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Relativistic Theory of Radiative Orbital Electron Capture

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A fully relativistic treatment of the radiation accompanying nuclear capture of orbital electrons is presented. All effects of the electrostatic field surrounding the nucleus are taken into account. As a preliminary step, convenient representations for the electron Green's function and initial state wave function in a Coulomb field are derived. These forms, involving Dirac operators applied to scalar functions and free-particle angular eigenfunctions, are developed from the second-order Dirac equation. They are particularly useful for calculations since the procedures which make use of the properties of traces can be employed with them.

With the aid of these representations the photon energy spectrum and polarization associated with allowed radiative K capture are computed. Relativistic Coulomb corrections are shown to decrease the expected photon intensity significantly at all energies. Since their effect is not sensitively dependent on energy, the predicted shape of the spectrum is not greatly altered. The Coulomb field also influences the degree of polarization of the photons emitted, but has an appreciable effect only near the lower end of the spectrum.

The influence of atomic screening on the capture from the K and L shells is also taken into account approximately. It is shown that screening considerably decreases the likelihood of radiative capture of all but the innermost electrons.

Finally, the existing experimental evidence is reviewed and shown to agree with the theory presented. Some additional experimental tests are proposed.

1. INTRODUCTION

In a previous paper,[1] an approximate theory of the radiation accompanying nuclear capture of orbital electrons has been developed. In that paper, the essential features of the gamma-ray energy spectra were explained with the aid of nonrelativistic and approximately relativistic wave functions. The calculations performed with these functions demonstrated that the Coulomb field played a most significant role. Taking it into account affected the predicted number of high-energy quanta emitted during the capture of S-state electrons. More important, it provided the mechanism by which the large number of low-energy quanta observed might be explained as quanta emitted in capture of P-state electrons. In view of the steadily increasing number of accurately measured radiative-capture spectra, a more exact theoretical treatment of the problem has become desirable.

A second and more immediate reason for performing this analysis is the recently discovered and hitherto unsuspected asymmetry of the beta interaction under spatial inversion.[2] It is easy to show that this asymmetry leads to a dominant circular polarization of the γ-ray spectrum. For example, the beta interaction most commonly assumed to date, taken together with the two-component neutrino theory,[3] predicts radiation, which, in the absence of a Coulomb field, would be completely circularly polarized.[4] It is therefore of interest to determine more generally the degree of polarization in the presence of a Coulomb field. The present paper, in providing this analysis, is devoted in part to the larger problem of simplifying relativistic calculations for electrons moving in a Coulomb field. It is hoped that the methods outlined will also be useful in other contexts.

The most important corrections to the radiation intensities previously calculated result from a more exact treatment of the relativistic effects and from taking into account the influence of atomic screening. The former corrections are the dominant ones for the innermost S-state electrons; the latter are more important for electrons further from the nucleus.

That relativistic corrections to the allowed radiative capture of S-state electrons should be large is easily made plausible by showing that these events must always involve the essentially relativistic electron spin. For radiative capture to occur, the electron must emit a photon during a transition to an intermediate state from which it may be captured. Since only spherically symmetric wave functions differ from zero at the nucleus, the intermediate state must be an S state. A spinless particle, however, cannot radiate a single quantum in the course of a transition from one spherically symmetric state to another. Hence radiative capture of electrons in spherically symmetric states must involve their spin. The simplest possible process open to an S-state electron, and the one most extensively treated in (I), is radiation of a quantum by a reorientation of the electron spin during capture. In this case,

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[1] R. J. Glauber and P. C. Martin, Phys. Rev. **104**, 158 (1956). Hereafter this paper will be referred to as (I). See also R. J. Glauber and P. C. Martin, Phys. Rev. **95**, 572 (1954).

[2] Wu, Ambler, Hayward, Hoppes, and Hudson, Phys. Rev. **105**, 1413 (1957); H. Frauenfelder *et al.*, Phys. Rev. **106**, 386 (1957).

[3] T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1671 (1957); also L. Landau, Nuclear Phys. **3**, 127 (1957); A. Salam, Nuovo cimento **5**, 299 (1957); and W. Pauli, *Handbuch der Physik* (Verlag-Julius Springer, Berlin, 1933), Vol. 24, pp. 226–227.

[4] This point has also been noted by R. E. Cutkosky, Phys. Rev. **107**, 300 (1957).

a large fraction of the energy released is carried off by the photon and the electron suffers a relativistic recoil. A second intrinsically relativistic mechanism, giving rise to radiative capture of S-state electrons, is emission of a single quantum when a virtual positron, ejected by the nucleus, annihilates the electron. Still a third process is the transition of S-state electrons to $P_{\frac{1}{2}}$ states from which direct capture may also take place. (Relativistic $P_{\frac{1}{2}}$ wave functions do not vanish at the origin since, as a result of spin-orbit coupling, they contain an admixture of S state.) These last two features have been only approximately included in (I) and will here be treated more exactly. While the latter two relativistic effects noticeably alter the $1S$-state spectrum, screening influences it only slightly. The $1S$ electrons, which are responsible for practically all the S-state radiation, lie so close to the nucleus that they are almost unshielded.

The importance of these relativistic and screening corrections to the intensity is interchanged in the capture of P-state electrons. As indicated in (I), these electrons emit predominantly low-frequency quanta while making electric dipole transitions to the intermediate S states from which they are captured. Relativistic effects may therefore be expected to be quite insignificant. On the other hand, all of the P-state electrons are on the average sufficiently far from the nucleus to make screening corrections sizeable.

Insofar as polarization is concerned, it is easily shown that the radiation emitted in a transition which involves no change of spatial parity will be completely polarized if the beta interaction consists of only scalar and tensor parts, or of only vector and axial vector parts, and if a two-component neutrino theory is assumed. Under the same assumptions the radiation emitted with a spatial parity change of the radiating system is unpolarized. This means that the radiation from S states will be polarized circularly while the radiation from P states will not. As discussed in (I) and above, however, there can be electric dipole radiation, that is, unpolarized radiation emitted during K capture because the S electron is partially in a $P_{\frac{1}{2}}$ state as a result of the atomic spin-orbit coupling. In this simple example, then, the lack of complete polarization of the radiation accompanying K capture is directly related to the fraction of radiation which is electric dipole in character. It will emerge that this fraction is quite small except at very low energies.

The use of the Dirac equation to treat the relativistic effects leads to some mathematical complications. As in the earlier presentation, it is convenient to find a function, the Green's function, which expresses the probability amplitude that an electron which has emitted a virtual quantum of a certain energy, propagates inwards to the nucleus. For the Dirac equation, this probability amplitude depends on the electron spin and hence the Green's function is an operator which acts on Dirac spinor wave functions. The determination of the transition amplitude for radiative capture and of the gamma-ray spectrum, by multiplying explicit matrix representations of such quantities, would be quite tedious. Considerable simplification is achieved by employing only the general algebraic properties of the Dirac operators in obtaining the Green's function and the wave functions and in carrying out the remainder of the calculation. In the second section of this paper, a method is developed which determines each wave function as a linear combination of Dirac operators applied to a free-particle spinor. By a similar technique, the Green's function is written as a sum of Dirac operators multiplied by a scalar radial function. The procedures which take advantage of the properties of traces to simplify calculations with free-particle Dirac spinors may then be used with these spinors and Green's function for the Coulomb field. In the following section, these procedures are employed to evaluate the intensity and polarization of radiation emitted by electrons in the K shell.

The effects of atomic screening which are the major correction for electrons further from the nucleus, are discussed in the fifth section. For these purposes, it is sufficient to assume that the electrons all move in the same average potential. Then, as in (I), the exclusion principle may be neglected in computing the matrix element for radiative capture of any one of them. The fact that the average potential in which the electrons move is different from the field of an isolated nucleus alters the predicted gamma-ray intensities in a manner which is easily understood. In the energy region of interest, the probability amplitude for capturing an electron which has undergone a virtual radiative transition decreases very rapidly as the distance of the electron from the nucleus increases. In other words, the Green's function has a short range. Since its range is short compared to the first Bohr radius, the Green's function is significantly different from zero only where the electric field has almost the pure Coulomb form. As a result, screening does not greatly affect the Green's function. The short range of the Green's function also means that the electron can be radiatively captured only when it is initially in the neighborhood of the nucleus. Since the electron wave functions are spread out as a result of screening, the likelihood of finding the electron near the nucleus, and hence, the probability of radiative capture is decreased. In atoms which are not too heavy, the probability of radiative capture of electrons in the L shell is considerably diminished and the likelihood of capture of electrons beyond this shell reduced to insignificance.

The results of the more refined treatment carried out here have been expressed as correction factors multiplying the simpler functions which occur in (I). Approximations for these correction factors have been derived which are quite accurate for elements with low charge (for example, A^{37} and V^{49}). With these approximations, observable departures from the less accurate treatment are predicted. The experimental evidence appears to be in agreement with these modifications.

2. RELATIVISTIC COULOMB WAVE FUNCTIONS AND GREEN'S FUNCTIONS

Calculations, which treat electrons in the presence of a Coulomb field relativistically, can in most cases be simplified by expressing the electron wave functions and the Green's function with the help of linear combinations of Dirac operators. The algebraic properties of the operators can then be used to expedite the calculations in the same way as they are employed to simplify relativistic problems involving free electrons. In this section expressions having the desired form will be obtained with the aid of the second-order Dirac equation. The solutions to this equation will be conveniently derived by introducing an operator which bears a resemblance to the orbital angular momentum. The angular eigenfunctions of this operator will be constructed from ordinary angular momentum eigenfunctions by applying simple projection operators. The wave functions and Green's function of the radial second-order equation corresponding to given angular eigenvalues will also be determined. From the solutions to the second-order equation so obtained, the wave functions and Green's function of the first-order equation will be constructed by applying appropriate projection operators.

If ψ_i is the wave function of an electron in the ith state and E_i is its energy, then ψ_i satisfies the Dirac equation[5]

$$[\boldsymbol{\gamma}\cdot\mathbf{p}-\gamma_0(E_i+Ze^2/r)+m]\psi_i=0. \tag{2.1a}$$

Its adjoint, $\psi_i^*\gamma_0=\bar{\psi}_i$, obeys the corresponding equation

$$\bar{\psi}_i[\boldsymbol{\gamma}\cdot\mathbf{p}-\gamma_0(E_i+Ze^2/r)+m]=0, \tag{2.1b}$$

in which $\mathbf{p}$ is assumed to act to the left. It is convenient to determine ψ_i by applying the projection operator, $\mathcal{P}_+$, for positive mass, to a function φ_i,

$$\begin{aligned}\psi_i&=(2m)^{-1}[-\boldsymbol{\gamma}\cdot\mathbf{p}+\gamma_0(E_i+Ze^2/r)+m]\varphi_i\\&=\mathcal{P}_+\varphi_i,\end{aligned} \tag{2.2}$$

which must then satisfy the second-order Dirac equation,

$$\begin{aligned}&[\boldsymbol{\gamma}\cdot\mathbf{p}-\gamma_0(E_i+Ze^2/r)+m]\\&\qquad\times[-\boldsymbol{\gamma}\cdot\mathbf{p}+\gamma_0(E_i+Ze^2/r)+m]\varphi_i=0,\end{aligned} \tag{2.3a}$$

$$[\mathbf{p}^2+m^2-(E_i+Ze^2/r)^2-i\alpha_rZe^2/r^2]\varphi_i=0. \tag{2.3b}$$

The operator $\alpha_r=\boldsymbol{\alpha}\cdot\mathbf{r}/r$ is the component of the Dirac matrix $\boldsymbol{\alpha}$ in the direction $\mathbf{r}$. The adjoints of these equations are easily seen to be

$$\begin{aligned}&\bar{\varphi}_j[\boldsymbol{\gamma}\cdot\mathbf{p}-\gamma_0(E_j+Ze^2/r)+m]\\&\qquad\times[-\boldsymbol{\gamma}\cdot\mathbf{p}+\gamma_0(E_j+Ze^2/r)+m]=0,\end{aligned} \tag{2.4a}$$

$$\bar{\varphi}_j[\mathbf{p}^2+m^2-(E_j+Ze^2/r)^2-i\alpha_rZe^2/r^2]=0, \tag{2.4b}$$

where $\bar{\varphi}_j=\varphi_j^*\beta$,

$$\bar{\psi}_j=\bar{\varphi}_j[-\boldsymbol{\gamma}\cdot\mathbf{p}+\gamma_0(E_j+Ze^2/r)+m](2m)^{-1}, \tag{2.5}$$

and $\mathbf{p}$ is again understood to operate to the left.

As in the Klein-Gordon equation, the energy appears quadratically and linearly in Eqs. (2.3) and (2.4). Hence, as in a Klein-Gordon equation, the orthogonality relations satisfied by solutions of different energies involve weight factors dependent on the states. These are easily found by the conventional procedure: Equation (2.3b) is multiplied by $\bar{\varphi}_j$ on the left, Eq. (2.4b) by φ_i on the right, both expressions integrated, and the results subtracted. The orthogonality relation which emerges has the form

$$\int d\mathbf{r}\,\bar{\varphi}_j\left(E_i+E_j+\frac{2Ze^2}{r}\right)\varphi_i=0, \tag{2.6a}$$

for $E_i\neq E_j$. The normalization of φ_i is most conveniently chosen to agree with the normalization of the first-order wave function, that is, so that

$$\begin{aligned}1&=\int d\mathbf{r}\psi_i^*\psi_i=\int d\mathbf{r}\,\bar{\varphi}_i\mathcal{P}_+\gamma_0\mathcal{P}_+\varphi_i\\&=\int d\mathbf{r}\,\bar{\varphi}_i\left(\frac{E_i+Ze^2/r}{m}\right)\mathcal{P}_+\varphi_i.\end{aligned} \tag{2.6b}$$

Equation (2.3b), which involves only one of the Dirac matrices, may be solved more easily than Eq. (2.1a). For this purpose, it is convenient to introduce spherical coordinates and express the square of the total momentum as

$$\mathbf{p}^2=r^{-1}p_r^2r+\mathbf{L}^2r^{-2},$$

where $p_r=-i\partial/\partial r$ is the radial momentum, and $\mathbf{L}$, the usual operator for orbital angular momentum. The operator, $\mathbf{L}^2$, may in turn be expressed as

$$\mathbf{L}^2=\mathcal{K}(\mathcal{K}-\beta),$$

where

$$\mathcal{K}=\beta(\boldsymbol{\sigma}\cdot\mathbf{L}+1). \tag{2.7}$$

This operator, $\mathcal{K}$, first introduced by Dirac, has several useful properties. It commutes with the first-order Dirac Hamiltonian and therefore identifies a constant of the motion. It also commutes with α_r. Further, its magnitude is related to the magnitude of the total angular momentum, $\mathbf{J}^2=(\mathbf{L}+\frac{1}{2}\boldsymbol{\sigma})^2$, by the equation

$$\mathcal{K}^2=(\boldsymbol{\sigma}\cdot\mathbf{L}+1)^2=(\mathbf{L}+\tfrac{1}{2}\boldsymbol{\sigma})^2+\tfrac{1}{4}=\mathbf{J}^2+\tfrac{1}{4}.$$

Since the eigenvalues of J^2 are $j(j+1)$ where $j=\frac{1}{2}$, $\frac{3}{2}$,

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[5] The notation used in this paper is the same as that defined in (I). Natural units are employed and the Dirac matrices satisfy the relations: $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=-2\delta_{\mu\nu}$, $\gamma_j=\beta\alpha_j$ $(j=1, 2, 3)$, and $\gamma_0=\beta=-i\gamma_4$.

$\cdots$, the eigenvalues of $\mathcal{K}^2$ must have the values $(j+\frac{1}{2})^2$. The eigenvalues of $\mathcal{K}$, which will be designated by K, are therefore restricted to

$$\mathrm{K}=\pm 1, \pm 2, \cdots. \tag{2.8a}$$

The absolute value of K will be denoted by κ and has the value

$$\kappa=|\mathrm{K}|=j+\tfrac{1}{2}. \tag{2.8b}$$

In terms of $\mathcal{K}$, Eq. (2.3b) may be written as

$$\{p_r^2+[\mathcal{K}(\mathcal{K}-\beta)-(Ze^2)^2-i\alpha_r Ze^2]/r^2 \\ -2E_iZe^2/r+m^2-E_i^2\}r\varphi_i=0. \tag{2.9}$$

The coefficient of $1/r^2$ takes the place of the operator for the square of the orbital angular momentum in the Schrödinger equation. This correspondence may be brought out by introducing the operator,[6]

$$\mathcal{L}=-\mathcal{K}\beta-iZe^2\alpha_r, \tag{2.10}$$

in terms of which, the second-order Dirac equation is

$$[p_r^2+\mathcal{L}(\mathcal{L}+1)/r^2-2E_iZe^2/r+m^2-E_i^2]r\varphi_i=0. \tag{2.11}$$

Since the operator $\mathcal{L}$ satisfies the equation $\mathcal{L}^2=\mathcal{K}^2-(Ze^2)^2$, its eigenvalues, Λ, are $\pm[(j+\frac{1}{2})^2-(Ze^2)^2]^{\frac{1}{2}}$. The positive square root will be designated by

$$\lambda=|\Lambda|=[\mathrm{K}^2-(Ze^2)^2]^{\frac{1}{2}}. \tag{2.12}$$

Eigenfunctions of the operator $\mathcal{L}$ which are at the same time eigenfunctions of Eq. (2.9) satisfy

$$[p_r^2+\Lambda(\Lambda+1)/r^2-2E_iZe^2/r+m^2-E_i^2]r\varphi_i=0. \tag{2.13}$$

The pair of uncoupled equations for $\Lambda=\pm\lambda$ replaces the coupled first-order equations of the usual treatment of the Dirac equation. The solutions to these equations may also be taken to be eigenfunctions of $\mathcal{K}$ and a component of the angular momentum, J_z, since $\mathcal{L}$, $\mathcal{K}$, and J_z all commute with one another.

The eigenfunctions and eigenvalues of Eq. (2.13) can be obtained by a variety of methods one of which is outlined in Appendix A. The eigenvalues, E_i, are given by

$$E_i=m[1+Z^2e^4/(n_r+\lambda+\tfrac{1}{2}\pm\tfrac{1}{2})]^{-\frac{1}{2}}, \tag{2.14}$$

where n_r the radial quantum number, takes on the values 0, 1, 2, $\cdots$, and where the alternative signs refer to $\Lambda=\pm\lambda$ respectively. The normalized eigenfunctions are

$$\varphi_i=c_{\pm\lambda}(2\mu_ir)^{-\frac{1}{2}-\lambda\mp\frac{1}{2}}\exp(\mu_ir)\chi_\Lambda{}^{\mathrm{K},m_z}(d/d2\mu_ir)^{n_r} \\ \times[(2\mu_ir)^{n_r+2\lambda\pm 1}\exp(-2\mu_ir)], \tag{2.15}$$

where

$$\mu_i^2=m^2-E_i^2, \tag{2.16}$$

and where $\chi_\Lambda{}^{\mathrm{K},m_z}$ is an angular eigenfunction of the operators $\mathcal{K}$, $\mathcal{L}$, and J_z, whose construction will be described presently. These stationary states of the second-order equation are displayed in the energy-level diagram of Fig. 1. They include the solutions to two first-order equations, the familiar first-order Dirac equation, and the first-order Dirac equation with the sign of the mass reversed. Equivalently, if φ_i is an eigenfunction of either of the operator factors occurring in Eq. (2.3a), it satisfies the second-order equation (2.13).

The eigenfunctions of the first-order equation (2.1) also may be taken to be eigenfunctions of $\mathcal{K}$ and J_z. These particular eigenfunctions of Eq. (2.1) must be expressible as combinations of the solutions to Eq. (2.13) which have the same eigenvalues E_i, K, and m_z. In general there are two such wave functions, one for each of the values $\mathcal{L}=\pm[\kappa^2-(Ze^2)^2]^{\frac{1}{2}}=\pm\lambda$. Neither of them need be an eigenfunction of the first-order equation since $\mathcal{L}$, unlike $\mathcal{K}$, does not commute with the ordinary Dirac Hamiltonian. The pairs of solutions of the second-order equation which must be combined to form eigenfunctions of the first-order equation are joined by the brackets in Fig. 1. The correct linear combinations are most simply determined by applying

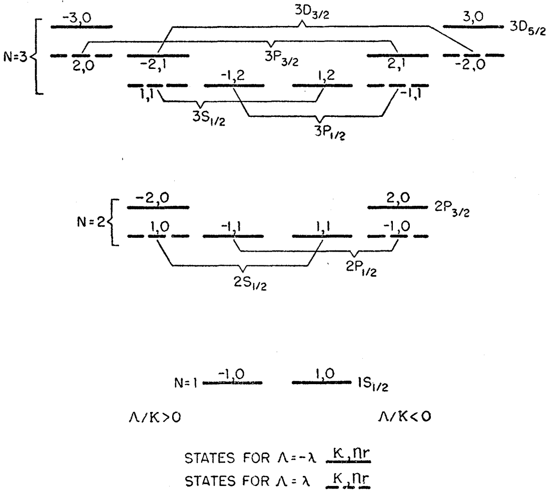

FIG. 1. Energy-level diagram for an electron bound in a Coulomb field according to the second-order Dirac equation, (2.3a) or (2.11). These levels are twice as numerous as the familiar ones for the first order equation, (2.1a). (The energy intervals, plotted vertically, are not drawn to scale.) Multiplets of levels are grouped according to the total quantum number N, shown at the left. Levels for negative values of the quantum number Λ are indicated as solid horizontal bars, while those for positive Λ are dashed. The individual levels are labeled by the pair of quantum numbers K, n_r appearing either above or below the bar.

The eigenstates of the first-order Dirac equation may be expressed as linear combinations of degenerate pairs of the states indicated above. The degenerate states superposed must have equal values of K, and $\Lambda=\pm\lambda$. Such pairings are indicated above by brackets which are labeled according to the hydrogenic levels they produce. The levels in the right half of the diagram are those for which Λ/K is negative, while those in the left half have positive values of Λ/K. Thus, the levels at the right are those for which the expectation value of β approaches plus unity as $Z\to 0$. In this limit they are the principal contributors to the linear combinations shown. For certain of the energy levels no degeneracy with respect to K exists. For this reason the hydrogenic states $1S_{\frac{1}{2}}$, $2P_{\frac{3}{2}}$, $3D_{\frac{5}{2}}$, $\cdots$ arise from the single states of the second-order equation indicated.

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[6] The introduction of the operator $\mathcal{L}$ was suggested to us by Dr. K. A. Johnson.

the projection operator occurring in Eq. (2.2) to remove the negative mass part of the functions, φ_i, in Eq. (2.15). Since the state specified by a given energy, K, m_z, and sign of the mass, is unique, the same function, apart from normalization, will be obtained by applying the positive-mass projection operator to either of the two solutions, φ_i, corresponding to $\Lambda=\pm\lambda$. Before doing this, it is necessary to express this projection operator in spherical coordinates. This may be accomplished by using the identity $\boldsymbol{\alpha}\cdot\mathbf{p}=r^{-2}(\boldsymbol{\alpha}\cdot\mathbf{r})(\boldsymbol{\alpha}\cdot\mathbf{r})(\boldsymbol{\alpha}\cdot\mathbf{p})=\alpha_r[p_r+i\boldsymbol{\sigma}\cdot\mathbf{L}/r]$, and Eqs. (2.7), (2.10), to write

$$m+\gamma_0\left(E_i+\frac{Ze^2}{r}\right)-\boldsymbol{\gamma}\cdot\mathbf{p} =m+\beta E_i+i\beta\alpha_r\left(\frac{\partial}{\partial r}+\frac{\mathcal{L}+1}{r}\right). \tag{2.17}$$

The previous discussion must be amended for a certain special subset of states of the second-order equation, those for which $n_r=0$ and $\Lambda=-\lambda$. As indicated in Fig. 1, there is no second eigenfunction in this case, which has the same eigenvalues for E_i and $\mathcal{K}$, but for which Λ is positive. The eigenfunctions of the first-order equation must therefore be determined by applying $\mathcal{P}_+$ to the state for which $\Lambda=-\lambda$. The states singled out in this way are the ones with maximal total angular momentum for a given principal quantum number; among them are the states $1S_{\frac{1}{2}}$, $2P_{\frac{3}{2}}$, $3D_{\frac{5}{2}}$. Each has a wave function for which the relations

$$E_i=m\lambda/\kappa \quad\text{and}\quad \left(\frac{\partial}{\partial r}+\frac{\mathcal{L}+1}{r}\right)\varphi_i=-\mu_i\varphi_i \tag{2.18}$$

follow from Eqs. (2.15) and (2.16). Consequently, the result of applying $\mathcal{P}_+$ to any of them is

$$(2m)^{-1}\{m+\beta E_i+i\beta\alpha_r[(\partial/\partial r)+(1-\lambda)/r]\}\varphi_i = (2\kappa)^{-1}(\kappa+\lambda\beta-iZe^2\beta\alpha_r)\varphi_i=(2\kappa)^{-1}(\kappa+\mathrm{K})\varphi_i. \tag{2.19a}$$

In a similar manner, the projection operator for negative mass operating on these states simplifies to

$$(2m)^{-1}\{m-\beta E_i-i\beta\alpha_r[(\partial/\partial r)+(1-\lambda)/r]\}\varphi_i = (2\kappa)^{-1}(\kappa-\mathrm{K})\varphi_i. \tag{2.19b}$$

Thus, when $n_r=0$ and Λ is negative, the eigenfunction for which $\mathrm{K}=\kappa$ does indeed satisfy the first-order Dirac equation. The solution for which $\mathrm{K}=-\kappa$ satisfies a Dirac equation in which the mass has the opposite sign. More generally, the solutions for which the ratio Λ/K is negative are the ones which correspond to expectation values of β near plus one rather than minus one. Thus these states are the principal contributors to the linear combinations which form the solutions to the first-order equation. For this reason it is convenient to generate the solutions to the first-order equation by applying $\mathcal{P}_+$ to solutions of the second-order equation for which Λ/K is negative.

The solutions to the first-order equation may be displayed in a more familiar form by using the canonical representation in which β and spin are diagonal. For the purpose of constructing eigenvectors of $\mathcal{L}$ and $\mathcal{K}$ in this representation, it is convenient first to determine the angular eigenvectors[7] $\chi_\pm^{\mathrm{K},m_z}$ for which

$$\mathcal{K}\chi_\pm^{\mathrm{K},m_z}=\mathrm{K}\chi_\pm^{\mathrm{K},m_z},\quad J_z\chi_\pm^{\mathrm{K},m_z}=m_z\chi_\pm^{\mathrm{K},m_z},$$

and

$$\beta\chi_\pm^{\mathrm{K},m_z}=\pm\chi_\pm^{\mathrm{K},m_z}. \tag{2.19c}$$

This may be accomplished by using the familiar properties of the angular momentum operators[8] together with the definition, Eq. (2.7), of $\mathcal{K}$. The angular eigenfunctions for which the eigenvalues of β and $\mathcal{K}$ have the same sign are

$$\chi_\pm^{\mathrm{K},m_z}(\Omega)=\left[\frac{\kappa+m_z-\frac{1}{2}}{2\kappa-1}\right]^{\frac{1}{2}}Y_{\kappa-1}^{m_z-\frac{1}{2}}(\Omega)\chi_\pm^{+} +\left[\frac{\kappa-m_z-\frac{1}{2}}{2\kappa-1}\right]^{\frac{1}{2}}Y_{\kappa-1}^{m_z+\frac{1}{2}}(\Omega)\chi_\pm^{-}; \tag{2.20a}$$

those for which the eigenvalues of β and $\mathcal{K}$ have opposite sign are

$$\chi_\pm^{\mathrm{K},m_z}(\Omega)=\left[\frac{\kappa-m_z+\frac{1}{2}}{2\kappa+1}\right]^{\frac{1}{2}}Y_{\kappa}^{m_z-\frac{1}{2}}(\Omega)\chi_\pm^{+} -\left[\frac{\kappa+m_z+\frac{1}{2}}{2\kappa+1}\right]^{\frac{1}{2}}Y_{\kappa}^{m_z+\frac{1}{2}}(\Omega)\chi_\pm^{-}. \tag{2.20b}$$

The functions, $Y_k^m(\Omega)$, occurring in these expressions are normalized spherical harmonics which are functions of the direction, Ω; the spinors, $\chi_\pm^\pm$, are eigenvectors for which $\beta\chi_\pm=\pm\chi_\pm$ and $\sigma_z\chi^\pm=\pm\chi^\pm$.

In the absence of the Coulomb field ($Z=0$), each eigenvector of β and $\mathcal{K}$ is also an eigenvector of $\mathcal{L}$ and $\mathcal{K}$. For nonvanishing Z, however, $\mathcal{L}$ and β do not commute and each of the two eigenvectors $\chi_{\pm\lambda}^{\mathrm{K},m_z}$, of $\mathcal{L}$ is a linear combination of the two eigenvectors, $\chi_\pm^{\mathrm{K},m_z}$, of β. As a result, each eigenvector, $\chi_\Lambda^{\mathrm{K},m_z}$, may be determined by projecting either eigenvector, χ_+^{K,m_z} or χ_-^{K,m_z}, onto the subspace in which $\mathcal{L}=\Lambda$. The operator which performs this projection is $(\mathcal{L}+\Lambda)/2\Lambda$. A convenient choice is to write each eigenvector of $\mathcal{L}$ as the projection of that eigenvector of β with which it would coincide if the Coulomb field vanished. With this choice, and the introduction of a normalization constant which will prove convenient, the expression,

$$\chi_{\pm\lambda}^{\mathrm{K},m_z}=\frac{\lambda\pm\mathcal{L}}{2\lambda}\left[\frac{2\lambda}{\kappa+\lambda}\right]^{\frac{1}{2}}\chi_\mp^{\mathrm{K},m_z}, \tag{2.21a}$$

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[7] The symbol χ is used generically for spinor functions of angle. Its superscripts and subscripts identify the particular function of angle referred to.

[8] These spherical harmonics have the standard phases defined, for example, in W. Magnus and F. Oberhettinger, *Special Functions* (Springer-Verlag, Berlin, 1943), p. 53.

results for the eigenvectors of $\mathcal{K}$, $\mathcal{L}$, and J_z for which K>0 and the corresponding expression,

$$\chi_{\pm\lambda}{}^{\mathrm{K},m_z}=\frac{\lambda\pm\mathcal{L}}{2\lambda}\left[\frac{2\lambda}{\kappa+\lambda}\right]^{\frac{1}{2}}\chi_{\pm}{}^{\mathrm{K},m_z}, \tag{2.21b}$$

occurs for the eigenvectors for which K<0. In the future, the index m_z will be suppressed.

In carrying out expansions in the eigenfunctions, $\chi_\Lambda{}^{\mathrm{K}}$, the non-Hermitian character of $\mathcal{L}$ must be taken into account. Orthogonality relations exist only in the sense that the scalar product of an eigenfunction of $\mathcal{L}$ having eigenvalue Λ with an eigenfunction of $\mathcal{L}^\dagger$ having an eigenvalue other than Λ^*, is zero. (Although $\mathcal{L}$ is not Hermitian, it has real eigenvalues, $\Lambda=\Lambda^*$.) Since β anticommutes with the skew-Hermitian part of $\mathcal{L}$, and commutes with its Hermitian part the operator $\mathcal{L}^\dagger$ is equal to $\beta\mathcal{L}\beta$. This means that the eigenfunctions of $\mathcal{L}^\dagger$ are Pauli adjoints, $(\chi_\Lambda{}^{\mathrm{K}})^*\beta$, of the eigenfunctions of $\mathcal{L}$. The orthonormality of these functions is explicitly demonstrated for K>0 by using the orthonormality of the eigenfunctions of the Hermitian operator, $\mathcal{K}$, and the idempotence of the projection operator $(\lambda\pm\mathcal{L})/2\lambda$:

$$\begin{aligned}\int d\Omega(\chi_\Lambda{}^{\kappa*}(\Omega)\beta\chi_{\Lambda'}{}^{\kappa'}(\Omega))\\ =\delta_{\kappa\kappa'}\int d\Omega(\chi_\Lambda{}^{\kappa*}(\Omega)\beta\chi_{\Lambda'}{}^{\kappa}(\Omega))\\ =-\frac{2\Lambda}{\kappa+\lambda}\int d\Omega\left(\chi_\mp{}^{\kappa*}(\Omega),\frac{\lambda\pm\mathcal{L}}{2\lambda}\chi_\mp{}^{\kappa'}(\Omega)\right)\delta_{\Lambda\Lambda'}\delta_{\kappa\kappa'}\\ =-(\Lambda/\lambda)\delta_{\Lambda\Lambda'}\delta_{\kappa\kappa'}.\end{aligned} \tag{2.22}$$

The more general orthonormality relation for arbitrary K is

$$\int d\Omega(\chi_\Lambda{}^{\mathrm{K}*}(\Omega)\beta\chi_{\Lambda'}{}^{\mathrm{K}'}(\Omega))=-(\Lambda/\lambda)(\mathrm{K}/\kappa)\delta_{\Lambda\Lambda'}\delta_{\mathrm{KK}'}. \tag{2.23}$$

With the choice of wave function normalization[9] made in Eqs. (2.21), the invariant integrals (2.22) and (2.23) reduce for the cases $\Lambda=\Lambda'$, K$=$K$'$ to the values ±1.

The solutions to the first-order equation, $\psi_i=\mathcal{P}_+\varphi_i$, are therefore given by

$$\begin{aligned}\psi_i=c_{-\lambda}(2m)^{-1}\{m+\beta E_i+i\beta\alpha_r[(d/dr)+(1-\lambda)/r]\}\\ \times(2\mu_ir)^{-\lambda}e^{\mu_ir}(d/d2\mu_ir)^{n_r}\\ \times[(2\mu_ir)^{n_r+2\lambda-1}e^{-2\mu_ir}]\chi_{-\lambda}{}^{\kappa}\end{aligned} \tag{2.24a}$$

for K>0, and

$$\begin{aligned}\psi_i=c_{+\lambda}(2m)^{-1}\{m+\beta E_i+i\beta\alpha_r[(d/dr)+(1+\lambda)/r]\}\\ \times(2\mu_ir)^{-\lambda-1}e^{\mu_ir}(d/d2\mu_ir)^{n_r}\\ \times[(2\mu_ir)^{n_r+2\lambda+1}e^{-2\mu_ir}]\chi_{\lambda}{}^{-\kappa}\end{aligned} \tag{2.24b}$$

for K<0. The normalization constants, $c_{\pm\lambda}$, occurring in them, are evaluated in Appendix B, and the spinors are given in the spin, β representation, by Eqs. (2.20) and (2.21).

Several of the statements made above are illustrated by the two $1S$ electron wave functions (one for each direction of the spin). Of the solutions to the second-order equation having $n_r=0$, $j=\frac{1}{2}$, and $\Lambda=-[1-(Ze^2)^2]^{\frac{1}{2}}$, the two for which K$=-1$ are annihilated by the positive-mass projection operator and the two for which K$=1$ are unchanged on applying this operator. The latter, which have the energy, $E_{1S}=m[1-(Ze^2)^2]^{\frac{1}{2}}$, are given by

$$\begin{aligned}\psi_{1S}=\left[\frac{\mu_{1S}{}^3}{\pi(\lambda+1)\Gamma(2\lambda+1)}\right]^{\frac{1}{2}}(2\mu_{1S}r)^{\lambda-1}\\ \times\exp(-\mu_{1S}r)(\lambda-\mathcal{L})\chi_+{}^{\pm},\end{aligned} \tag{2.25}$$

where Eqs. (2.20) and (2.21) have been employed to replace $\chi_+{}^{\kappa}$ for $\kappa=1$ by $(4\pi)^{-\frac{1}{2}}\chi_+{}^{\pm}$. In this expression, the quantity μ_{1S} equals Ze^2m or $1/a$ where a is the Bohr radius. These wave functions have been normalized so that $\int\psi^*\psi d\mathbf{r}=1$ as can be verified directly. Normalization of more complicated bound state wave functions is most conveniently accomplished by using certain properties of the Green's function to be derived.

The Green's function, $G(E;\mathbf{r},\mathbf{r}')$ of the first-order Dirac equation, like each wave function of that equation, is easily found by applying the projection operator, (2.2), to the corresponding function $\mathcal{G}(E;\mathbf{r},\mathbf{r}')$, of the second-order equation. The latter equation, when expressed in spherical coordinates, is the inhomogeneous counterpart of Eq. (2.9),

$$\begin{aligned}[p_r{}^2-2EZe^2/r+m^2-E^2+\mathcal{L}(\mathcal{L}+1)/r^2]r\mathcal{G}(E;\mathbf{r},\mathbf{r}')r'\\ =rr'\delta(\mathbf{r}-\mathbf{r}').\end{aligned} \tag{2.26}$$

The Green's function, $\mathcal{G}(E;\mathbf{r},\mathbf{r}')$ may be expanded as a bilinear series of angular eigenfunctions and their adjoints, multiplied by coefficients, $g_{\Lambda\Lambda'}(E;r,r')$, which depend on the radial coordinates

$$\begin{aligned}\mathcal{G}(E;\mathbf{r},\mathbf{r}')=\sum_{\Lambda',\Lambda''}-(\mathrm{K}\Lambda/\kappa\lambda)g_{\Lambda',\Lambda''}(E;r,r')\\ \times\chi_{\Lambda'}(\Omega)\bar{\chi}_{\Lambda''}(\Omega').\end{aligned} \tag{2.27}$$

Taking the scalar product of Eq. (2.26) with $\bar{\chi}_\Lambda$ on the left and with $\chi_{\Lambda'}$ on the right leads to the conclusion that $g_{\Lambda\Lambda'}=g_\Lambda\delta_{\Lambda\Lambda'}$.

In dimensionless notation, in which

$$\begin{aligned}\mu^2=m^2-E^2,\qquad &\eta=Ze^2E/\mu,\\ x=2\mu r,\qquad &y_\Lambda(\eta;x,x')=2\mu rg_\Lambda(E;r,r')r',\end{aligned} \tag{2.28}$$

the function $y_\Lambda(\eta;x,x')$ satisfies the equation

$$\left[\frac{d^2}{dx^2}+\frac{\eta}{x}-\frac{1}{4}-\frac{\lambda(\lambda\pm1)}{x^2}\right]y_{\pm\lambda}(\eta;x,x')=-\delta(x-x'). \tag{2.29}$$

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[9] While this normalization is convenient in applications requiring expansions of invariant functions, the positive-definite normalization $\int d\Omega\chi^*(\Omega)\chi(\Omega)$ may also be used. In that case the factor multiplying the Kronecker delta in (2.23) would be $-\Lambda/\mathrm{K}$ instead of $-\Lambda\mathrm{K}/\lambda\kappa$.

The boundary conditions imposed on the Green's function, $\mathcal{G}$, require that each radial Green's function, y_Λ, be finite at infinity and obey appropriate regularity conditions at the origin.[10] These functions, y_Λ, may be simply obtained from the solutions of the homogeneous equation, $f_\Lambda(x)$, which fulfill the boundary condition at infinity, and the others, $h_\Lambda(x)$, which behave properly at the origin. In terms of them, as may be verified by integrating over a small region containing x', the functions, y_Λ, satisfy

$$y_\Lambda(x,x')=f_\Lambda(x_>)h_\Lambda(x_<)/[f_\Lambda h_\Lambda'-h_\Lambda f_\Lambda']. \quad (2.30)$$

In this expression, $x_<$ represents the smaller, and $x_>$ the larger, of the two arguments of y_Λ. The denominator, the Wronskian of f_Λ and h_Λ, has the same value for all x. For large x, the second derivative and constant terms dominate Eq. (2.29), and two solutions behave as $e^{\frac{1}{2}x}$ and $e^{-\frac{1}{2}x}$. The function, f_Λ, must therefore be the Whittaker function[11] $W_{\eta,\,\lambda\pm\frac{1}{2}}$. For small x, the $1/x^2$ and second derivative terms are the dominant ones in Eq. (2.29); therefore the solutions behave as $x^{\lambda+\frac{1}{2}\pm\frac{1}{2}}$ and $x^{-\lambda+\frac{1}{2}\mp\frac{1}{2}}$. Only the former is admissible. Since one of the arguments of the Green's function is confined, in discussing electron capture, to values smaller than the nuclear radius, it is sufficient to use this small-distance approximation for the interior solution, h_Λ, and to write

$$g_{\pm\lambda}(E;r,r') =\frac{\Gamma(\frac{1}{2}-\eta+\lambda\pm\frac{1}{2})}{\Gamma(2\lambda+1\pm1)}(2\mu r_<)^{\lambda-\frac{1}{2}\pm\frac{1}{2}}\frac{W_{\eta,\,\lambda\pm\frac{1}{2}}(2\mu r_>)}{r_>}. \quad (2.31)$$

In problems in which the arguments of the Green's function are unrestricted, the exact solution to Eq. (2.29) which behaves as $x^{\lambda+\frac{1}{2}\pm\frac{1}{2}}$ near the origin must be used. The resulting generalization of Eq. (2.31), involving the confluent hypergeometric function, ${}_1F_1$, is

$$\begin{aligned} g_{\pm\lambda}(E;r,r')=&[\Gamma(\tfrac{1}{2}-\eta+\lambda\pm\tfrac{1}{2})/\Gamma(2\lambda+1\pm1)](2\mu r_<)^{\lambda-\frac{1}{2}\pm\frac{1}{2}} \\ &\times\exp(-\mu r_<)\ {}_1F_1(\lambda+\tfrac{1}{2}-\eta\pm\tfrac{1}{2},\ 2\lambda+1\pm1,\ 2\mu r_<) \\ &\times W_{\eta,\,\lambda\pm\frac{1}{2}}(2\mu r_>)/r_>. \quad (2.32)\end{aligned}$$

The Green's function, $G(E)$, which satisfies the first-order equation

$$[\boldsymbol{\gamma}\cdot\mathbf{p}-\gamma_0(E+Ze^2/r)+m]G(E;\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'),$$

is related to the second-order Green's function by

$$G(E)=\mathcal{P}_+2m\mathcal{G}(E). \quad (2.33)$$

The fact that the first-order Green's function, $G(E)$, is infinite when E is in the discrete spectrum of the Hamiltonian and that the residues at these points are bilinear combinations of the corresponding eigenfunctions may be used to deduce the normalization constants for both the first- and second-order wave functions. This procedure is carried out in Appendix B.

3. RADIATIVE *K* CAPTURE

In view of the recently discovered asymmetry of the beta-decay interaction under spatial inversion, it is desirable to examine the polarization as well as the intensity of the radiation predicted by a fully relativistic theory of radiative K capture. This parity asymmetry is taken into account by replacing the capture Hamiltonian considered in (I) by a more general one, for which the matrix element, N_{fi}, takes the form[12]

$$N_{fi}=\sum_{(s),\lambda}\langle T_\lambda^{(s)}\rangle_{fi}[C^{(s)}+i\gamma_5C^{(s)\prime}]T_\lambda^{(s)}. \quad (3.1)$$

In this matrix element, the primed constants refer to the parity nonconserving interactions; $i\gamma_5=i\gamma_0\gamma_1\gamma_2\gamma_3$ is Hermitian; the index (s) takes on the values S, V, T, A, and P, for the scalar, vector, tensor, axial vector, and pseudoscalar parts of the interaction; and the quantity $\langle T_\lambda^{(s)}\rangle_{fi}$ represents the corresponding nuclear matrix element.

An additional, but minor, modification of the procedure discussed in (I) is caused by the singularity at the origin of relativistic Coulomb S-state wave functions. Because of this singularity, it is necessary to consider the fact that the capture interaction occurs over a finite nuclear volume. The details of the nuclear density distribution within this volume, however, do not significantly affect a quantity like the ratio of nonradiative to radiative capture. It is therefore sufficient, in determining this ratio, to use proton and neutron wave functions which are constant inside the nuclear radius, r_N, and to employ the Dirac electron wave functions appropriate to a point charge. If the variation in the neutrino wave function inside the nucleus is also neglected, this procedure is equivalent to averaging the wave function of the electron undergoing capture over the small nuclear volume. Apart from these modifications, the required matrix element has the same form as the one derived in Eq. (2.30) of (I),

$$\begin{aligned} M_i(\mathbf{k},\mathbf{p})=&e(2\pi/k)^{\frac{1}{2}}(4\pi r_N{}^3/3)^{-1} \\ &\times\int_{r'<r_N}d\mathbf{r}\int d\mathbf{r}(\bar{\phi}_\mathbf{p}(0)N_{fi}\mathcal{G}(E_i-k;\mathbf{r}',\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}} \\ &\times[2\mathbf{e}\cdot\mathbf{p}+ie_\mu\sigma_{\mu\nu}k_\nu]\psi_i(\mathbf{r})). \quad (3.2)\end{aligned}$$

The function $\phi_\mathbf{p}(0)$ is the neutrino spinor evaluated at the origin, k_μ is the photon four-vector, and e_μ is its polarization vector. It is convenient to employ the two complex polarization vectors which describe circularly polarized gamma radiation, that is to introduce vectors, $\mathbf{e}_+$ and $\mathbf{e}_-$, which satisfy

$$\mathbf{e}_\pm\times(\mathbf{k}/k)=\pm i\mathbf{e}_\pm,\quad \mathbf{e}_-\times\mathbf{e}_+=i(\mathbf{k}/k),\quad \mathbf{e}_\pm{}^*\cdot\mathbf{e}_\pm=1. \quad (3.3)$$

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[10] The regularity conditions on $\mathcal{G}$ near the origin are the same as those imposed on the wave function, and discussed in Appendix A.

[11] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis* (The Macmillan Company, New York, 1943), p. 337.

[12] These constants $C^{(s)}$, agree with those used by Lee and Yang, reference 3, in discussing beta decay.

These vectors can be chosen as $\mathbf{e}_{\pm}=(\mathbf{e}_x\pm i\mathbf{e}_y)/\sqrt{2}$ in a right-handed coordinate system in which the z axis lies along $\mathbf{k}$. Radiation associated with the vector $\mathbf{e}_+$ will be designated as right handed and that associated with $\mathbf{e}_-$ will be called left handed.[13] By means of the identities

$$ie_\mu\sigma_{\mu\nu}k_\nu=i\boldsymbol{\sigma}\cdot(\mathbf{e}\times\mathbf{k})-(\mathbf{e}\cdot\boldsymbol{\alpha})k,$$

and

$$\alpha_1=i\gamma_5\sigma_{23}=i\gamma_5\sigma_1, \tag{3.4}$$

the term $ie_{\pm\mu}\sigma_{\mu\nu}k_\nu$ occurring in Eq. (3.2) may be reduced to

$$\begin{aligned} ie_{\pm\mu}\sigma_{\mu\nu}k_\nu &= -[(\boldsymbol{\alpha}\cdot\mathbf{e}_\pm)\pm(\boldsymbol{\sigma}\cdot\mathbf{e}_\pm)]k \\ &= \mp(1\pm i\gamma_5)(\boldsymbol{\sigma}\cdot\mathbf{e}_\pm)k. \end{aligned} \tag{3.5}$$

As has been indicated in (I), this term of Eq. (3.2) provides nearly all the intensity in capture from S states while the $2\mathbf{e}\cdot\mathbf{p}$ term is principally responsible for the photons emitted in capture from P states. The $2\mathbf{e}\cdot\mathbf{p}$ term will be shown to lead to radiation in S-state capture only because the relativistic S states contain a small P-state admixture due to spin-orbit coupling. Thus, when the matrix element (3.2) is calculated for right and left polarized radiation, the important term, (3.5), contains as a factor, the projection operator, $1\pm i\gamma_5$, which selects neutrinos with a given sense of spin relative to their direction of motion. It may be expected, and is easily verified, that a beta interaction which produces neutrinos with only one of these two spin directions, or equivalently, a two-component neutrino theory, will produce polarized radiation in consequence of this asymmetry. In particular it will be shown that there would be complete polarization except for Coulomb effects if the interaction were entirely scalar, tensor, and pseudoscalar, or entirely vector and axial vector.

In order to compute the matrix element (3.2) in the presence of the Coulomb field, it is only necessary to use $\mathcal{G}(E_i-k;\mathbf{r}',\mathbf{r})$ for $r'<r_N$. In this region the simplified form given in Eq. (2.31) may be employed in the expansion of $\mathcal{G}$ in terms of its angular eigenfunctions. Since $\mathbf{r}'$ is confined to the small nuclear volume, all terms in the summation which vanish when $\mathbf{r}'$ goes to zero, may be neglected. The only terms which survive are those for which $\Lambda=-[1-(Ze^2)^2]^{\frac{1}{2}}$. When the angular eigenfunctions, $\chi_{-\lambda}{}^{K}(\Omega)$ and $\chi_{-\lambda}{}^{K}(\Omega')$, which multiply this radial function are introduced, and the summation over $K=\pm1$ carried out, the Green's function becomes

$$\begin{aligned} \mathcal{G}(E_i-k;\mathbf{r},\mathbf{r}') &= \frac{\Gamma(\lambda-\eta)}{\Gamma(2\lambda)}\frac{(2\mu r')^{\lambda-1}}{4\pi}\left(1+\frac{iZe^2\alpha_{r'}}{\lambda+1}\right) \\ &\quad\times\left(1+\frac{iZe^2\alpha_r}{\lambda+1}\right)\frac{\lambda+1}{2\lambda}\frac{W_{\eta,\lambda-\frac{1}{2}}(2\mu r)}{r}, \end{aligned} \tag{3.6}$$

where

$$\mu^2=m^2-(E_i-k)^2 \quad\text{and}\quad \eta=Ze^2(E_i-k)/\mu.$$

In this formula and hereafter λ is understood to have the value appropriate to a state for which $j=\frac{1}{2}$,

$$\lambda=[1-(Ze^2)^2]^{\frac{1}{2}}.$$

When the integral representation,

$$\Gamma(\lambda-\eta)W_{\eta,\lambda-\frac{1}{2}}(2\mu r) = (2\mu r)^{1-\lambda}e^{-\mu r}\int_0^\infty t^{-\eta+\lambda-1}(2\mu r+t)^{\eta+\lambda-1}e^{-t}dt,$$

is introduced, and the first argument, $\mathbf{r}'$, of $\mathcal{G}(E_i-k;\mathbf{r}',\mathbf{r})$ averaged over the nuclear volume, Eq. (3.6) becomes

$$\begin{aligned} \frac{3}{4\pi r_N{}^3}\int_{r'<r_N} d\mathbf{r}'\,\mathcal{G}(E_i-k;\mathbf{r},\mathbf{r}') &= \frac{3}{\lambda+2}\left(\frac{r_N}{r}\right)^{\lambda-1}\frac{\lambda+1}{\Gamma(2\lambda+1)}\frac{e^{-\mu r}}{4\pi r}\left(1+\frac{iZe^2\alpha_r}{\lambda+1}\right) \\ &\quad\times\int_0^\infty t^{-\eta+\lambda-1}(2\mu r+t)^{\eta+\lambda-1}e^{-t}dt. \end{aligned} \tag{3.7}$$

(When averaged over directions of $\mathbf{r}'$ the Dirac matrix, $\alpha_{r'}$, vanishes.)

Some properties of this Green's function warrant discussion. Outside of a region whose range may be characterized by the distance, $\mu^{-1}=[\mu_i{}^2+2kE_i-k^2]^{-\frac{1}{2}}$, the Green's function has a negligible magnitude. This distance varies with the photon energy. From a value of the Bohr radius for zero-energy photons, it decreases rapidly with increasing photon energy to the order of magnitude of the electron Compton wavelength. It remains this size through photon energies $k\sim m$ and then increases, becoming infinite at a photon energy equal to the total energy of the initial electron plus the rest energy of a positron. When the photon spectrum extends above this energy, real positron emission, as well as K capture, can occur. The variation in the range just described can be qualitatively explained. According to the uncertainty principle, a process which involves a large fluctuation in energy must be a short-lived one and conversely. The virtual emission of a low-energy gamma ray by an electron can almost conserve energy; so can the emission of a gamma ray having an energy of the order of, or greater than, two electron masses, provided that the gamma ray is due to the annihilation of the electron and a virtual positron emitted by the nucleus. Therefore, if the frequency of the radiation lies in either of these regions, the intermediate state can persist long enough for the quantum to be emitted far from the nucleus. If not, the radiation and nuclear process must take place at positions within a distance of the order of the electron Compton wavelength of each other.

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[13] These designations of circular polarization are suggested, in contrast to the older ones of classical theory, by the angular momenta carried by the quanta.

The likelihood of capture of any virtual electron is enhanced markedly by the attractive nature of the Coulomb field. The Green's function exhibits this property by increasing in magnitude with increasing charge. This dependence of magnitude on charge occurs primarily through the parameter η of the parametric integral in Eq. (3.7). The enhancement is most extreme for initial states, i, lying above the $1S$ state. If the photon energy equals E_i-E_j, where the state j has lower energy than i, energy can be conserved in the intermediate state and the Green's function then becomes infinite. This behavior of the Green's function occurs, for example, at the resonance of the $2P$ spectrum discussed in (I).

A third property of the Green's function which deserves mention is the fact that it sums over all states from which direct capture can take place, both $S_{\frac{1}{2}}$ and $P_{\frac{1}{2}}$ states. An advantage of working with the second-order equation is that both are automatically included in the Green's function for which $\Lambda=-[1-(Ze^2)^2]^{\frac{1}{2}}$.

The replacement $s=t/2\mu r$ in the integral in Eq. (3.7) and the introduction of this equation into the capture matrix element, (3.2), yields

$$M_{1S}(\mathbf{k},\mathbf{p})=e\left(\frac{2\pi}{k}\right)^{\frac{1}{2}}\langle\psi\rangle_N\Bigg(\bar{\phi}_{\mathrm{p}}(0),\,N_{fi}\frac{\lambda+1}{\Gamma(2\lambda+1)}$$
$$\times\int rdr\left(\frac{r}{a}\right)^{1-\lambda}e^{-\mu r}\int\frac{d\Omega}{4\pi}\left(1+\frac{iZe^2\alpha_r}{\lambda+1}\right)(2\mu r)^{2\lambda-1}$$
$$\times\int ds s^{-\eta+\lambda-1}(1+s)^{\eta+\lambda-1}e^{-2\mu rs}(2\mathbf{e}\cdot\mathbf{p}+ie_\mu\sigma_{\mu\nu}k_\nu)e^{-i\mathbf{k}\cdot\mathbf{r}}$$
$$\times\left(1+\frac{iZe^2\alpha_r}{\lambda+1}\right)\left(\frac{r}{a}\right)^{\lambda-1}e^{-r/a}\chi_+{}^{\pm}\Bigg).\tag{3.8}$$

In this equation, and henceforth, the parameters, μ and η are understood to have values appropriate to capture from the $1S$ state,

$$\mu^2=m^2-(E_{1S}-k)^2,\quad \eta=Ze^2(E_{1S}-k)/\mu.$$

The abbreviation,

$$\langle\psi\rangle_N=\left[\frac{\lambda+1}{\pi\Gamma(2\lambda+1)a^3}\right]^{\frac{1}{2}}\frac{3}{\lambda+2}(2r_N/a)^{\lambda-1},\tag{3.9}$$

represents the average value of the $1S$ electron wave function within the nuclear volume.

It is convenient to simplify first the dependence on angles and Dirac matrices of the expression X which is defined by

$$X=\int\frac{d\Omega}{4\pi}\left(1+\frac{iZe^2\alpha_r}{\lambda+1}\right)(2\mathbf{e}\cdot\mathbf{p}+ie_\mu\sigma_{\mu\nu}k_\nu)e^{-i\mathbf{k}\cdot\mathbf{r}}$$
$$\times\left(1+\frac{iZe^2\alpha_r}{\lambda+1}\right)\chi_+{}^{\pm}.$$

Because X operates on a spherically symmetric function, the differential operator, $2\mathbf{e}\cdot\mathbf{p}$, may be averaged by itself. With the aid of the anticommutation relation $\{\boldsymbol{\alpha}\cdot\mathbf{k},e_\mu\sigma_{\mu\nu}k_\nu\}=0$, X may be rewritten as

$$X=\int\frac{d\Omega}{4\pi}\Bigg[ie_\mu\sigma_{\mu\nu}k_\nu-\left(1+\frac{iZe^2\alpha_r}{\lambda+1}\right)^2 2ie_r\frac{\partial}{\partial r}$$
$$+\frac{2Ze^2}{(\lambda+1)r}\left(1+\frac{iZe^2\alpha_r}{\lambda+1}\right)(\mathbf{e}\cdot\boldsymbol{\alpha}-e_r\alpha_r)$$
$$-\left(\frac{Ze^2}{\lambda+1}\right)^2 i\alpha_r e_\mu\sigma_{\mu\nu}k_\nu\alpha_r\Bigg]e^{-i\mathbf{k}\cdot\mathbf{r}}\chi_+{}^{\pm},$$

where e_r is the component of $\mathbf{e}$ along $\mathbf{r}$. The angular integrations may be performed with the aid of the relations

$$\int\frac{d\Omega}{4\pi}e^{-i\mathbf{k}\cdot\mathbf{r}}=j_0(kr),\quad \int\frac{d\Omega}{4\pi}e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{r}_\mu=-ij_1(kr)\hat{k}_\mu,$$
$$\int\frac{d\Omega}{4\pi}e^{-i\mathbf{k}\cdot\mathbf{r}}(\hat{r}_\mu\hat{r}_\nu-\tfrac{1}{3}\delta_{\mu\nu})=-(\hat{k}_\mu\hat{k}_\nu-\tfrac{1}{3}\delta_{\mu\nu})j_2(kr),$$

where $\hat{r}=\mathbf{r}/r$, $\hat{k}=\mathbf{k}/k$, and the functions $j_l(kr)$ are spherical Bessel functions. After the Dirac matrices have been simplified with frequent use of the anticommutation relation $\{\mathbf{e}\cdot\boldsymbol{\alpha},\,\mathbf{k}\cdot\boldsymbol{\alpha}\}=0$, X takes the form

$$X=\Bigg\{j_0(kr)\Bigg[\left(1+\frac{1}{3}\left(\frac{Ze^2}{\lambda+1}\right)^2\right)(i\boldsymbol{\sigma}\cdot\mathbf{e}\times\mathbf{k}-\mathbf{e}\cdot\boldsymbol{\alpha}k)$$
$$+\frac{4}{3}\left(\frac{Ze^2}{\lambda+1}\right)\mathbf{e}\cdot\boldsymbol{\alpha}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\Bigg]$$
$$-\frac{j_1(kr)}{kr}2\left(\frac{Ze^2}{\lambda+1}\right)^2 i\boldsymbol{\sigma}\cdot\mathbf{e}\times\mathbf{k}$$
$$+j_2(kr)\Bigg[\frac{4}{3}\left(\frac{Ze^2}{\lambda+1}\right)\mathbf{e}\cdot\boldsymbol{\alpha}\left(\frac{\partial}{\partial r}-\frac{1}{2r}\right)$$
$$-\frac{2}{3}\left(\frac{Ze^2}{\lambda+1}\right)^2(i\boldsymbol{\sigma}\cdot\mathbf{e}\times\mathbf{k}-\mathbf{e}\cdot\boldsymbol{\alpha}k)\Bigg]\chi_+{}^{\pm}.\tag{3.10}$$

For purposes of comparison and calculation it is convenient to introduce the polarization vectors $\mathbf{e}_+$ and $\mathbf{e}_-$ explicitly and to write the matrix element obtained by substituting Eqs. (3.4) and (3.10) into Eq. (3.8), as

$$M_{1S}(\mathbf{k},\mathbf{p})=(e/2m)(2\pi/k^3)^{\frac{1}{2}}\langle\psi\rangle_N(\bar{\phi}_{\mathrm{p}}(0)N_{fi}$$
$$\times[A_{1S}i\boldsymbol{\sigma}\cdot\mathbf{e}\times\mathbf{k}-B_{1S}\mathbf{e}\cdot\boldsymbol{\alpha}k]\chi_+{}^{\pm})\tag{3.11a}$$
$$=-(e/2m)(2\pi/k)^{\frac{1}{2}}\langle\psi\rangle_N(\bar{\phi}_{\mathrm{p}}(0)N_{fi}$$
$$\times[i\gamma_5B_{1S}\pm A_{1S}](\boldsymbol{\sigma}\cdot\mathbf{e}_\pm)\chi_+{}^{\pm}),\tag{3.11b}$$

where

$$A_{1S}=\frac{\lambda+1}{\Gamma(2\lambda+1)}\frac{mk}{\mu}\int dr\int ds\biggl\{j_0(kr)\biggl[1+\frac{1}{3}\biggl(\frac{Z\alpha}{\lambda+1}\biggr)^2\biggr]$$

$$-\frac{j_1(kr)}{kr}2\biggl(\frac{Z\alpha}{\lambda+1}\biggr)^2-j_2(kr)\frac{2}{3}\biggl(\frac{Z\alpha}{\lambda+1}\biggr)^2\biggr\}$$

$$\times s^{-\eta+\lambda-1}(1+s)^{\eta+\lambda-1}(2\mu r)^{2\lambda}e^{-\mu r(2s+1)}e^{-r/a}, \quad (3.12\text{a})$$

and

$$B_{1S}=\frac{\lambda+1}{\Gamma(2\lambda+1)}\frac{mk}{\mu}\int dr\int ds\biggl\{j_0(kr)$$

$$\times\biggl[1-\frac{4}{3}\biggl(\frac{Z\alpha}{\lambda+1}\biggr)\biggl(\frac{\lambda}{kr}-\frac{1}{ka}\biggr)+\frac{1}{3}\biggl(\frac{Z\alpha}{\lambda+1}\biggr)^2\biggr]$$

$$+j_2(kr)\biggl[\frac{4}{3}\biggl(\frac{Z\alpha}{\lambda+1}\biggr)\biggl(\frac{1}{ka}+\frac{3-2\lambda}{2kr}\biggr)-\frac{2}{3}\biggl(\frac{Z\alpha}{\lambda+1}\biggr)^2\biggr]\biggr\}$$

$$\times s^{-\eta+\lambda-1}(1+s)^{\eta+\lambda-1}(2\mu r)^{2\lambda}e^{-\mu r(2s+1)}e^{-r/a}. \quad (3.12\text{b})$$

The fine-structure constant, $\alpha=e^2=1/137$, has been introduced in these expressions since the Dirac matrix $\boldsymbol{\alpha}$ no longer appears.

All the terms which A_{1S} and B_{1S} do not have in common may be traced to the term $2\mathbf{e}\cdot\mathbf{p}$ in Eq. (3.2). The terms which occur in both and arise from the $e_\mu\sigma_{\mu\nu}k_\nu$ term in the equation are the ones which would give rise to completely polarized radiation in a two-component neutrino theory with a pure S–T–P or A–V interaction.

The total probability $w_{\pm 1S}(k)$ for the capture of a $1S$ electron with the emission of a photon polarized in the $\mathbf{e}_\pm$ sense, is determined by squaring the matrix element $M_{1S}(\mathbf{p},\mathbf{k})$, summing over electron spins, neutrino spins and momenta, photon momenta, and final states of the nucleus. For unaligned nuclei, the calculation yields, in analogy with Eqs. (2.33) and (4.5) of I,

$$w_{1S}=\sum_{\mathbf{e}_\pm}\int dk\, w_{\pm 1S}(k),$$

$$w_{\pm 1S}(k)=\alpha(2m\pi)^{-2}\langle\psi\rangle_N{}^2k[(k_{\max})_{1S}-k]^2$$

$$\times\sum_f \tfrac{1}{4}\,\mathrm{Tr}\{(N_{if}{}^\dagger N_{fi})[(A_{1S}\pm i\gamma_5B_{1S})^2$$

$$+\beta(A_{1S}{}^2-B_{1S}{}^2)]\}, \quad (3.13)$$

where $(k_{\max})_{1S}=E_{1S}-\Delta E$.

For allowed transitions, a nonrelativistic reduction of the nuclear matrix elements may be used in evaluating the trace in Eq. (3.13). In terms of the coupling constants introduced in Eq. (3.1) and the nonrelativistic nuclear matrix elements, $\langle 1\rangle_{fi}$ and $\langle\boldsymbol{\sigma}\rangle_{fi}$, the expressions for the differential circularly polarized photon spectra are:

$$w_{\pm 1S}(k)=\alpha(2\pi m)^{-2}\langle\psi\rangle_N{}^2k((k_{\max})_{1S}-k)^2\textstyle\sum_f\{(A_{1S}{}^2$$

$$+B_{1S}{}^2)[|\langle 1\rangle_{fi}|^2(|C^S|^2+|C^{S'}|^2+|C^V|^2+|C^{V'}|^2)$$

$$+|\langle\boldsymbol{\sigma}\rangle_{fi}|^2(|C^T|^2+|C^{T'}|^2+|C^A|^2+|C^{A'}|^2)]$$

$$\pm 2A_{1S}B_{1S}[|\langle 1\rangle_{fi}|^2 2\,\mathrm{Re}(C^{S*}C^{S'}-C^{V*}C^{V'})$$

$$+|\langle\boldsymbol{\sigma}\rangle_{fi}|^2 2\,\mathrm{Re}(C^{T*}C^{T'}-C^{A*}C^{A'})]$$

$$+(A_{1S}{}^2-B_{1S}{}^2)[|\langle 1\rangle_{fi}|^2 2\,\mathrm{Re}(C^{S*}C^V+C^{S'*}C^{V'})$$

$$+|\langle\boldsymbol{\sigma}\rangle_{fi}|^2 2\,\mathrm{Re}(C^{T*}C^A+C^{T'*}C^{A'})]\}. \quad (3.14\text{a})$$

The term containing $A_{1S}{}^2-B_{1S}{}^2$ is the Fierz interference term and appears to be absent from β-decay spectra. Its contribution in Eq. (3.14a) would presumably be negligible in any case since $A_{1S}{}^2-B_{1S}{}^2$ is small $[O(Z\alpha)]$ at most energies. For a two-component neutrino theory, each pair of coupling constants is effectively restricted to satisfy $C^{(S)\prime}=C^{(S)}$. If both these simplifications are introduced, the spectra described by Eq. (3.14a) can also be written

$$w_{\pm 1S}(k)=(\alpha/2\pi^2m^2)\langle\psi\rangle_N{}^2k((k_{\max})_{1S}-k)^2$$

$$\times\textstyle\sum_f\{|\langle 1\rangle_{fi}|^2[(A_{1S}\pm B_{1S})^2|C^S|^2$$

$$+(A_{1S}\mp B_{1S})^2|C^V|^2]+|\langle\boldsymbol{\sigma}\rangle_{fi}|^2[(A_{1S}\pm B_{1S})^2|C^T|^2$$

$$+(A_{1S}\mp B_{1S})^2|C^A|^2]\}, \quad (3.14\text{b})$$

where either C^S or C^V, and either C^T or C^A vanishes.

The total intensity, obtained by summing Eq. (3.14a) over polarizations, is

$$w_{1S}=(\alpha/\pi^2m^2)\langle\psi\rangle_N{}^2\sum_f\biggl\{[|\langle 1\rangle_{fi}|^2(|C^S|^2+|C^{S'}|^2$$

$$+|C^V|^2+|C^{V'}|^2)+|\langle\boldsymbol{\sigma}\rangle_{fi}|^2(|C^T|^2+|C^{T'}|^2$$

$$+|C^A|^2+|C^{A'}|^2)]\int dk\, k((k_{\max})_{1S}-k)^2R_{1S}(k)$$

$$+[|\langle 1\rangle_{fi}|^2 2\,\mathrm{Re}(C^{S*}C^V+C^{S*\prime}C^{V'})$$

$$+|\langle\boldsymbol{\sigma}\rangle_{fi}|^2 2\,\mathrm{Re}(C^{T*}C^A+C^{T*\prime}C^{A'})]$$

$$\times\int dk\, k((k_{\max})_{1S}-k)^2F_{1S}(k)\biggr\}, \quad (3.15\text{a})$$

where the abbreviations,

$$R_{1S}(k)=\tfrac{1}{2}(A_{1S}{}^2+B_{1S}{}^2) \quad (3.16\text{a})$$

and

$$F_{1S}(k)=\tfrac{1}{2}(A_{1S}{}^2-B_{1S}{}^2) \quad (3.16\text{b})$$

have been introduced. For the two-component theory with no Fierz interference, Eq. (3.15a) reduces to

$$w_{1S}=(2\alpha/\pi^2m^2)\langle\psi\rangle_N{}^2\sum_f[|\langle 1\rangle_{fi}|^2(|C^S|^2+|C^V|^2)$$

$$+|\langle\boldsymbol{\sigma}\rangle_{fi}|^2(|C^T|^2+|C^A|^2)]$$

$$\times\int dk\, k((k_{\max})_{1S}-k)^2R_{1S}(k). \quad (3.15\text{b})$$

The equations, (3.15), are similar in form to the ones obtained as Eq. (9.15) of the earlier calculation, I.

With the approximations employed there, A_{1S} reduced to unity so that $R_{1S}(k)$ was represented in Eq. (9.16) of I by $\frac{1}{2}(1+B_{1S}^2)$. Since the Fierz interference term was assumed to vanish in that discussion the term involving $F_{1S}(k)$ was omitted.

In order to compute the fraction of capture transitions in which radiation is emitted, it is necessary also to determine the probability of radiationless capture from the $1S$ state. This may be accomplished in the standard manner, that is, by squaring the amplitude, summing over initial electron spins, final neutrino spins and directions, and final states of the nucleus. With the $1S$ wave function of Eq. (2.25), this summation is immediately carried out and the probability

$$w_K=(4\pi)^{-1}\langle\psi\rangle_N^2(k_{\max})_{1S}^2\,\textstyle\sum_f \mathrm{Tr}(N_{if}{}^\dagger N_{fi}(1+\beta)) \quad (3.17)$$

obtained. By using the nonrelativistic approximations appropriate to allowed capture, and the coupling constants of Eq. (3.1), this expression may be written in the form

$$\begin{aligned} w_K=(1/\pi)\langle\psi\rangle_N^2(k_{\max})_{1S}^2 \\ \times\textstyle\sum_f\{|\langle1\rangle_{fi}|^2(|C^S+C^V|^2+|C^{S'}+C^{V'}|^2) \\ +|\langle\boldsymbol{\sigma}\rangle_{fi}|^2(|C^T+C^A|^2+|C^{T'}+C^{A'}|^2)\}. \quad (3.18\text{a}) \end{aligned}$$

In the two-component neutrino theory with no Fierz interference Eq. (3.18a) becomes

$$\begin{aligned} w_K=(2/\pi)\langle\psi\rangle_N^2(k_{\max})_{1S}^2 \\ \times\textstyle\sum_f\{|\langle1\rangle_{fi}|^2(|C^S|^2+|C^V|^2) \\ +|\langle\boldsymbol{\sigma}\rangle_{fi}|^2(|C^T|^2+|C^A|^2)\}. \quad (3.18\text{b}) \end{aligned}$$

A comparison with the expression for the P-state spectrum obtained in Eq. (8.6) of I shows that the trace appearing in the nonrelativistic treatment of the P-state term is the same as the one appearing in the probability of ordinary capture, Eq. (3.17), but differs from the one in Eq. (3.13). The relative magnitudes of the various spectra associated with allowed capture will therefore depend on nuclear properties unless the terms involving $\mathrm{Tr}(N_{if}{}^\dagger N_{fi}\beta)$, that is, the Fierz interference terms, vanish. In the remainder of this paper that assumption, which appears to be supported by experimental evidence, will be made. Should it prove not to be rigorously true, the more general formulas [(3.14a), (3.15a), and (3.18a)] would have to be employed.

4. EVALUATION OF RELATIVISTIC COULOMB CORRECTIONS

In this section, approximate forms will be obtained for the quantities, A_{1S} and B_{1S}, which describe the intensity and polarization of the radiative spectra. According to the approximations made in the previous paper, the integral, A_{1S}, which is associated with spin-flip radiative capture, is equal to one in the nonrelativistic region, except for terms of order $(Z\alpha)^2$ and $k/2m$. In the relativistic region, except for terms of order $Z\alpha$ not calculated there, it was shown also to have unit value. The factor B_{1S}, on the other hand, has already been shown in Sec. 9 of (I) to vary considerably with energy. At high energies, B_{1S}, like A_{1S}, differs from unity by terms of order $Z\alpha$. In the binding-energy region, however, B_{1S} decreases rapidly, tending to zero with vanishing photon energy. The decrease has been shown to result from a cancellation between terms representing capture through intermediate P states and through intermediate positron states. Again, the errors in the low-energy region were of order $(Z\alpha)^2$ and $k/2m$.

The calculations which will now be performed confirm the calculation of (I) in the low-energy and medium-energy region. However, the terms of order $Z\alpha$ omitted at high energies are not small, and for heavy elements, higher order corrections are necessary. For argon, in particular, the corrections amount to forty percent at high energies although $Z\alpha$ is only equal to 0.13.

The exact calculation of the integrals A_{1S} and B_{1S} in closed form for arbitrary charge and energy is possible only in a restricted sense. If the functions $j_l(kr)$ are written in the form $\mathrm{Im}(\sum \tau_n r^{-n}e^{-ikr})$, where each τ_n is either real or imaginary, then the integration over r may be performed and leads to a series of integrations over the variable s, each of which has the form

$$I_\nu=\Gamma(2\lambda-\nu)\int_0^\infty \frac{s^{-\eta+\lambda-1}(1+s)^{\eta+\lambda-1}ds}{[\mu+\mu_{1S}+2\mu s+ik]^{2\lambda-\nu}}, \quad (4.1)$$

where $\nu=0$, 1, 2, or 3. When ν is not equal to zero, these integrals diverge, but the imaginary parts of those linear combinations of integrals occurring in A_{1S} and B_{1S} have finite values. The integral for $\nu=0$ has the value

$$I_0=\frac{\Gamma(2\lambda)}{(2\mu)^{2\lambda}(\lambda-\eta)}\,{}_2F_1(2\lambda,\,1,\,\lambda+1-\eta;\,(\mu-\mu_{1S}-ik)/2\mu).$$

The other integrations may be expressed by series

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{(n+\nu)!},$$

where $(a)_n=a(a+1)\cdots(a+n-1)$. For elements which are not too heavy it is possible to approximate the functions without employing these rather slowly convergent series.

In the integrals I_ν, the terms of order $(Z\alpha)^2$ relative to unity, may be neglected, that is, λ may be set equal to one. With that approximation, the integrals may be transformed to ones of the type

$$K_q(\rho)=\rho\int_0^1\frac{y^{-\eta}dy}{(\rho y+1)^{q+1}}, \quad (4.2)$$

by means of the substitution, $s=y/(1-y)$. Methods for deriving rapidly convergent series for integrals of this form have already been discussed in (I), Sec. 8.

For the parts of A_{1S} and B_{1S} involving $j_0(kr)$, only the integrals for which $q=0$ and $q=1$ are required. By introducing the expansions of these integrals, developed in (I), the formulas

$$A_{1S}=\mathrm{Im}[2m/(\mu+\mu_{1S}-ik)+(\eta\mu/km)(E_{1S}+i\mu_{1S})\mathcal{Q}], \quad (4.3a)$$

$$B_{1S}=A_{1S}(1+\tfrac{2}{3}\mu_{1S}^2/km)-\tfrac{2}{3}(\mu_{1S}/k)\,\mathrm{Im}\,\mathcal{Q}, \quad (4.3b)$$

$$\mathcal{Q}=2\left[\ln\left(\frac{\mu+\mu_{1S}+ik}{2\mu}\right)+\eta\sum_{n=1}^{\infty}\frac{1}{n(n-\eta)}\left(\frac{\mu_{1S}+ik-\mu}{\mu_{1S}+ik+\mu}\right)^n\right] \quad (4.4)$$

are obtained. In the region where these series must be used for the terms of Eqs. (3.12) which contain $j_0(kr)$, the terms containing $j_2(kr)$, which would vanish if retardation were neglected, are negligible. It is possible to put a bound on them by replacing $j_2(kr)$ by the first term in its power-series expansion, $(1/15)(kr)^2$. The resulting integral is found to contribute no more than a few percent of the intensity in argon at photon energies below m.

For photons with energies comparable with the rest energy of the electron, intermediate states in which the electron is bound become less important and an expansion of the Coulomb Green's function in powers of $Z\alpha$ becomes feasible. When the computation is carried out with the initial wave function also expanded to first order in $Z\alpha$ at small distances from the nucleus, the result is:

$$A_{1S}=1-Z\alpha\{(\mu/k)+2[1-(m/k)]\tan^{-1}(k/\mu)\}, \quad (4.5a)$$

$$B_{1S}=1-Z\alpha\{(\mu/k)[1+(m/k)]+2[1-(m/k)^2]\tan^{-1}(k/\mu)\}. \quad (4.5b)$$

[In these expressions, the terms of Eqs. (3.12) which involve $j_2(kr)$ have been retained, since for photon energies larger than m, they become significant.] The values obtained from Eqs. (4.3) used at low energies and from Eqs. (4.5) have been found to agree in light elements for photon energies in the neighborhood of half the electron rest energy; the spectrum has therefore been determined by the latter formulas at higher energies. The values of A_{1S}, B_{1S}, and R_{1S}, for argon, are plotted against photon energy in Fig. 2.

For three particular photon energies, better approximations to A_{1S} and B_{1S} for moderately heavy elements can be obtained directly from Eqs. (3.12). The first of these is vanishing photon energy ($k=0$); the second, a photon energy equal to the total energy of a $1S$ electron ($\eta=0$ and $k=\lambda m$); and the third, a photon energy equal to that liberated when a $1S$ electron is annihilated by a stationary positron [$\mu=0$ and $k=(1+\lambda)m$]. In each case, simplifications occur which make it possible to determine more accurate expressions conveniently.

In the neighborhood of $k=0$, the integrals, A_{1S} and B_{1S} may be expanded in a power series in the photon energy. The integrations over r may then be performed, and the resulting integrations over s may be reduced to beta functions. The first terms in these series, exact to all orders of $Z\alpha$, are

$$A_{1S}=\tfrac{1}{3}(2\lambda+1)[1-(k/m)], \quad (4.6a)$$

$$B_{1S}=0+O(k/Z\alpha m). \quad (4.6b)$$

When $\eta=0$, the integrals in Eqs. (3.15) also simplify and it is possible to evaluate them to second order in the charge. The result of this calculation is

$$A_{1S}=1-Z\alpha+\tfrac{1}{4}\pi(Z\alpha)^2, \quad (4.7a)$$

$$B_{1S}=1-2Z\alpha+(4-\tfrac{1}{2}\pi)(Z\alpha)^2. \quad (4.7b)$$

The simplifications which occur when $\mu=0$ and $\eta=-\infty$ are somewhat less extensive. In this limit, the Whittaker function occurring in the Green's function becomes a modified Bessel function of the second kind. The integrals, A_{1S} and B_{1S}, are then to second order in $Z\alpha$,

$$A_{1S}=1-\tfrac{1}{2}\pi Z\alpha+2(Z\alpha)^2, \quad (4.8a)$$

$$B_{1S}=1-\tfrac{3}{4}\pi Z\alpha+(9/2)(Z\alpha)^2. \quad (4.8b)$$

Although the first-order effect of the Coulomb field is generally quite large, the fairly small magnitude of the second-order terms in Eqs. (4.7) and (4.8) suggests that the first-order results, (4.5), are fairly reliable, at least for the lighter elements.

To complete the relativistic treatment of radiative capture, it would be necessary to determine the spectra of all shells as accurately as the spectrum of the K electrons. For the other most important states, the $2P$ states, which contribute independently the majority of the low-energy photons, relativistic corrections would, however, presumably be less important. It is perhaps worth noting that these calculations need not be performed in order to make careful comparison between the $1S$ spectrum and its predicted form. The spectra associated with capture of electrons from different shells can be separated experimentally by means of coincidence techniques. By analyzing only those hard

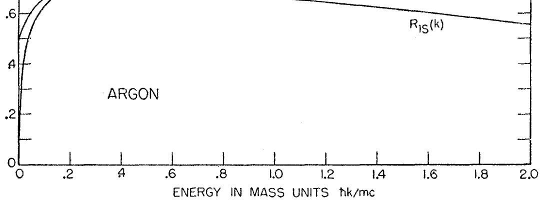

FIG. 2. The values of the amplitude functions $A_{1S}(k)$ and $B_{1S}(k)$ defined by (3.12a,b) for argon-37 ($Z=18$). The relativistic correction factor $R_{1S}(k)$ defined by (3.16a) is also shown.

γ rays occurring in coincidence with characteristic K-shell x-rays, a direct test of the predictions of this section can be made.

5. SCREENING

Screening affects the rate of radiative capture both by changing the initial configurations of the electrons being captured and by altering the probability amplitude than an electron reach the nucleus after virtually emitting a quantum. In order to study these effects, an approximate model may be used. In this model, correlations are ignored and each electron is taken to occupy some stationary state of a fixed external potential. The potential, in turn, is chosen to resemble the one the electrons themselves give rise to, in addition to the nuclear Coulomb field. With this approximation, as proven in (I), all the other electrons may be neglected in computing the transition probability for a single one.

The first and more important effect of screening is to alter the wave function of each initial state. The correction to the rate of radiative capture which results from this change is quite similar to the one occurring in ordinary capture. Ordinary capture only takes place when the electron is initially within the small nuclear volume. Radiative capture is relatively likely to occur when the electron is initially within the region where the Green's function is sizable. For the photon frequencies of interest, lying above the binding-energy region and below the threshold for positron production ($k\sim 2m$), the range of this Green's function has been shown to be comparable with the electron Compton wavelength. Although the latter is considerably larger than the nuclear radius, it is still very small on an atomic scale. This suggests that the radiative probabilities may be corrected for screening in the same way as the ordinary capture transition probabilities. In that case the matrix element for capture from a given initial electron state may be determined merely by multiplying the unscreened result by the ratio of the values near the origin of the screened and unscreened wave functions for that state.

That the second effect of screening, the alteration of the Green's function, is quite small, may also be understood qualitatively. Over the short distance in which the Green's function is sizable, the electric field may be well approximated by the Coulomb field of the nucleus. In this region the presence of the electron shells acts only to add a constant, Δ, to the electrostatic nuclear potential, or equivalently, to displace the energy variable in the Green's function by that amount. This energy occurs most sensitively in the Green's function, Eqs. (3.5)–(3.6), through the variable $\mu^2=m^2-(\bar{E}_i-k-\Delta)^2$ where $\bar{E}_i=m-\bar{\epsilon}_i$ is the total energy of the ith state in the screened field and $\bar{\epsilon}_i$ its binding energy. By neglecting terms of relative order $(Z\alpha)^2$, this expression may be reduced to $2m(k+\epsilon_i+\Delta-\Delta_i)-k^2$ where $\epsilon_i=m-E_i$ has been introduced for the binding energy of the corresponding state in the unscreened field, and Δ_i for the difference $\epsilon_i-\bar{\epsilon}_i$. Since both Δ and Δ_i are very small with respect to $k+\epsilon_i$ the difference between the screened Green's function $\mathcal{G}'^{\mathrm{sc}}(\bar{E}_i-k)\cong\mathcal{G}'(E_i-k-\Delta+\Delta_i)$ and $\mathcal{G}'(E_i-k)$ is quite small. Actually, the approximation $\mathcal{G}'^{\mathrm{sc}}(\bar{E}_i-k)\cong\mathcal{G}'(E_i-k)$ is even better than this argument indicates since the two shifts in energy due to screening, Δ and Δ_i, tend to cancel. To the extent that the electronic charge cloud can be considered to lie outside the region in which the wave function of the captured electron is appreciable, the energy shift, Δ_i, must equal Δ. In this limit the shells of atomic electrons play no real role, serving only to redefine the zero of energy.

The arguments thus far presented depend on the short-range nature of the Green's function. Consequently they need not be accurate for photons of very low energy. Such quanta may be emitted far from the nucleus and an accurate theory for them might require a more elaborate treatment of both the wave functions and the Green's function. Since part of the energy region of experimental interest is not very far from the binding-energy region, it is necessary to check quantitatively the validity of the simple approximation scheme previously suggested. The methods by which it has been tested and shown to be sufficiently accurate, will now be indicated.

To make the screening problem tractable, the screened Coulomb potential has been approximated by a Hulthén potential,[14]

$$A_0^{\mathrm{sc}}=\frac{Ze\gamma e^{-\gamma r}}{1-e^{-\gamma r}}. \tag{5.1}$$

This potential behaves like $(Ze/r)-\frac{1}{2}\gamma Ze$ near the origin, and decreases exponentially when r is larger than the atomic dimension, $1/\gamma$. By neglecting the terms which contribute the fine structure, the equation for $\mathcal{G}'^{\mathrm{sc}}(E)$ corresponding to Eq. (I, 3.2) may be written

$$[\nabla^2+E^2-m^2+2eEA_0^{\mathrm{sc}}(r)]\mathcal{G}'^{\mathrm{sc}}(E;0,\mathbf{r})=-\delta(\mathbf{r}). \tag{5.2}$$

The S-state wave functions for this potential have simple forms (Jacobi polynomials), and the Green's function, $\mathcal{G}'^{\mathrm{sc}}(E;0,r)$, may also be expressed in closed form as the hypergeometric function

$$\mathcal{G}'^{\mathrm{sc}}(E;0,r)$$

$$=\frac{e^{-\mu r}}{4\pi r}\frac{\Gamma(1+2\mu/\gamma+\eta/\xi)\Gamma(1+2\mu(1-\xi)/\gamma)}{\Gamma(1+2\mu/\gamma)}$$

$$\times{}_2F_1(-\eta/\xi,\,2\mu\xi/\gamma,\,1+2\mu/\gamma;\,e^{-\gamma r}) \tag{5.3a}$$

$$=\frac{e^{-\xi\mu r}}{4\pi r}\int_0^\infty dte^{-t}\left[\frac{1-e^{-\frac{1}{2}\gamma t/\mu\xi}}{e^{\frac{1}{2}\gamma r}-e^{-\frac{1}{2}\gamma(r+t/\mu\xi)}}\right]^{-\eta/\xi}. \tag{5.3b}$$

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[14] The Hulthén potential is discussed in L. Rosenfeld, *Nuclear Forces* (Interscience Publishers, Inc., New York, 1948), Vol. I, p. 79.

In these expressions, the parameters

$$\mu^2 = m^2 - E^2, \quad \eta = Z\alpha E/\mu, \qquad \xi = \tfrac{1}{2}[1 + (1 + 2\eta\gamma/\mu)^{\frac{1}{2}}] \tag{5.4}$$

have been introduced. At the origin, the difference between the Hulthén and pure Coulomb potential energies is the constant $\Delta = \frac{1}{2}Z\alpha\gamma$. The pure Coulomb Green's function for the displaced energy, $E' = E - \frac{1}{2}Z\alpha\gamma$, is

$$\mathcal{G}'(E') = \frac{e^{-\mu' r}}{4\pi r} \int_0^\infty dt e^{-t} \left(\frac{t + 2\mu' r}{t} \right)^{\eta'}, \tag{5.5}$$

where, consistent with the approximations made in originally deriving Eq. (5.2),

$$\mu' = E'^2 - m^2 \rightarrow \mu(1 + \eta\gamma/\mu)^{\frac{1}{2}}, \qquad \eta' = Z\alpha E'/\mu' \rightarrow Z\alpha E/\mu'. \tag{5.6}$$

A comparison of Eq. (5.3) and Eq. (5.5), which assumes that the electrons contribute a constant potential shows that the latter representation is correct to order $(\eta\gamma/\mu)^2$. Since the effective range, $1/\gamma$, of the Hulthén potential is of atomic size (larger than the Bohr radius of an inner electron and therefore larger than the range of the Green's function, $1/\mu$, at even the lowest energies) and since, in addition, the parameter η decreases rapidly with increasing k, from its value, unity, for zero-frequency photons, the approximation of Eq. (5.3b) by Eq. (5.5) is indeed a very good one. The error made in replacing the screened Green's function, $\mathcal{G}'^{\mathrm{sc}}(\bar{E}_i - k)$, by the unscreened one, $\mathcal{G}'(\bar{E}_i - k - \Delta_i)$, is somewhat larger than the difference between the former and $\mathcal{G}'(\bar{E}_i - k - \Delta)$. It is smaller, however, than the difference between $\mathcal{G}'(\bar{E}_i - k - \Delta)$ and $\mathcal{G}'(\bar{E}_i - k)$. An overestimate of the error may therefore be obtained by subtracting from the integral in Eq. (5.5) the same integral with μ' replaced by μ. The result shows that the difference between the screened and unscreened Green's function is at worst proportional to the first power of $\eta\gamma/\mu$ and hence is still quite small.

As discussed earlier, the corrections to the simplest approximation for screening (multiplication by the ratio of screened to unscreened electron densities), should be greatest for low-energy photons and for electrons in shells outside the K shell. It was therefore decided to test the approximations by calculating the corrections of order $\eta\gamma/\mu$ for the $2P$ state in iron. The screening parameter occurring in the Hulthén potential was determined by two independent methods. First, since the Hartree calculations[15] had been done for iron, the screening constant was directly fitted to the computed potential. Secondly, the screening constant was chosen to fit the observed S-state energy levels. Both methods yielded about the same value, $\gamma \cong 0.23 Z\alpha m$.

In order to determine the matrix element in the screened potential, it was also necessary to make use of screened P-state wave functions. Since these wave functions of the Hulthén potential could not be expressed in closed form, they were approximated by wave functions Coulombic in shape but with a scale factor determined by variationally minimizing the energy. The energy bound derived from this wave function was close to the observed energy, and more significantly, the wave function agreed fairly well with the $2P$ Hartree and Hartree-Fock functions. Using this wave function and the Green's function correct to order $\eta\gamma/\mu$, the probability of radiative capture was evaluated at a photon energy roughly equal to the K-electron binding energy in iron and at another energy three times as large. The predicted intensity was also computed for these energies on the basis of the unscreened wave function and Green's function. The difference between the latter result multiplied by the ratio of screened to unscreened probability densities at the origin, and the former answer determined with the screened wave function and Green's function amounted to less than twenty percent at the lower energy and about two percent at the higher one. In the present calculations, therefore, screening has always been included by multiplying the unscreened probabilities by the factor

$$S_i = |\psi_i^{\mathrm{sc}}(0)|^2 / |\psi_i(0)|^2. \tag{5.7}$$

The quantities, S_i, which this procedure requires, may be found from various sources. In obtaining them for the K and L shells, use was made of graphs computed by Rose and Brysk and contained in an unpublished report.[16] For convenience, their graphs are reproduced here in Fig. 3. In the case of more distant electron

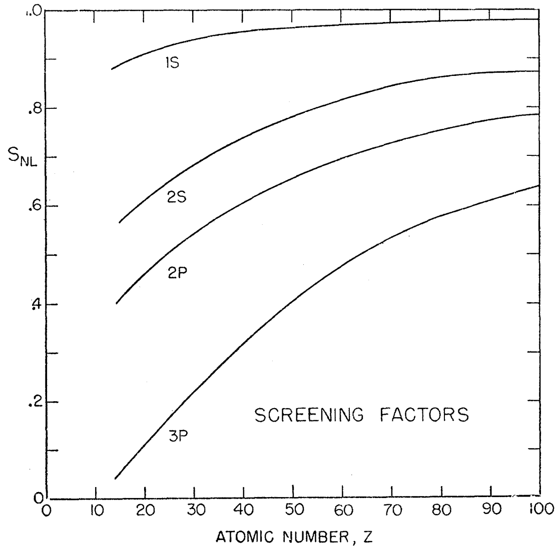

FIG. 3. Approximate values of the screening factors S_{nl} defined by (5.7). These have been derived from the work of Rose and Brysk[16] and Hartree self-consistent field calculations.[17]

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[15] M. Manning and L. Goldberg, Phys. Rev. **53**, 662 (1938).

[16] M. Rose and H. Brysk, Oak Ridge National Laboratory Report 1830 (unpublished).

shells such graphs were not available, and instead, recourse was made to various Hartree self-consistent field calculations.[17] Fortuitously, both argon and iron are among the elements for which these calculations have actually been performed. For other elements it has been possible to interpolate between the various available calculations to obtain rough values of the desired ratio. Generally, these rough values are quite sufficient for the *M* and higher shells. Their spectra, which were previously less intense than those of the lower shells, are so greatly diminished by screening that they become quite negligible.

6. CONCLUSIONS AND COMPARISON WITH EXPERIMENT

With the aid of the approximate forms derived in Secs. 3 and 4, it is possible to determine both the polarization and intensity of inner bremsstrahlung in the presence of the Coulomb field. The polarization, P, is defined as the difference in intensity of right and left polarized radiation, divided by their sum:

$$P_{1S}(k)=\frac{w_{+1S}(k)-w_{-1S}(k)}{w_{+1S}(k)+w_{-1S}(k)}. \tag{6.1}$$

The required expressions for the intensities of polarized radiation are given by Eq. (3.14). For a two-component neutrino theory the expression for the polarization reduces to

$$P_{1S}(k)=p_{1S}(k) \times\frac{\sum_f\{|\langle 1\rangle_{fi}|^2(|C^S|^2-|C^V|^2)+|\langle\boldsymbol{\sigma}\rangle_{fi}|^2(|C^T|^2-|C^A|^2)\}}{\sum_f\{|\langle 1\rangle_{fi}|^2(|C^S|^2+|C^V|^2)+|\langle\boldsymbol{\sigma}\rangle_{fi}|^2(|C^T|^2+|C^A|^2)\}}, \tag{6.2}$$

where

$$p_{1S}(k)=\frac{(A_{1S}+B_{1S})^2-(A_{1S}-B_{1S})^2}{(A_{1S}+B_{1S})^2+(A_{1S}-B_{1S})^2}. \tag{6.3}$$

At very low energies, the polarization vanishes since $B_{1S}\approx 0$ while $A_{1S}\approx 1$. As the energy rises, A_{1S} and B_{1S} both approach unity except for terms of order $Z\alpha$. At the higher energies, $(A_{1S}-B_{1S})^2$ is therefore of order $(Z\alpha)^2$ relative to $(A_{1S}+B_{1S})^2$ and the polarization is not appreciably affected by the Coulomb field. In particular, the fraction, p, which multiplies the ratio of nuclear matrix elements is given approximately by

$$p_{1S}(k)\approx 1-\tfrac{1}{2}(Z\alpha)^2(m/k)^2\{(\mu/k) +2[1-(m/k)]\tan^{-1}(\mu/k)\}^2, \tag{6.4}$$

(where $\mu^2=m^2-(E_{1S}-k)^2$) for photon energies which are not very small compared to the electron rest mass.

If the two-component neutrino theory is assumed, and in addition the coupling is either entirely scalar and tensor, or entirely vector and axial vector, then the ratio of nuclear matrix elements is unity and the polarization of high-energy gamma rays is essentially complete and of one sense for all nuclei. On the other hand, if the coupling were purely vector and tensor, for example, the polarization of pure Fermi ($\langle\boldsymbol{\sigma}\rangle_{fi}=0$) and pure Gamow-Teller ($\langle 1\rangle_{fi}=0$) transitions would both be complete but the senses would be opposite. If this combination were to prove correct, the extent of polarization would provide a method for determining the relative strength of Gamow-Teller and Fermi nuclear matrix elements of nuclei which undergo allowed *K*-capture transitions.

For convenience the intensities predicted by the calculations in this paper have been expressed as correction factors multiplying the formulas obtained by simpler methods. Thus, the ratio of the probability of radiative capture of electrons in the shell *nl* to the ordinary capture probability of 1*S* electrons is given by the expression

$$\frac{w_{nl}}{w_K}=\frac{\alpha}{\pi}\int_0^{(k_{\max})_{nl}/\frac{1}{2}Z^2\alpha^2m}(\tfrac{1}{2}Z^2\alpha^2)^2 I_{nl}\left(\frac{k}{\frac{1}{2}Z^2\alpha^2m}\right)d\left(\frac{k}{\frac{1}{2}Z^2\alpha^2m}\right) \times\left[\frac{(k_{\max})_{1S}-(k+\epsilon_{nl}-\epsilon_{1S})}{(k_{\max})_{1S}}\right]^2\frac{S_{nl}}{S_{1S}}. \tag{6.5}$$

Here the functions, I_{nl}, are the intensity functions for the various shells which were discussed in (I) and defined to contain the relativistic correction factors R_{nl} as in [I, (10.2)]. The constants ϵ_{nl} are the (positive) binding energies of the electron shells, and the quantities $(k_{\max})_{nl}=m-\Delta E-\epsilon_{nl}$ are the maximum photon frequencies which can accompany capture from them. For the *K* shell Eq. (6.5) reduces to[18]

$$\frac{w_{1S}}{w_K}=\frac{\alpha}{\pi m^2}\int_0^{(k_{\max})_{1S}}k\left(1-\frac{k}{(k_{\max})_{1S}}\right)^2R_{1S}(k)dk. \tag{6.6}$$

The factor S_{nl} in (6.5) represents the correction due to screening. For the energies of interest it is given by the ratio $|\psi_{nl}^{sc}(0)|^2/|\psi_{nl}(0)|^2$, that is, by the relative likelihood of finding the electron in the state *nl* at the origin with and without taking screening into account. Graphs of these factors S_{nl} for the most important shells are reproduced in Fig. 3. For the remaining shells, for which these factors are much smaller, they may be interpolated from available Hartree calculations.[17] The most notable effect of the screening corrections is to reduce the 2*P*-state spectrum relative to that of the 1*S* state, and to do so to a larger extent when the charge is small.

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[17] The elements on which D. R. Hartree and W. Hartree have performed calculations reported in the Proceedings of the Royal Society (London), Section A, include: K and Cs, **A143**, 506 (1934); F, Al, and Rb, **A151**, 96 (1935); Be, Ca, and Hg, **A149**, 210 (1935); Be, **A150**, 9 (1935); Cl, **A156**, 45 (1936); Cu, **A157**, 490 (1936); Ca, **A164**, 167 (1938); K and A, **A166**, 450 (1938).

[18] Except for the factor R_{1S} this term is just the result of P. Morrison and L. I. Schiff, Phys. Rev. **58**, 24 (1940).

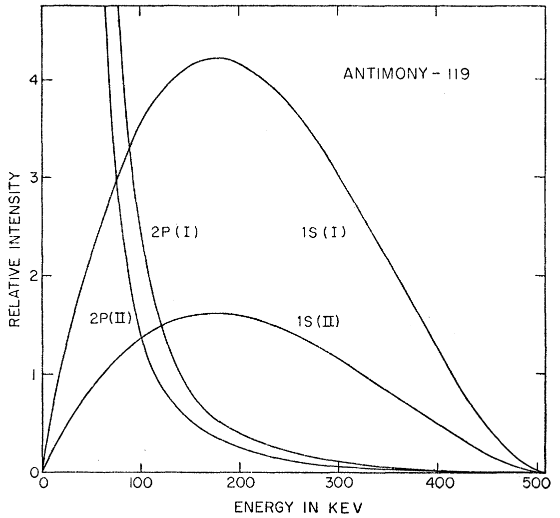

FIG. 4. Illustration of the effects of relativistic and screening corrections on the intensities of the internal bremsstrahlung spectra accompanying orbital electron capture. The curves labeled I show the uncorrected theoretical spectra for capture from the $1S$ and $2P$ states in Sb^{119}, while those labeled II show the intensity reductions brought about by the corrections.

The relativistic factor, $R_{1S}(k)$ has been evaluated in the high-energy region only for the $1S$ state, where it is most important. It leads to a correction which varies relatively slowly with energy. Expressions for it have been obtained at all energies for light nuclei and at certain energies for somewhat heavier ones. These results are contained respectively, in Eqs. (4.3)–(4.5) and Eqs. (4.6)–(4.8). For a given energy, these corrections tend to decrease the intensity ratio given in Eq. (6.5) from the estimate with no Coulomb field $[R(k)=1]$. The decrease becomes more significant as the charge increases.

The calculations carried to completion are most accurate for fairly light elements but are probably qualitatively correct for heavier ones. They have been performed with the experiments on allowed transitions in A^{37}, V^{49}, and Sb^{119} in mind.[19–21] Figures 4 and 5 compare the spectra obtained by using the simple and more accurate theories. The absolute intensities predicted by the two theories differ considerably, as Fig. 4 indicates. (The example of antimony has been taken since the P-state spectrum is more conspicuous in this case, but comparable changes are produced in most elements.) Since the relativistic correction varies more slowly with photon energy than the intensity functions do, the shapes of the spectra predicted for the individual electronic states are not so appreciably altered. In Fig. 5, where the two total spectra for A^{37} have been normalized to produce the same integrated intensity for the $1S$-state portion of the spectrum, the shapes of the curves differ only slightly. This is due partially to a compensation of the relativistic reduction ($\approx 30\%$) of the $1S$ spectrum and the screening reduction ($\approx 60\%$) of the $2P$ spectrum, and partially just to the small region in which the P intensity is significant in a light element with a large energy release.

Some other general features of radiative capture spectra, exemplified by these figures, bear mention. The shapes of the $1S$-state spectra are primarily determined by the available energy, $k_{\max}$, of the nuclear reaction. The elementary shape, $k(k_{\max}-k)^2$, which has a maximum at $\frac{1}{3}k_{\max}$, is reduced in intensity, and may be altered in shape by the relativistic factor, $R(k)$, which in light elements, first increases and then decreases with increasing photon energy. In argon, where the maximum of the elementary form coincides with the maximum of the factor $R(k)$, this factor tends to slightly accentuate the maximum; in antimony, the intensity at all energies is reduced to about 0.4 the value predicted without taking relativistic Coulomb effects into account.

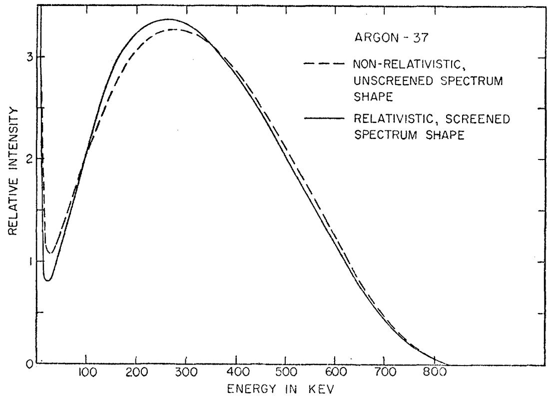

FIG. 5. Comparison of the uncorrected and corrected shapes of the total spectral intensity distribution ($1S+2S+2P$) for A^{37}. To compare the shapes alone the spectra have been normalized to yield the same integrated intensities for the S-state contributions.

The shapes of the P-state spectra, which must be added to these S-state spectra are primarily dependent, not on the maximum energy but on the nuclear charge. These spectra increase in absolute magnitude with increasing charge and begin to rise steeply as the γ-ray energy decreases to a few times the K-shell binding energy[22] (that is, an energy proportional to the square of the charge). When screening is taken into account, the intensities are cut down, but to a lesser extent with increasing charge. Hence the corrections reenforce the tendency for the P-state spectra to become more dominant with increasing charge. This effect is apparent in the argon spectrum and is more pronounced in the spectrum of antimony. It is probably most graphically

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[19] T. Lindqvist and C. S. Wu, Phys. Rev. **100**, 145 (1955).

[20] R. W. Hayward and D. D. Hoppes, Phys. Rev. **104**, 183 (1956).

[21] Olsen, Mann, and Lindner, Phys. Rev. **106**, 985 (1957).

[22] This effect was first observed and reported in the experiments of L. Madansky and F. Rasetti, Phys. Rev. **94**, 407 (1954).

illustrated by the spectrum[20,23] of cesium-131, where the low-energy release and large charge combine to give a *P*-state spectrum which dominates the *S*-state spectrum at all energies.

Comparison of the experimental results for A^{37}, V^{49}, and Sb^{119} with those computed theoretically are contained in Figs. 6–8. The experiment on argon was performed by Lindqvist and Wu,[19,24] the vanadium experiment by Hayward and Hoppes,[20] and that on antimony by Olsen, Mann, and Lindner.[21]

The experiments which have been discussed thus far do not discriminate among the spectra of the various initial electron states, measuring only the sum of these spectra. A number of the theoretical predictions may be checked in greater detail and with greater accuracy by determining which electron is captured when a given photon is emitted. An experiment of this kind, where the gamma rays are observed in coincidence with the characteristic x-rays which follow capture, has recently been performed by Michalowicz.[25] This experiment indicates the correctness of qualitative features of the theory described in (I) and here. If only gamma rays which occur in coincidence with *K*-shell x-rays are observed, there is no precipitous increase of gamma-ray intensity as the photon energies decrease.[26] The steep rise observed when the total intensity is measured is therefore presumably explained by *P*-state capture.

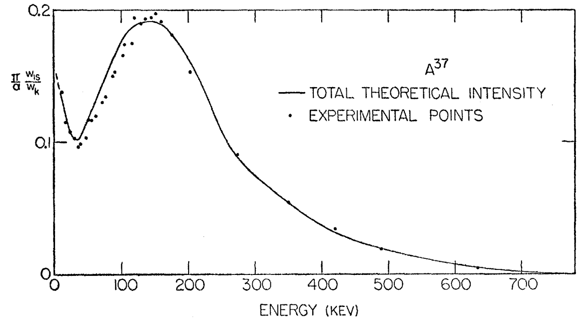

FIG. 6. Comparison of the theoretical and experimental spectra for A^{37}. The corrected theoretical spectrum of Fig. 5 has been modified to allow for the efficiency of the scintillation counter in comparing with the observations of Lindqvist and Wu.[19]

As Figs. 4 and 5 indicate, the relativistic and screening corrections may affect the absolute intensity of a spectrum more noticeably than they affect its shape. A second kind of useful experiment, therefore, is one which measures the intensity of the radiative capture spectrum relative to the nonradiative process. In the antimony experiment, this relative intensity has actually been measured and found to agree approximately with the reduction predicted here. The theoretically determined reduction factor to be applied to the Born approximation[18] is about 0.4±0.1; the experimentally determined value is 0.48±0.10. The theoretically determined energy at which the two spectra should have equal intensity is 103±5 kev; the experimentally inferred value is the same.

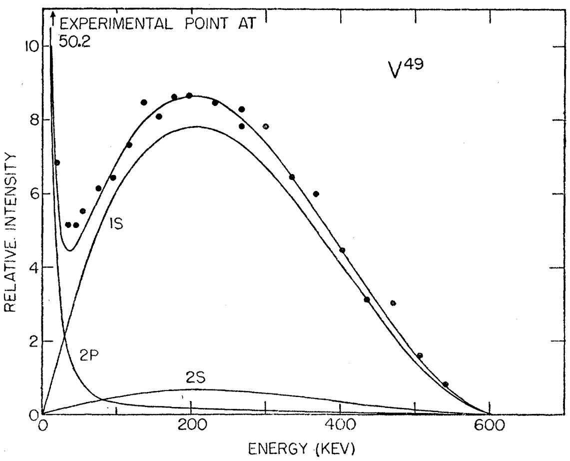

FIG. 7. The theoretical γ-ray spectra for electron capture in V^{49} and a comparison of their total with the intensity observed by Hayward and Hoppes.[20]

The calculations presented in these papers treat in complete detail allowed transitions from the *K* shell in moderately light elements. Although only allowed transitions have been treated, the large magnitude of the corrections found casts some doubt on the existing Born-approximation treatments of forbidden radiative transitions. For quantitative predictions about the *K*-shell spectra in heavier elements, the spectra of other shells in allowed transitions, and all the spectra accompanying unallowed transitions, the same techniques developed here may be employed. It is to be

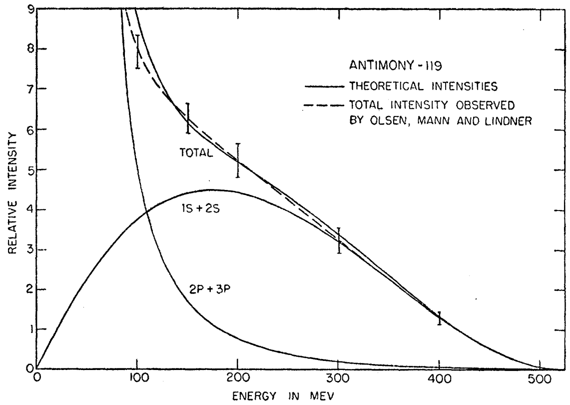

FIG. 8. Comparison of the total γ-ray intensities for Sb^{119} as predicted theoretically, and as observed experimentally by Olsen, Mann, and Lindner.[21] The separate theoretical contributions of the *S* and *P* states are also shown.

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[23] B. Saraf, Phys. Rev. **94**, 642 (1954); **95**, 97 (1954).

[24] The comparison of theory and experiment for A^{37} has been previously reported by Glauber, Martin, Lindqvist, and Wu, Phys. Rev. **101**, 905 (1956).

[25] A. Michalowicz, Compt. rend. **242**, 108 (1956).

[26] A partial separation of the spectra by coincidence techniques has also been carried out by N. Rasmussen (private communication).

hoped that some of these additional calculations will also be undertaken.

ACKNOWLEDGMENTS

The authors are grateful to Dr. K. A. Johnson for his suggestion about the treatment of the Dirac equation, and to Dr. P. O. Löwdin for providing simple expressions for the Hartree wave functions in iron. They would like to thank Drs. M. Rose and H. Brysk for permission to reproduce their graphs of screening corrections and Dr. L. Madansky, Dr. F. Rasetti, Dr. T. O. Lindqvist, Dr. C. S. Wu, Dr. R. W. Hayward, Dr. D. D. Hoppes, Dr. J. L. Olsen, Dr. L. G. Mann, and Dr. M. Lindner for providing them with experimental results prior to publication. Finally, they wish to express their appreciation to the Brookhaven National Laboratory, the University of California Radiation Laboratory, the Institute of Theoretical Physics in Copenhagen, and the Lorentz Institute in Leiden, for hospitality extended them while they wrote this paper.

APPENDIX A

This appendix describes briefly the solution of the radial part of the second-order Dirac equation (2.13). This equation may be written in the dimensionless form

$$\left[\frac{d^2}{dx^2}+\frac{\eta_i}{x}-\frac{1}{4}-\frac{\lambda(\lambda\pm1)}{x^2}\right]U_i(x)=0, \tag{A.1}$$

by introducing the variables

$$\begin{aligned}\mu_i^2&=m^2-E_i^2, &\eta_i&=Ze^2E_i/\mu_i,\\ x&=2\mu_i r, &\varphi_i(r)&=[U_i(x)/x]\chi_{\pm\lambda}{}^{K}.\end{aligned} \tag{A.2}$$

The solution of Eq. (A.1) is a linear combination of Whittaker functions

$$U_i(x)=\tilde{c}_{\pm\lambda}W_{\eta_i,\lambda\pm\frac{1}{2}}(x)+\tilde{d}_{\pm\lambda}W_{-\eta_i,\lambda\pm\frac{1}{2}}(-x). \tag{A.3}$$

A necessary condition for φ_i to be an admissible wave function may be obtained from its relation to the physically meaningful function, ψ_i. For the latter to be admissible, it must predict a finite charge, $e\int\psi^*\psi d\mathbf{r}$, in every volume of space. This restriction may be imposed on a dimensionless form of the first-order wave function,

$$V_i(x)=\frac{1}{2m}\left[m+\beta E_i+i\beta\alpha_r 2\mu_i\left(\frac{d}{dx}+\frac{1\pm\lambda}{x}\right)\right]\frac{U_i(x)}{x}. \tag{A.4}$$

The restriction that V_i vanish at infinity eliminates from Eq. (A.3) the term with coefficient $\tilde{d}_{\pm\lambda}$. The requirement that V_i be square-integrable in the neighborhood of the origin is equivalent to the restriction, $x^{\frac{3}{2}}V_i(x)\rightarrow0$ as $x\rightarrow0$. The Whittaker function, U_i of Eq. (A.3), may be represented as a contour integral in the positive sense about a branch cut extending from zero to infinity

$$U_i(x)=\tilde{c}_{\pm\lambda}(i/2\pi)\Gamma(\eta_i+\tfrac{1}{2}-\lambda\mp\tfrac{1}{2})e^{-\frac{1}{2}x}x^{\frac{1}{2}-\lambda\mp\frac{1}{2}}$$
$$\times\int_{\infty}^{0+}(-t)^{-\eta_i-\frac{1}{2}+\lambda\pm\frac{1}{2}}(x+t)^{\eta_i-\frac{1}{2}+\lambda\pm\frac{1}{2}}e^{-t}dt. \tag{A.5}$$

Since the expression in front of the integral behaves as $x^{-\lambda+\frac{1}{2}\mp\frac{1}{2}}$, $x^{\frac{3}{2}}V_i(x)$ will behave as $x^{-\lambda\mp\frac{1}{2}}$, and hence be inadmissible, unless the coefficient multiplying this power vanishes,

$$\begin{aligned}(i/2\pi)\Gamma(\eta_i+\tfrac{1}{2}-\lambda\mp\tfrac{1}{2})&\int_{\infty}^{0+}(-t)^{-\eta_i-\frac{1}{2}+\lambda\pm\frac{1}{2}}t^{\eta_i-\frac{1}{2}+\lambda\pm\frac{1}{2}}e^{-t}dt\\ &=(\pi)^{-1}\sin[\pi(\eta_i+\tfrac{1}{2}-\lambda\mp\tfrac{1}{2})]\Gamma(\eta_i+\tfrac{1}{2}-\lambda\mp\tfrac{1}{2})\Gamma(2\lambda\pm1)\\ &=\Gamma(2\lambda\pm1)/\Gamma(\tfrac{1}{2}-\eta_i+\lambda\pm\tfrac{1}{2})=0,\end{aligned} \tag{A.6}$$

that is, unless $\eta_i+\frac{1}{2}-\lambda\mp\frac{1}{2}$ is a positive integer. This condition is equivalent to the requirement

$$\eta_i=n_r+\lambda+\tfrac{1}{2}\pm\tfrac{1}{2},$$
or
$$E_i=m[1+Ze^2/(n_r+\lambda+\tfrac{1}{2}\pm\tfrac{1}{2})^2]^{-\frac{1}{2}}, \tag{A.7}$$

where n_r is a non-negative integer and the alternative signs refer to $\Lambda=\pm\lambda$. For these energies the functions, u_i, may be considerably simplified. The contour integrals, (A.5), reduce to integrals about poles at the origin, which may be evaluated by the Cauchy formula,

$$\oint t^{-n-1}f(t)dt=\frac{2\pi i}{n!}\left[\left(\frac{d}{dt}\right)^n f(t)\right]_{t=0}. \tag{A.8}$$

Since the function, $f(t)$, in question, has the form,

$$\begin{aligned}f(t)&=e^x g(x+t),\\ g(\zeta)&=e^{-\zeta}\zeta^{n_r+2\lambda\pm1},\end{aligned} \tag{A.9}$$

the derivatives with respect to t may be replaced by derivatives with respect to x, and t set equal to zero. Apart from normalization, the functions for which Eqs. (A.6)–(A.7) are satisfied, are therefore given alternatively by

$$\begin{aligned}U_i(x)&=\tilde{c}_{\pm\lambda}W_{\eta_i,\lambda\pm\frac{1}{2}}(x)=c_{\pm\lambda}(-1)^{n_r}W_{\eta_i,\lambda\pm\frac{1}{2}}(x),\\ &=c_{\pm\lambda}e^{\frac{1}{2}x}x^{\frac{1}{2}-\lambda\mp\frac{1}{2}}(d/dx)^{n_r}(x^{n_r+2\lambda\pm1}e^{-x})\\ &=c_{\pm\lambda}[\Gamma(n_r+2\lambda+1\pm1)/\Gamma(2\lambda+1\pm1)]e^{-\frac{1}{2}x}\\ &\qquad\times x^{\lambda+\frac{1}{2}\pm\frac{1}{2}}\,{}_1F_1(-n_r,2\lambda+1\pm1;x).\end{aligned} \tag{A.10}$$

These functions behave as $x^{\lambda+\frac{1}{2}\pm\frac{1}{2}}$ near the origin, and consequently, $x^{\frac{3}{2}}V_i(x)\sim x^{\lambda+\frac{1}{2}}$ for both $\Lambda=\pm\lambda$. Each of these functions, U_i, therefore leads to an eigenfunction of the Dirac equation.

APPENDIX B

The normalization of the eigenfunctions derived in Appendix A, and of the first-order Dirac wave functions, is simply accomplished by virtue of the relation of the latter to the Green's function, $G(E)$,

$$G(E;\mathbf{r},\mathbf{r}')=\sum_i \frac{\psi_i(\mathbf{r})\bar{\psi}_i(\mathbf{r}')}{E_i-E}. \tag{B.1}$$

In particular, the residue at E_i of the function $G(E)$ is equal to the sum of bilinear combinations of eigenfunctions with that energy. In practice, it is convenient to express ψ, $\bar{\psi}$, and G in terms of the corresponding functions of the second-order equation. On account of these definitions, Eqs. (2.2), (2.5), and (2.33), the relation between the Green's function and wave functions takes the form

$$\operatorname*{Res}_{E=E_i} \mathcal{G}(E;\mathbf{r},\mathbf{r}')=-\sum_{\mathrm{K}'\Lambda'} \varphi_{\Lambda',E_i}{}^{\mathrm{K}'}(\mathbf{r})\,\bar{\varphi}_{\Lambda',E_i}{}^{\mathrm{K}'}(\mathbf{r}')$$
$$\times[m+\beta E_i+i\beta\alpha_{r'}(ip_{r'}+(1+\Lambda')/r')]/(2m)^2. \tag{B.2}$$

When Eq. (B.2) is multiplied on the left by $\bar{\chi}_\Lambda{}^{\mathrm{K}}(\Omega)$ and on the right by $\chi_\Lambda{}^{\mathrm{K}}(\Omega')$, and the products integrated over Ω and Ω', an equation for the radial variables,

$$\operatorname*{Res}_{E=E_i} g_\Lambda(E;r,r')=\frac{-1}{(2m)^2}\frac{U_{\Lambda,E_i}(2\mu_ir)}{(2\mu_ir)}\int d\Omega'$$
$$\times\left(\bar{\chi}_\Lambda{}^{\mathrm{K}}(\Omega')\left[m+\beta E_i-i\beta\alpha_{r'}\left(\frac{\partial}{\partial r'}-\frac{1+\Lambda}{r'}\right)\right]\right.$$
$$\left.\times\frac{U_{\Lambda,E_i}(2\mu_ir')}{2\mu_ir'}\chi_\Lambda{}^{\mathrm{K}}(\Omega')\right), \tag{B.3}$$

results. The radial Green's function occurring here has been defined in Eq. (2.27) and the radial wave functions in Eq. (A.2). On performing the indicated angular integrations, the term involving $i\beta\alpha_{r'}$ is eliminated and β replaced by κ/λ, so that

$$\operatorname*{Res}_{E=E_i} g_\Lambda(E;r,r')$$
$$=-\frac{1}{4m}\left(1+\frac{\kappa}{\lambda}\frac{E_i}{m}\right)\frac{U_{\Lambda,E_i}(2\mu_ir)}{2\mu_ir}\frac{U_{\Lambda,E_i}(2\mu_ir')}{2\mu_ir'}. \tag{B.4}$$

In order to determine the normalized wave functions $U_{\Lambda,E}$ it is sufficient to identify this expression with the residue of the radial Green's function computed explicitly in Sec. 2. For this purpose it is simplest to use the form of $g_\Lambda(E;r,r')$ given by Eq. (2.31) in which r is confined to small values. The factor in the latter expression which becomes singular as $E\rightarrow E_i$ is the function $\Gamma(\frac{1}{2}\pm\frac{1}{2}+\lambda-\eta)$, whose residue at $E=E_i$ is easily computed,

$$\operatorname*{Res}_{E=E_i}\Gamma(\tfrac{1}{2}\pm\tfrac{1}{2}+\lambda-\eta)$$
$$=\operatorname*{Res}_{E=E_i}\Gamma\left(-n_r-Ze^2\left(\frac{E}{\mu}-\frac{E_i}{\mu_i}\right)\right)$$
$$=(-1)^{n_r+1}\left\{Ze^2n_r!\frac{\partial}{\partial E}\left[\frac{E}{(E^2-m^2)^{\frac{1}{2}}}\right]_{E=E_i}\right\}^{-1}$$
$$=(-1)^{n_r+1}\mu_i{}^3/Ze^2m^2n_r!. \tag{B.5}$$

Use of this relation in finding the residue of Eq. (2.31) followed by identification of the result with (B.4) leads to a direct identification of the radial functions (A.10), together with their normalization constants,[27]

$$c_{\pm\lambda}=\left[\frac{n_r!}{8\mu_i{}^3}\left(\frac{m}{E_i}+\frac{\kappa}{\lambda}\right)(n_r+\lambda+\tfrac{1}{2}\pm\tfrac{1}{2})\right.$$
$$\left.\times\Gamma(n_r+2\lambda+1\pm1)\right]^{-\frac{1}{2}}. \tag{B.6}$$

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[27] Another procedure for normalizing the Dirac wave functions without integrating them has been brought to the attention of the authors by W. H. Furry and W. H. Lamb. This method is due to H. A. Kramers and is contained in his book, *Quantentheorie des Electrons und der Strahlung* (Akademische Verlagsgesellschaft, Leipzig, 1938), p. 312.