Covariant Quantum Statistics of Fields*

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Relativistic formulas are derived for energy, momentum and number densities and distributions for systems of bosons and fermions from a covariant formulation of the statistics of fields. Charge, spin, and angular momentum statistics are discussed.

1. INTRODUCTION

A COVARIANT statistical mechanics of fields, to our knowledge, has not been developed. This work was undertaken to formulate the Fermi-Landau statistical theory of multiple boson production in relativistic covariant form by using the formalism of the field theory. Such a formulation brings some promising improvements into the theory and also provides a bridge to the more direct field-theoretical treatment of the problem. Since an exact treatment of the phenomena does not exist owing to the difficulties in meson theory, it seems worthwhile to develop the statistical theory to its logical end and to try to find approximation methods from there on. The present paper, however, is devoted entirely to a convariant field-theoretical formulation of statistical mechanics, which in itself is of some interest.[1] We derive the covariant form of the Stefan-Boltzmann law and the energy and angular distributions for a system of bosons and fermions. This includes as a special case a new derivation of the relativistic form of the blackbody radiation. It seems that the field theory is the more logical framework for this kind of problem rather than the ordinary statistical mechanics. Further we discuss spin, charge, momentum, and angular momentum statistics by introducing corresponding "temperatures" for these quantities. The angular momentum statistics is of some importance in the theory of superfluidity.[2]

2. GENERAL THEORY

We start from a general covariant formulation of the field theory, consisting of an invariant Lagrangian density L constructed from the field operators $\psi^s(x)$ and of an invariant dynamical principle. The invariance of L with respect to general transformations defines the constants of the motion of the system. These constants (charge, energy-momentum, total angular momentum, number of heavy particles, strangeness, etc.) determine the statistical behavior of the system in equilibrium, for a system with no constants of the motion cannot reach any sort of equilibrium. In general there will be an equilibrium state associated with each constant of the motion.

The general form of the Boltzmann factor in the presence of many constants of the motion has been already given by Bergmann.[3] The meaning of the equilibrium in the relativistic case and the conditions under which a relativistic system reaches equilibrium are by no means obvious. These conditions are discussed in Appendix I. The density operator of a system of fields corresponding to the aforementioned constants is taken to be (Appendix I)

$$\rho=\exp(\alpha Q-\beta_j P^j+\lambda_{ij}J^{ij}+\cdots),\quad (i,j=0,1,2,3),\quad (1)$$

where Q is the operator of the total charge, P^j is that of the energy-momentum four-vector, J^{ij} is that of the total angular momentum tensor, etc. α, β_j, λ_{ij}, $\cdots$ are constant c numbers. ρ is constant in time and invariant under Lorentz transformations. The partition function of the system of fields is

$$Z=\mathrm{Tr}\rho=\sum_\nu\langle\nu|\rho|\nu\rangle,\quad (2)$$

where $\{|\nu\rangle\}$ is any complete orthonormal basis of the underlying Hilbert space. The probability of any situation corresponding to eigenvalues a' of a set of operators A is $\langle a'|\rho|a'\rangle/Z$ and the expectation value of the set A is given by $\bar{A}=\mathrm{Tr}(A\rho)/Z$. In particular, the expectation values of the constants of the motion in equilibrium are

$$\bar{Q}=\frac{\partial}{\partial\alpha}\ln Z,\quad \bar{P}^j=-\frac{\partial}{\partial\beta_j}\ln Z,\quad \bar{J}^{ij}=\frac{\partial}{\partial\lambda_{ij}}\ln Z,\ \cdots.\quad (3)$$

Equation (1) is valid for arbitrary interacting fields since the constancy of Q, P^j, $J^{ij}\cdots$ follow from very general requirements like gauge invariance, Lorentz invariance, etc. These constants of the motion satisfy the following commutation relations:

$$[Q,P^j]=[Q,J^{ij}]=0,\quad (4)$$

$$[P^i,P^j]=0.\quad (5)$$

But

$$[P^i,J^{jk}]=i(g^{ij}P^k-g^{ik}P^j),\quad (6)$$

$$[J^{ij},J^{km}]=i(g^{jk}J^{im}+g^{im}J^{kj}+g^{jm}J^{ki}+g^{ik}J^{jm}),\quad (7)$$

$$(g_{ik}\text{: } 1,-1,-1,-1).$$

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[1] For previous treatments of quantum statistics of fields see A. E. Scheidegger and C. D. McKay, Phys. Rev. **83**, 125 (1951); Ezawa, Tomozowa, and Umezawa, Nuovo cimento **4**, 810 (1957).

[2] Blatt, Butler, and Schafroth, Phys. Rev. **100**, 481 (1955).

[3] P. G. Bergmann, Phys. Rev. 84, 1026 (1951).

Therefore we can always disentangle Q from the exponent in Eq. (1), but in general it is not possible to disentangle other operators from the exponent. In this case one has to use one of the expansion forms of $\exp(A+B)$. J^{ij} is composed of a spin and an orbital part: $J_{ij}=J_{ij}^{\text{orbit}}+J_{ij}^{\text{spin}}$. If the average value of the orbital angular momentum is zero, we need only to take in Eq. (1) the spin angular momentum which commutes with P. From Eqs. (6) and (7) it follows that in a reference system in which the total system is at rest, P^0 commutes with the total angular momentum. These are the cases which can be treated without invoking the commutation relations.

To evaluate Z in Eq. (2) it is convenient to choose as basis the eigenvectors of Q, P^i, and spin for free fields which are states with definite numbers of *bare* particles having definite momentum and spin. In the case of interacting fields the eigenvalues of Q are again integral multiples of e and that of the total angular momentum are integral or half-integral numbers. In particular, if the interaction Lagrangian density L' is invariant under the gauge transformation, Q is identical with the charge operator of free fields. However, the eigenstates of P are no longer states with definite numbers of bare particles. If the constants of the motion of interacting fields are connected by a unitary transformation to those of free fields—and this is usually the case in scattering situations, i.e., no bound state formation—then the partition function of interacting fields is the same as that of the corresponding noninteracting fields, since $\mathrm{Tr}\rho$ in Eq. (2) is invariant under unitary transformations. The contributions of all graphs in a perturbation expansion cancel out. As long as there are no bound states, the statistical properties of interacting fields are independent of the interaction and are the same as those of free fields. We can see this result also in another way. When one introduces the eigenstates of P^i which are states of definite "clothed" particles of various kinds, the eigenvalues of P^i are the sums of the energy-momentum vectors of the various particles in those states, and these sums are constants (while the individual p^i are no longer constants). By the adiabatic hypothesis the sum is equal to the total momentum of the free fields, so that Eq. (2) gives in the normalized basis of "clothed" particles the same result as in the basis of bare particles. This fact may be the main reason for the success of the Fermi-Landau statistical theory of multiple meson production mentioned in the Introduction. In this paper we shall not consider bound state problems of interacting fields.

3. ENERGY-MOMENTUM STATISTICS

Let us consider first the case where the average value of the total angular momentum is equal to zero, i.e., $\lambda_{ij}=0$. If there is only one kind of charge present, then charge statistics is equivalent to the use of grand canonical ensembles, α being then related to the thermodynamical potential. The same is true for strangeness and, in the case of heavy particles, for hyperon charge. For light fermions or bosons of different charge, only the difference of the numbers of positive and negative particles will be constant.

We denote the eigenstates in Eq. (2) by $|\{n_r^s k^{(r)}\}\rangle$, representing states with n_r^s particles of kind s (spin, charge, $\cdots$) and of four-momentum $k^{(r)}$, for all values of s and r. This designation is covariant since in the case of free fields the field variables $\psi^s(x)$ can be separated in covariant form into positive- and negative-frequency parts:

$$\psi^s(x)=\psi^{(+)s}+\psi^{(-)s}=(2\pi)^{-4}\int dk e^{-ikx} \times[b^{(+)s}(k)+b^{(-)s}(k)]; \quad (8)$$

then the above-mentioned states are

$$|\{n_r^s k^{(r)}\}\rangle=\prod_{s,r}[b^{(-)s}(k^{(r)})]^{n_r^s}|0\rangle, \quad (9)$$

where $|0\rangle$ is the vacuum state. If one quantizes with commutators, the n's can have all possible integral non-negative values, whereas the quantization with anticommutators gives

$$[b^{(-)}(k)]^2|0\rangle=0, \quad (10)$$

so that in this case n can have only the values 0 and 1. Equation (2) becomes—we write the equation first for particles of the same charge—

$$Z=\sum_{n_1^1,\cdots,n_r^s,\cdots}\exp(\alpha\sum_{s,r}n_r^s-\beta_j\sum_{s,r}n_r^s k^{(r)j}) \times\langle\{n_r^s k^{(r)}\}|\{n_r^s k^{(r)}\}\rangle. \quad (11)$$

We normalize the states in Eq. (12) by taking a finite volume V as seen from a given frame of reference. In the following, V stands for $\int d^4x$ taken over an invariant finite part of a space-like surface. Thus $(1/V)\ln Z$ is the true Lorentz-covariant quantity, not $\ln Z$ itself. Then

$$\frac{1}{V}\ln Z=\sum_{s,r}\ln Z_r^s\equiv\sum_{s,r}\ln\sum_{n_r^s}\exp[n_r^s(\alpha-\beta_j k^{(r)j})] =\sum_{s,r}\ln[1\pm\exp(\alpha-\beta_j k^{(r)j})]^{\pm 1}. \quad (12)$$

The r summation can be converted into an integral and the s summation gives a factor S if there are S kinds of particles, and one gets finally

$$\frac{1}{V}\ln Z=(2\pi)^{-3}S\int dk\ln[1\pm\exp(\alpha-\beta_j k^j)]^{\pm 1} \times\delta[k_0-(\mathbf{k}^2+m^2)^{\frac{1}{2}}], \quad (13)$$

where the upper sign is for Fermi particles and the lower sign is for Bose particles. The δ function means an integration over positive energies. To evaluate the last equation we expand the logarithm:

$$\ln(1\pm q)=-\sum_{n=1}^{\infty}\frac{(\mp q)^n}{n}.$$

This expansion is valid for

$$\alpha-\beta_j k^j\leq 1. \quad (14)$$

Then Eq. (13) becomes, after the k_0 integration has been carried out,

$$\frac{1}{V}\ln Z=\mp(2\pi)^{-3}S\int d\mathbf{k}\sum_{n=1}^{\infty}\frac{(\mp 1)^n}{n}e^{n\alpha}\times\exp\{n[\boldsymbol{\beta}\cdot\mathbf{k}-\beta_0(\mathbf{k}^2+m^2)^{\frac{1}{2}}]\}.$$

Let $K=|\mathbf{k}|$, $\beta=|\boldsymbol{\beta}|$, then

$$\frac{1}{V}\ln Z=\mp(2\pi)^{-3}S\sum_{n=1}^{\infty}\frac{(\mp 1)^n}{n}e^{n\alpha}\int dK\,K^2\times\exp[n\beta K\cos\theta-n\beta_0(\mathbf{k}^2+m^2)^{\frac{1}{2}}]\times\sin\theta d\theta d\varphi \quad (15)$$

$$=\mp(2\pi)^{-2}2S\sum_{n=1}^{\infty}\frac{(\mp 1)^n}{n}e^{n\alpha}\int dK\,K^2\times\exp[-n\beta_0(K^2+m^2)^{\frac{1}{2}}]\frac{\sinh(n\beta K)}{n\beta K}. \quad (16)$$

(a) $m=0$ (photons, neutrinos)

The integral in (16) in this case can be easily evaluated and we get, with $\alpha=0$ (total charge is zero),

$$\frac{1}{V}\ln Z=\frac{2}{\pi^2}\frac{\beta_0}{(\beta_0^2-\beta^2)^2}\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{2}{\pi^2}\zeta(4)\frac{\beta_0}{(\beta_0^2-\beta^2)^2}, \quad (17)$$

where we have introduced Riemann's ζ functions. The average energy density $\bar{E}$, the momentum density $\bar{P}$, and the number of photons are given by

$$\bar{E}=\frac{6}{\pi^2}\frac{\beta^2+3\beta_0^2}{3(\beta_0^2-\beta^2)^3}\zeta(4), \quad (18)$$

$$\bar{P}=\frac{8}{\pi^2}\frac{\beta}{(\beta_0^2-\beta^2)^3}\zeta(4), \quad (19)$$

$$\bar{N}=\frac{2}{\pi^2}\frac{\beta_0}{(\beta_0^2-\beta^2)^2}\zeta(3). \quad (20)$$

Let us consider, as a special case, a system of reference moving with a relative velocity v in the x_1 direction; then

$$\beta/\beta_0=v/c, \quad (21)$$

and Eq. (18) becomes

$$\bar{E}=\bar{E}_0\frac{3+(v/c)^2}{3(1-v^2/c^2)}, \quad (22)$$

where $\bar{E}_0$ is the energy density in the rest frame of reference,

$$\bar{E}_0=aT^4=\frac{8\pi^5}{15h^3c^3}(kT)^4. \quad (23)$$

Equation (22) agrees with the known form of relativistic blackbody energy density found by invariance considerations.[4]

The condition (14) means that $\beta_0 k_0>\beta K$ which is always satisfied.

(b) $m\neq 0$, total charge $=\bar{N}e$

Equation (16) can be written (when one uses the expansion of $\sinh x/x$) as

$$\frac{1}{V}\ln Z=\mp\frac{2S}{(2\pi)^2}\sum_{n=1}^{\infty}\frac{(\mp 1)^n}{n}e^{n\alpha}\sum_{l=0}^{\infty}\frac{(n\beta)^{2l}}{(2l+1)!}m^{2l+3}\times B_{2l}(n\beta_0 m), \quad (24)$$

where

$$B_{2l}(z)=\int_0^{\infty}dt\,t^{2l+2}\exp[-z(t^2+1)^{\frac{1}{2}}]. \quad (25)$$

It is shown in Appendix II that $B_{2l}(z)$ can be expressed in terms of Schläfli's Bessel functions $K_\nu(z)$ and their derivatives as

$$B_{2l}(z)=\sum_{\nu=0}^{l}(-1)^l\binom{l}{\nu}B_0^{(2l-2\nu)}(z), \quad (26)$$

where

$$B_0(z)=K_2(z)/z. \quad (27)$$

If β is zero, i.e., in a rest frame of reference, Eq. (24) simplifies:

$$\frac{1}{V}\ln Z=\mp\frac{2S}{(2\pi)^2}\sum_{n=1}^{\infty}\frac{(\mp 1)^n}{n}e^{n\alpha}m^3\frac{K_2(n\beta_0 m)}{n\beta_0 m}. \quad (28)$$

Energy-momentum and particle number densities are given then from (24):

$$\bar{N}=\mp 2S(2\pi)^{-2}\sum_{n=1}^{\infty}(\mp 1)^n e^{n\alpha}\sum_{l=0}^{\infty}\frac{(n\beta)^{2l}}{(2l+1)!}m^{2l+3}\times B_{2l}(n\beta_0 m), \quad (29)$$

$$\bar{P}=\mp 2S(2\pi)^{-2}\sum_{n=1}^{\infty}(\mp 1)^n\frac{e^{n\alpha}}{n}\sum_{l=0}^{\infty}\frac{2l}{(2l+1)!}n^{2l}\beta^{2l-1}\times m^{2l+3}B_{2l}(n\beta_0 m), \quad (30)$$

$$\bar{E}=\mp 2S(2\pi)^{-2}\sum_{n=1}^{\infty}(\mp 1)^n e^{n\alpha}\sum_{l=0}^{\infty}\frac{(n\beta)^{2l}}{(2l+1)!}m^{2l+4}\times B_{2l}'(n\beta_0 m). \quad (31)$$

[4] M. v. Laue, *Relativitatstheorie* (Friedrich Vieweg und Sohn, Braunschweig, 1952), Vol. 1, p. 178; R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Clarendon Press, Oxford, 1946), p. 161.

At this point we can also give the energy and angular distributions of the particles, which are defined by $\int E(K)dK=\bar{E}$ and $\int N(\theta)\sin\theta d\theta d\varphi=\bar{N}$, respectively. From (15),

$$N(\theta)=\mp S(2\pi)^{-3}\sum_{n=1}^{\infty}(\mp1)^{n}e^{n\alpha}\sum_{l=0}^{\infty}\frac{(n\beta_0\cos\theta)^l}{l!}m^{l+3}\times B_l(n\beta_0 m), \quad (32)$$

where

$$B_{2l+1}(z)=\sum_{\nu=0}^{l}(-1)^{\nu}\binom{l}{\nu}B_1^{(2l-2\nu)}(z), \quad (33)$$

with

$$B_1(z)=2e^{-z}(z^{-2}+3z^{-3}+3z^{-4}). \quad (34)$$

(See Appendix II.)

If β is zero, the angular distribution is isotropic (in the present case in which the total angular momentum of the system has been assumed to be zero). The angular distribution for $m=0$ becomes

$$N_0(\theta)=\mp S(2\pi)^{-3}\sum_{n=1}^{\infty}(\mp1)^{n}\frac{e^{n\alpha}}{n^3}\frac{2}{(\beta_0-\beta\cos\theta)^3}. \quad (35)$$

The energy distribution in terms of the momentum four-vector $K=|\mathbf{k}|$, $k_0^2=K^2+m^2$, is, from (16),

$$E(K)=\mp S(2\pi)^{-2}\sum_{n=1}^{\infty}(\mp1)^{n}e^{n\alpha}K^2(K^2+m^2)^{\frac{1}{2}}\times\exp[n\beta_0(K^2+m^2)^{\frac{1}{2}}]\frac{\sinh(n\beta K)}{n\beta K}. \quad (36)$$

In the very special case of photons ($S=2$, bosons, i.e., lower sign, $\alpha=0$, $m=0$), Eq. (36) reduces to Planck's law:

$$E(K)dK=\frac{K^3}{\pi^2}\frac{\exp(-\beta_0 K)}{1-\exp(-\beta_0 K)}dK,$$

or (37)

$$E(\nu)d\nu=8\pi\nu^3\frac{\exp(-2\pi\nu/kT)}{1-\exp(-2\pi\nu/kT)}2\pi\nu d\nu,$$

since

$$K=k_0=2\pi\nu \qquad (\hbar=c=1).$$

From Eq. (31) we can also calculate the specific heat in the general relativistic form:

$$C_v=\frac{\partial\bar{E}}{\partial T}=\pm S(2\pi)^{-2}\sum_{n=1}^{\infty}(\mp1)^{n}e^{n\alpha}n\sum_{l=0}^{\infty}\frac{(n\beta)^{2l}}{(2l+1)!}\times m^{2l+5}\frac{B_l''(nm/kT)}{kT^2}. \quad (38)$$

4. CHARGE STATISTICS

For a system of three types of charges with the numbers n_r^{s+}, n_r^{s-}, and n_r^{s0}, the eigenvalues of the total charge Q are $\sum_{s,r}(n_r^{s+}-n_r^{s-})$, whereas the eigenvalues of P are again $\sum_{s,r}(n_r^{s+}+n_r^{s-}+n_r^{s0})k^{(r)}$. In this case, Z in Eq. (12) splits up into three parts:

$$Z=Z^+Z^-Z^0, \quad (39)$$

or

$$\ln Z=\ln Z^+ +\ln Z^- +\ln Z^0,$$

where

$$\ln Z^+=\ln Z(\alpha),\quad \ln Z^-=\ln Z(-\alpha),\quad \ln Z^0=\ln Z(0), \quad (40)$$

and $\ln Z(\alpha)$ is given by Eq. (13) or (16) or (24). Hence the total charge, momentum, and energy densities are given by

$$\bar{Q}=e\frac{\partial}{\partial\alpha}[\ln Z(\alpha)+\ln Z(-\alpha)]=e[\bar{N}(\alpha)-\bar{N}(-\alpha)], \quad (41)$$

$$\bar{P}=-\frac{\partial}{\partial\beta}\ln[Z(\alpha)Z(-\alpha)Z(0)]=\bar{P}(\alpha)+\bar{P}(-\alpha)+\bar{P}(0), \quad (42)$$

$$\bar{E}=-\frac{\partial}{\partial\beta_0}\ln[Z(\alpha)Z(-\alpha)Z(0)]=\bar{E}(\alpha)+\bar{E}(-\alpha)+\bar{E}(0), \quad (43)$$

where $\bar{N}(\alpha)$, $\bar{P}(\alpha)$, and $\bar{E}(\alpha)$ are given by Eqs. (29), (30), and (31).

In problems like the multiple production of mesons mentioned in Sec. 1, one is interested in the number $\bar{N}$ as a function of energy and momentum. This is obtained by eliminating α, β, and β_0 among the three equations (29), (30), (31) or (41), (42), (43). It is seen that this number is markedly different depending on whether one uses the simple statistics or the charge statistics.

The same type of analysis can be made for a mixture of fermion and boson systems. Here again one finds $Z=Z^{\text{fer}}Z^{\text{bos}}$ and $\bar{Q}$, $\bar{P}$, and $\bar{E}$ are additive in fermion and boson expressions, $\bar{E}=\bar{E}^{\text{ferm}}+\bar{E}^{\text{boson}}$, etc.

5. SPIN AND ANGULAR MOMENTUM STATISTICS

We now consider the situation when the total angular momentum of the system is not zero. The system may be spin-polarized or have an angular momentum in some direction, say due to a primary collision. It is perhaps not very difficult, for instance, to produce a polarized blackbody radiation with photons or electrons.

We consider the cases where the J^{ij} commute with P^j and among themselves (see Sec. 2). Then we will have, instead of Eq. (12),

$$Z=\sum_{n_1^1,\cdots,n_r^s,\cdots}\exp(\alpha\sum_{s,r}n_r^s-\beta_j\sum_{s,r}n_r^s k^{(r)j})\times\langle\{n_r^s k^{(r)}\}|\exp(\lambda_{ij}J^{ij})|\{n_r^s k^{(r)}\}\rangle. \quad (44)$$

To evaluate the matrix elements in this equation, we expand $\psi^{(-)}(x)$ in the expression

$$b^{(-)s}(k^{(r)})=\int dx\exp(ik^{(r)}x)\psi^{(-)s}(x)$$

into a set of eigenfunctions of J^{ij}:

$$\psi^{(-)s}(x)=\sum_\mu a_\mu^{(-)s}Y_\mu(x).$$

Hence

$$b^{(-)s}(k^{(r)})|0\rangle=\sum_\mu a_\mu^{(-)s}|0\rangle dx\int \exp(ik^{(r)}x)Y_\mu(x)$$
$$=\sum_\mu Y_\mu(k^{(r)})a_\mu^{(-)s}|0\rangle,$$

where the $a_\mu^{(-)s}|0\rangle$ are eigenvectors of J^{ij}, and the $Y_\mu(k^{(r)})$ are Fourier transforms of the spherical harmonics for orbital angular momentum alone. The corresponding eigenvalues μ_{ij} take integral values in the case of orbital angular momentum, and the values $\mp\frac{1}{2}$ for a spinor field. The matrix elements in question are then of the following form:

$$\sum_{ij}\sum_\mu Y_\mu^{ij}(k^{(r)})\exp(\lambda_{ij}\mu_{ij}^s). \tag{45}$$

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APPENDIX I

It is of interest to state the conditions under which the density matrix ρ is given by Eq. (1). Let us denote the constants of the motion of the system collectively by C and consider a Gibbsian ensemble. Perhaps the simplest set of postulates are the following:

1. The density matrix of a system in equilibrium is equivalent to a distribution over the Gibbsian ensemble of identical, statistically independent systems.

2. The density matrix ρ depends only on the set of Hermitian operators C. Since the quantities C for arbitrary subsets of the ensemble are additive, the essential idea is that one can also group the members of the ensemble in pairs or in triples, etc., without changing the statistical character of it. Thus, if we group the systems in the ensemble in pairs, we get

$$\rho(C^{(1)}+C^{(2)})=\rho(C^{(1)})+\rho(C^{(2)}); \tag{I,1}$$

i.e., the same function ρ but now of argument $C^{(1)}+C^{(2)}$ describes the newly obtained ensemble. Since ρ and C are Hermitian, the unique solution of the operator functional equation (I,1) is (generalizing the c-number result)

$$\rho=e^{\alpha C}=\exp(\alpha_iC_i), \tag{I,2}$$

which reduces to Eq. (1) for the case considered.

These two conditions characterize the ensembles studied here. The actual mechanism of how a system reaches the equilibrium defined by Eq. (1) is another question which will probably be different from the usual nonrelativistic approach to equilibrium.

APPENDIX II

We want to evaluate the functions

$$B_l(z)=\int_0^\infty dt\, t^{l+2}\exp[-z(t^2+1)^{\frac{1}{2}}]. \tag{II,1}$$

B_0 and B_1 are known integrals and are given in Eqs. (27) and (34). For odd l the functions can be reduced to elementary functions. For l even but not zero, the integral (II,1) does not seem to be manageable directly. By differentiating $B_l(z)$ twice under the integral, we obtain the following recursion relation:

$$B_l''(z)=B_l(z)+B_{l+2}(z).$$

Since $B_0(z)$ is known, we get then the results quoted in the text, Eqs. (26) and (33).