high Reynolds numbers modes of sufficiently high $k$ do not undergo significant direct dynamical interaction with the strongly excited low-$k$ part of the motion, but are in a statistical state determined solely by the rate of energy-cascade. An inspection of (3.1) shows that the coefficients of the bilinear terms giving interactions with wave numbers $\ll k$ actually are not smaller than the coefficients of terms giving interactions with wave numbers $\sim k$, but it may be reasoned that the effect of the interactions with low wave numbers should be essentially to convect local regions of fluid bodily without significantly disturbing their small-scale (high-$k$) internal dynamics. A serious flaw in this argument would appear to be the failure to recognize that the $\mathbf{u}(\mathbf{k},t)$ are *collective* coordinates whose values are determined jointly by *all* the local regions which compose the flow. It was pointed out in Sec. 5.1 that in an extensive turbulent flow the contributions from different regions should interfere to produce rapid variation of phase with $\mathbf{k}$. The convection by the low-$k$ motion produces relative motion of local regions which then may be expected to give rise to rapid and complicated changes of the phases with time. This should affect profoundly the triple phase correlations which determine the energy-transfer among the high $\mathbf{k}$'s.[36]

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[36] It may be seen from (3.1) that the action of the low-$k$ part of the motion on the high-$k$ part is to couple and induce energy-exchange between modes of nearly equal but *distinct* $\mathbf{k}$'s; it does not couple modes to themselves. On the present theory of the infinite domain, modes of arbitrarily close but distinct $\mathbf{k}$'s are but weakly dependent, and this action properly is described as mixing rather than convection.

As a counter-consideration to Kolmogorov's original argument in the $x$-space representation, it may be noted that the fine-scale structure of high Reynolds-number turbulence consists typically not of compact blobs but of a complicated tangle of *extended* vortex filaments and sheets.

The empirical discrimination between the present theory and the Kolmogorov theory will be discussed in a future paper.[33]

It was noted previously that if the statistical hypotheses of Sec. 5 are sound, the treatment developed in Secs. 6–9 should constitute an exact theory of stationary turbulence in an infinite domain supported by homogeneous Gaussian-distributed modal driving forces.[37,38] No assumption has been made or implied concerning the size of the Reynolds number based on macrovelocity and the macroscale fixed by the driving forces; it could even be less than unity. Work in progress indicates that the theory can be extended to provide an exact description of some cases in which energy is supplied to the turbulence by a steady shear flow rather than by the physically unrealistic external forces invoked in the present paper. In particular, it appears possible to give an exact treatment of fully developed turbulence in an infinitely long pipe. The present theory also can be extended to decaying turbulence, to compressible fluids, and to the mixing of a passive field by turbulence.

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### APPENDIX

Consider any conservative systems **q**, **p** with equations of motion

$$\dot{q}_\alpha = Q_\alpha(q,t), \quad \dot{p}_\alpha = P_\alpha(p,t) \tag{A.1}$$

such that

$$\textstyle\sum_\alpha \partial\dot{q}_\alpha/\partial q_\alpha = 0, \quad \sum_\alpha \partial\dot{p}_\alpha/\partial p_\alpha = 0, \tag{A.2}$$

and with energies

$$E_q = \tfrac{1}{2}\textstyle\sum_\alpha q_\alpha^2, \quad E_p = \tfrac{1}{2}\sum_\alpha p_\alpha^2. \tag{A.3}$$

Consider a M-B distribution of the composite system; the Gibbs density depends only on $E_q+E_p$ and is time-invariant. Let us introduce a conservative coupling so that (A.1) is altered to

$$\dot{q}_\alpha = Q_\alpha + \epsilon\textstyle\sum_\beta A_{\alpha\beta}(t)p_\beta, \quad \dot{p}_\alpha = P_\alpha - \epsilon\sum_\beta A_{\beta\alpha}(t)q_\beta, \tag{A.4}$$

where $\epsilon$ is infinitesimal and $\mathbf{A}(t)$ is an arbitrary function which vanishes for $t<0$. Equation (A.2) is unchanged and the Gibbs density is invariant under the perturbation. Therefore,

$$\delta\langle q_\mu(t)p_\lambda(t)\rangle = \langle\delta q_\mu(t)p_\lambda(t)\rangle + \langle q_\mu(t)\delta p_\lambda(t)\rangle = 0 \tag{A.5}$$

for all $\mu$ and $\lambda$, where $\delta\mathbf{q}$ and $\delta\mathbf{p}$ are the perturbations in **q** and **p**. Noting the statistical independence of the unperturbed systems, we find from (A.4), (A.5),

$$\sum_{\alpha,\beta}\int_0^t A_{\alpha\beta}(t')[g_{\mu\alpha}(t-t')U_{\lambda\beta}(t-t') - h_{\lambda\beta}(t-t')R_{\mu\alpha}(t-t')]dt' = 0, \tag{A.6}$$

where $g_{\mu\alpha}(\tau)$, $R_{\mu\alpha}(\tau)$ are elements of the averaged infinitesimal-impulse-response and covariance matrices for the $q$'s and $h_{\lambda\beta}(\tau)$, $U_{\lambda\beta}(\tau)$ for the $p$'s. This can hold for arbitrary $t$, $\mathbf{A}$ and for all $\mu$ and $\lambda$ only if

$$g_{\mu\alpha}(\tau)/R_{\mu\alpha}(\tau) = h_{\lambda\beta}(\tau)/U_{\lambda\beta}(\tau) = \text{const}, \quad (\tau\geq 0). \tag{A.7}$$

Since (A.1) are first-order equations, $g_{\mu\lambda}(+0)$, $h_{\mu\lambda}(+0) = \delta_{\mu\lambda}$. Then the solution of (A.7) is

$$g_{\mu\lambda}(\tau) = R_{\mu\lambda}(\tau)/R, \quad h_{\mu\lambda}(\tau) = U_{\mu\lambda}(\tau)/R, \quad (\tau\geq 0), \tag{A.8}$$

where $R$ is twice the mean energy/mode in the M-B distribution and we note $R_{\mu\lambda}(0) = \delta_{\mu\lambda}R$, $U_{\mu\lambda}(0) = \delta_{\mu\lambda}R$.[39]

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[37] It is of interest to note that even if for some reason the direct-interaction hypothesis of Sec. 5.2 should not prove strictly valid, the infinite domain covariance and impulse-response equations could still be exactly correct because the contributions to the skew moments from the indirect interaction might be of such character that they do not contribute, in the limit, to the extended sums over modes in which the skew moments enter these equations.

[38] In general, the first-order perturbation theory cannot be expected to give exactly correct results for turbulence confined to a finite volume. It seems likely that it should give good accuracy for sufficiently high-$k$ modes but might lead to substantial error for low-$k$ modes.

[39] For a linear system, (A.8) is anticipated by the work of Callen and his co-workers, who use a different approach [H.B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951); Callen, Borasch, and Jackson, Phys. Rev. **88**, 1382 (1952)]. The relation between the correlation and response matrices associated with small fluctuations of macroscopic thermodynamic variables from equilibrium is examined by Callen and Greene [H. B. Callen and R. F. Greene, Phys. Rev. **88**, 1387 (1952)].