Properties of Nuclear Matter*

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The properties of nuclear matter have been determined by the solution of the nuclear many-body problem, using the reaction matrix theory of Brueckner. The nonlinear integral equations characteristic of the theory have been solved with the aid of the fast electronic computer IBM 704. The two-body interaction assumed is the phenomenological potential of Gammel, Christian, and Thaler.

It is found that the binding energy of nuclear matter, neglecting Coulomb forces, is 14.6 Mev per particle at a density corresponding to a radius parameter $r_0=1.00\times10^{-13}$ cm. The Coulomb repulsion in a heavy nucleus is shown to drop the density by approximately 15%. The tensor force is shown to give approximately 6-Mev binding energy.

The results are found to be very sensitive to the self-consistency requirements of the theory, the binding energy shifting from 14.6 Mev to 34.4 Mev if the velocity dependence of the single-particle potential is neglected. The actual solutions were made self-consistent by an iteration procedure which converged in five or six iterations, the final results being self-consistent to one part in 10^5 or 10^6.

The effective mass for nucleon motion in the Fermi sea is found to vary from $0.56M$ for slow particles to $0.66M$ for particles near the Fermi surface. This velocity dependence of the potential is closely related to the symmetry energy which also depends, however, on the shifting in the spin populations as the neutron-proton ratio is changed from unity. The symmetry energy computed is 10 to 15% larger than that deduced from experiment.

I. INTRODUCTION

IN previous papers,[1,2] one of us (K. A. B.) and others have developed a method for determining the properties of extended nuclear matter. This theory was first used to make an approximate study of the equilibrium density and binding energy of nuclear matter[3–6] and to develop a theory of high-energy nuclear reactions,[7] energy-level fine structure, and configuration mixing,[8] and neutron reactions with nuclei at low energy.[9,10] Later studies,[11–19] particularly that by Bethe,[20] have made further analyses of the theoretical foundation of the method and also examined the problems which arise in applying the method to finite systems. Thus in this paper, it is not necessary to review the foundations of the method.

The purpose of this paper is to give the details[21] of accurate numerical solutions of the equations of the theory and results obtained for the properties of extended nuclear matter. We shall give a description of the techniques used in obtaining solutions which is sufficiently detailed to serve as a suitable reference for future work. The difficulties peculiar to this formulation of the many-body problem will be exhibited by the development of the method of solution which we give.

The numerical solutions have been obtained using the IBM 704 computing machines of the Los Alamos Scientific Laboratory.

II. STATEMENT OF FORMALISM

According to the many-body theory to be used in this paper,[22] neglecting what are called *cluster terms* in the earlier papers, the energy of a nucleon of momentum m_0 (we suppress spin and isotopic spin indices for the present) propagating in an infinite nuclear medium in its lowest energy state is

$$E_0(m_0)=\frac{m_0^2}{2M}+\sum_{n_0}[(m_0n_0|K_0|m_0n_0)-(m_0n_0|K_0|n_0m_0)]. \quad (1)$$

The last term arises from exchange of all coordinates (space, and also spin and isotopic spin). In Eq. (1), M is the nucleon mass, and the sum over n_0 runs over all occupied states (here and in the rest of this section we refer to plane-wave states). K is a *reaction matrix* which determines the interaction energy of a pair of particles; K is determined from the two body potential V by the integral equation

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[22] We follow closely here the methods described in reference 2.

$$(m'n'|K_0|m_0n_0)=(m'n'|V|m_0n_0)+\sum_{m_1s_1}\frac{(m'n'|V|m_1s_1)(m_1s_1|K_0|m_0n_0)}{D}, \quad (2)$$

$$D=E_0(m_0)+E_0(n_0)-E_1(m_1,s_1;m_0,n_0)-E_1(s_1,m_1;m_0,n_0),$$

where the sum over m_1 and s_1 is over all unoccupied states. In Eq. (2), the energy $E_1(s_1,m_1;m_0,n_0)$ is the energy of a particle which has momentum s_1 as a consequence of the interaction of particles whose momenta were m_0 and n_0 resulting in a configuration of two particles whose momenta are m_1 and s_1. [$E_1(m_1,s_1;m_0,n_0)$ is the energy of the particle whose momentum is m_1.]

The energy $E_1(m_1,s_1;m_0,n_0)$ which appears in Eq. (2) is given by an equation similar to Eq. (1):

$$E_1(m_1,s_1;m_0,n_0)=\frac{m_1^2}{2M}+\sum_{n_1}(m_1n_1|K_1(s_1;m_0,n_0)|m_1n_1)-\text{exchange}, \quad (3)$$

where

$$(m'n'|K_1(s_1;m_0,n_0)|m_1n_1)=(m'n'|V|m_1n_1)+\sum_{m_2s_2}\frac{(m'n'|V|m_2s_2)(m_2s_2|K_1(s_1;m_0,n_0)|m_1n_1)}{D}$$

$$D=E_0(m_0)+E_0(n_0)+E_0(n_1)-E_1(s_1,m_1;m_0,n_0)-E_2(m_2s_1s_2;m_1s_1n_1;m_0n_0n_1)-E_2(s_2s_1m_2;m_1s_1n_1;m_0n_0n_1). \quad (4)$$

In Eq. (4), $E_2(s_2s_1m_2;m_1s_1n_1;m_0n_0n_1)$ is the energy of a particle whose momentum is s_2 as a consequence of (A) interaction of particles whose momenta were m_0 and n_0 resulting in a configuration of two particles whose momenta are m_1 and s_1 (in the meanwhile the particle whose momentum is n_1 does not interact), followed by (B) interaction of particles whose momenta are m_1 and n_1 resulting in a configuration of two particles whose momenta are m_2 and s_2. [The particle whose momentum is s_1 does not interact in state (B).]

The E_2's are defined in terms of K_2's which satisfy integral equations similar to Eq. (4) except that E_3's appear in D; and so on without end. The complicated structure of this infinite ladder of equations[23] which occurs in the theory is discussed in detail in Appendix A where it is shown that the ladder can be reduced to a single equation without approximation by introducing a parameter with an infinite range in the denominator of the integral equations for the K matrices (all K matrices then have a similar structure). Were we to redo the problem using computing machines comparable in speed to the IBM 704, we would not make the approximation described in this section again, but we would proceed in the manner outlined in Appendix A.

In the calculations to be described in the remainder of this paper, we have not utilized the results of Appendix A. Instead we have terminated the sequence of equations for the K matrices by approximating to the correct energy denominator. To do this, we make use of the fact that the dependence of $E_1(m_1,s_1;m_0,n_0)$ on s_1 is weak since s_1 does not appear in the kinetic energy term $m_1^2/2M$ in Eq. (3) or in the Born approximation term $(m'n'|V|m_1n_1)$ in the integral equation (4) for the K matrix. $E_1(m_1,s_1;m_0,n_0)$ depends on s_1 only through the appearance of s_1 in the energy denominator of the integral equation [Eq. (4)] for the K matrix. The approximation we have used is to replace

$$E_0(n_1)-E_1(m_1,s_1;m_0,n_0)\rightarrow-\Delta \quad (5)$$

in the energy denominator D in Eq. (4), where Δ is some average excitation energy. This replacement makes D independent of s_1, consequently makes $K_1(s_1;m_0,n_0)$ independent of s_1, and finally makes $E_1(m_1,s_1;m_0,n_0)$ independent of s_1.

The energy denominator which occurs in the integral equation for $K_2(s_1s_2;m_1s_1n_1;m_0n_0n_1)$ is:

$$D=E_0(m_0)+E_0(n_0)+E_0(n_1)+E_0(n_2)-E_1(s_1,m_1;m_0,n_0)-E_2(s_2s_1m_2;m_1s_1n_1;m_0n_0n_1)-E_3(m_3s_1s_2s_3;m_2s_1s_2n_2;m_1s_1n_1n_2;m_0n_0n_1n_2)-E_3(s_3s_1s_2m_3;m_2s_1s_2n_2;m_1s_1n_1n_2;m_0n_0n_1n_2). \quad (6)$$

In Eq. (6) we replace

$$E_0(n_1)+E_0(n_2)-E_1(s_1,m_1;m_0,n_0)-E_2(s_2s_1m_2;m_1s_1n_1;m_0n_0n_1)\rightarrow-\Delta, \quad (7)$$

and so on in all steps in the ladder of equations. This leads to the replacement

$$E_2(m_2s_1s_2;m_1s_1n_1;m_0n_0n_1)\rightarrow E_1(m_2,s_2;m_0,n_0), \quad (8)$$

the right-hand side being independent of s_2 as already mentioned.

Thus the energy denominator in Eq. (4) becomes

$$D=E_0(m_0)+E_0(n_0)-E_1(m_2;m_0,n_0)-E_1(s_2;m_0,n_0)-\Delta, \quad (9)$$

and the ladder is terminated.

In the calculations, the parameter Δ was varied between 0 and

$$\Delta_{\max}=E_0(p_f)-E_0(0); \quad (10)$$

which is the energy difference of particles lying at the top of the Fermi sea and the bottom of the Fermi sea. The weak dependence of the results on the value of Δ justifies the approximations made here.

We restate the reduced equations here for convenience. We now write for the on-the-energy-shell propagation

$$E_0(m_0)=\frac{m_0^2}{2M}+\sum_{n_0}[(m_0n_0|K|m_0n_0)-\text{exchange}], \quad (11)$$

[23] The problems peculiar to propagation in excited states or "off-the-energy-shell" are discussed in references 2, 13, and 20.

where

$$(m'n'|K|m_0n_0)=(m'n'|V|m_0n_0)+\sum_{m''n''}\frac{(m'n'|V|m''n'')(m''n''|K|m_0n_0)}{\Sigma-E_1(m'',\Sigma)-E_1(n'',\Sigma)}, \quad (12)$$

and

$$\Sigma=E_0(m_0)+E_0(n_0). \quad (13)$$

For off-the-energy-shell propagation the energy is

$$E_1(m_1,\Sigma)=\frac{m_1^2}{2M}+\sum_{n1}[(m_1n_1|K(\Sigma)|m_1n_1)-\text{exchange}], \quad (14)$$

where

$$(m'n'|K(\Sigma)|m_1n_1)=(m'n'|V|m_1n_1)+\sum_{m''n''}\frac{(m'n'|V|m''n'')(m''n''|K(\Sigma)|m_1n_1)}{\Sigma-\Delta-E_1(m'',\Sigma)-E_1(n'',\Sigma)}. \quad (15)$$

In Eq. (15), Σ is arbitrary but needed only for the range from $2E_0(0)$ to $2E_0(p_f)$ since this is the maximum range for $E_0(m_0)+E_0(n_0)$. The equations for K and $K(\Sigma)$ are of similar structure and can be solved by the same procedure. The similarity of Eqs. (11)–(15) to Eqs. (A-8)–(A-8″) of Appendix A should be noted.

III. DETAILS OF THE INTEGRAL EQUATIONS FOR THE K MATRICES

In this section we discuss a transformation to coordinate space, a treatment of the exclusion principle, a removal of difficulties associated with the presence of a repulsive hard core in the potential, and a treatment of the tensor force. These are required to reduce the integral equations for the K matrices to a form suitable for numerical calculation.

A. Transformation to Coordinate Space

A transformation of the integral equations for the K matrices to coordinate space is necessary to avoid difficulties associated with the presence of a repulsive hard core in the potential. The transformation to coordinate space proceeds in a manner described in the work of Møller[24] and Lippmann and Schwinger.[25]

The matrix elements of the potential between plane wave basis states appropriate to an infinite medium are diagonal in the total momentum, so that in Eq. (12), for example

$$\mathbf{m}'+\mathbf{n}'=\mathbf{m}_0+\mathbf{n}_0=\mathbf{p}, \quad (16)$$

where p is the total momentum. We also define the relative momentum

$$\tfrac{1}{2}(\mathbf{m}_0-\mathbf{n}_0)=\mathbf{k}. \quad (17)$$

Other total and relative momenta are defined similarly. Since the K's will conserve total momentum if V does, the total momenta appear as parameters in the integral equations. Equation (12) for example becomes

$$(k'|K(p)|k)=(k'|V|k)+\sum_{\mathbf{k}''}\frac{(k'|V|k'')(k''|K(p)|k)f(p,k'')}{\Sigma-E_1(p_+,\Sigma)-E_1(p_-,\Sigma)}. \quad (18)$$

The sum over $\mathbf{k}''$ runs over momenta,

$$\mathbf{p}_+=\tfrac{1}{2}\mathbf{p}+\mathbf{k}'', \qquad \mathbf{p}_-=\tfrac{1}{2}\mathbf{p}-\mathbf{k}'', \quad (19)$$

and $f(p,k'')$ is a step function included to take account of the exclusion principle. $f(p,k'')$ is defined by

$$f(p,k'')=1 \text{ if } p_+>p_F \text{ and } p_->p_F; \quad =0 \text{ otherwise.} \quad (20)$$

To proceed with the transformation to coordinate space, we introduce a wave matrix Ω defined by

$$K=V\Omega, \quad (21)$$

or in more detail by

$$(k'|K(p)|k)=\sum_{k''}(k'|V|k'')(k''|\Omega(p)|k). \quad (22)$$

The sum in Eq. (22) runs over all momenta k''. Substituting this definition of Ω into the integral equation (18), we obtain an integral equation for Ω:

$$(k'|\Omega(p)|k)=(k'|1|k)+\sum_{\mathbf{k}''}\frac{(k'|V|k'')f(p,k'')(k''|\Omega(p)|k)}{\Sigma-E_1(p_+,\Sigma)-E_1(p_-,\Sigma)}. \quad (23)$$

We now go to coordinate space by introducing a complete orthogonal set of basis wave functions (plane waves) $\varphi_k(r)$:

$$1|k)=\varphi_k(r), \quad (24)$$

and the coordinate space wave function $\psi_{p,k}(r)$.

$$\Omega(p_0)|k)=\psi_{p,k}(r). \quad (25)$$

In addition, for any operator

$$O|k)=\int dr'(r|O|r')\varphi_{k'}(r'), \quad (26)$$

and

$$(k'|O|k)=\int dr'\varphi_k(r')O|k). \quad (27)$$

Since V is diagonal in coordinate space, Eq. (26) takes on a simple form for V; namely,

$$V|k)=V(r)\varphi_k(r). \quad (28)$$

Since Eq. (23) is valid for all k', it can be abbreviated as follows [or since the $\varphi_k(r)$ for a complete orthogonal

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set, Eq. (23) gives]:

$$\Omega(p)|k)=1|k)+\sum_{k''}\frac{V|k'')f(p,k'')(k''|\Omega(p)|k_0)}{\Sigma-E_1(p_+,\Sigma)-E_1(p_-,\Sigma)}, \quad (29)$$

or, according to Eqs. (24)–(27)

$$\psi_{p,k}(r)=\varphi_k(r)+\int d\mathbf{r}'G_p(r,r')V(r')\psi_{p,k}(r'), \quad (30)$$

where

$$G_p(r,r')=\sum_{k''}\frac{\varphi_{k''}(r)\varphi_{k''}(r')f(p,k'')}{\Sigma-E_1(p_+,\Sigma)-E_1(p_-,\Sigma)}. \quad (31)$$

Equation (22) gives

$$(k'|K(p)|k)$$

$$=\sum_{k''}\int d\mathbf{r}'\varphi_{k'}(r')V(r')\varphi_{k''}(r')\int dr''\varphi_{k''}(r'')\psi_k(r'')$$

$$=\int d\mathbf{r}'\varphi_{k'}(r')V(r)\psi_k(r'), \quad (32)$$

which is a recognizable result.

In solving Eq. (15) instead of Eq. (12), everything is the same except for the appearance of Δ in the energy denominator which now appears in the Green's function of Eq. (31).

B. Partial Wave Expansion

In the absence of the exclusion principle [that is, when $f(p,k)=1$ for all values of its arguments] and in the absence of self-consistency requirements [that is, when $E_1(p,\Sigma)=p^2/2M$ for all Σ], Eqs. (30) and (31) reduce to the usual two-body scattering equations. It is natural, therefore, to attempt a reduction of the problem by making a partial wave expansion analogous to the partial wave expansion made in the two-body problem. However, this cannot be done in a simple and rigorous way for the following reason. In the two-body problem, Eq. (31) becomes

$$G_p(r,r')=\sum_{k''}\frac{\exp[i\mathbf{k}''\cdot(\mathbf{r}-\mathbf{r}')]}{\Sigma-(p^2/4M)-(k''^2/M)}. \quad (33)$$

The summand does not depend on the angle between $\mathbf{p}$ and $\mathbf{k}''$ so that G depends on $|\mathbf{r}-\mathbf{r}'|$ and not on the angle between $\mathbf{p}$ and $\mathbf{r}-\mathbf{r}'$. The summand in Eq. (31) depends on the angle between $\mathbf{p}$ and $\mathbf{k}''$ through both $f(p,k'')$ and the energies $E_1(p_+,\Sigma)$ and $E_1(p_-,\Sigma)$. In order to make the usual partial wave expansion, it is necessary to make approximations which remove this coupling between the directions of $\mathbf{p}$ and $\mathbf{k}''$. We have made the following approximations:

(A) In Eq. (31) we replace $f(p,k'')$ by its angular average

$$f(p,k'')\rightarrow\frac{1}{4\pi}\int d\hat{p}\,f(p,k'')=0,\ (k''^2+\tfrac{1}{4}p^2)^{\frac{1}{2}}<p_F$$
$$=1,\ k''-\tfrac{1}{2}p>p_F \quad (34)$$
$$=\frac{k''^2+\frac{1}{4}p^2-p_F^2}{k_p''}\ \text{otherwise.}$$

Here $\hat{p}$ is a unit vector in the direction of $\mathbf{p}$, and $\int d\hat{p}$ means integration over angles. This approximation derives its validity from the weak dependence of the K matrix on the details of the treatment of the exclusion effect. We believe the approximation Eq. (34) to be good, since most of the effect of the exclusion principle in the Green's function [Eq. (31)] is included correctly.

(B) Approximation A does not yet remove the dependence of the summand in Eq. (31) on the angle between $\mathbf{p}$ and $\mathbf{k}''$. This angle still occurs in $E_1(p_+,\Sigma)$ and $E_1(p_-,\Sigma)$. We eliminate the dependence of $E_1(p_+,\Sigma)$ and $E_1(p_-,\Sigma)$ on the angles in essentially the same way as we eliminated the dependence of $f(p,k'')$ on this angle. Consider the polynomial expansion of $E_1(p_+,\Sigma)$ $E_1(p_-,\Sigma)$:

$$E_1(p_+,\Sigma)+E_1(p_-,\Sigma)$$
$$=A+B(p_+^2+p_-^2)+C(p_+^4+p_-^4)+\text{etc.} \quad (35)$$

Substituting Eq. (19) into Eq. (35), we see that the angle between $\mathbf{p}$ and $\mathbf{k}''$ first occurs in the quartic term. This we eliminate from the quartic term by replacing $(\mathbf{p}\cdot\mathbf{k}'')^2$ by its angular average

$$(\mathbf{p}\cdot\mathbf{k}'')^2\rightarrow(1/4\pi)\int d\hat{p}(\mathbf{p}\cdot\mathbf{k}'')^2f(p,k'') \quad (36)$$

[compare this Eq. (36) to Eq. (34)]. To fourth order in the energy expansion, this replacement Eq. (36) is equivalent to the replacements

$$p_\pm^2\rightarrow\tfrac{1}{4}p^2+k''^2\pm(1/\sqrt{3})f^2(p,k'')pk'', \quad (37)$$

where $f(p,k'')$ is the averaged $f(p,k'')$.

To justify Eq. (37) we make the following observations. The angular coupling is absent in the dominant constant and quadratic terms in $E_1(p_+,\Sigma)+E_1(p_-,\Sigma)$. We expect the quartic term to be important only for $k''\gg p$, but in this case p_+ and p_- will be large so that $E_1(p_+,\Sigma)+E_1(p_-,\Sigma)=(p_+^2+p_-^2)/2M=(\frac{1}{4}p^2+k''^2)/M$ exactly, an equality which also follows from Eq. (37). Thus Eq. (37) is correct for $k''\gg p$ also.

We have attempted no further justification of approximations (A) or (B). We believe them to have an effect which is negligible compared with uncertainties resulting from our treatment of off-the-energy-shell propagation and our neglect of the linked-cluster corrections. The partial wave expansion now proceeds exactly as it does in the usual two-body scattering problem. We repeat the derivations here for completeness. The plane wave functions φ_k of Eq. (24), which

are the natural basis functions to use in this problem, have the partial wave expansion (we now include the spin functions explicitly)

$$\varphi_k(r)=\exp(i\mathbf{k}\cdot\mathbf{r})\chi_s^{m_s}$$
$$=\sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}Y_l^0(\hat{k},\hat{r})\chi_s^{m_s}. \quad (38)$$

We mean here and in the rest of this section by $f(\hat{k},\hat{r})$ a function of the angles between $\hat{k}$ and $\hat{r}$ and the arbitrary reference plane chosen. In the presence of the tensor force,[26] the orbital angular momentum l is not a constant of the motion and it is necessary to re-express this expansion in terms of the eigenfunctions of total angular momentum $F_L{}^{Jm_Js}$. These are related to the $Y_L{}^{m_L}\chi_s{}^{m_s}$ by the equation

$$F_L{}^{Jm_Js}=\sum_{m_s=-1}^{1}Y_L{}^{m_J-m_s}\chi_s{}^{m_s}C(Jm_j;L,m_j-m_s;s,m_s), \quad (39)$$

where the C's are Clebsch-Gordan coefficients. We also need the inverse transformation

$$Y_L{}^{m_L}\chi_s{}^{m_s}=\sum_{J=L-1}^{L+1}F_L{}^{Jm_L+m_s s}C(Jm_L+m_s;Lm_L,sm_s). \quad (40)$$

The functions $F_L{}^{Jm_Js}$ are eigenfunctions of J and L; they are defined only for $L=J$ for the singlet state with $s=0$, and only for $L=J$ or $J\pm1$ for the triplet state with $s=1$. The tensor force couples the triplet states with the same J and parity, that is, the states with $s=1$ and $L=J\pm1$. The state with $s=1$ and $L=J$ does not couple with any other state since the triplet states with $L=J\pm1$ have opposite parity.

Putting Eq. (40) with $m_L=0$ in Eq. (38) gives

$$\exp(i\mathbf{k}\cdot\mathbf{r})\chi_s{}^{m_s}=\sum_{J=0}^{\infty}\sum_{l=J-1}^{J+1}(2l+1)i^l j_l(kr)\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}$$
$$\times F_l{}^{Jm_s s}(\hat{k},\hat{r})C(Jm_s;l0;sm_s). \quad (41)$$

We also introduce a corresponding expansion for the coordinate space wave function ψ. To do this we must treat the coupled states with $L=J\pm1$ with particular care. We note that due to the coupling introduced by the tensor force, the unperturbed state with $L=J-1$ goes over into a mixture of states with $L=J\pm1$, and similarly for the unperturbed state with $L=J+1$. Thus it is necessary to distinguish the two solutions generated from the two orthogonal unperturbed solutions. The notation we adopt for the two solutions generated by the two channels $L=J\pm1$ is that under the action of the tensor force the unperturbed solution $j_{J-1}F_{J-1}{}^{Jm_Js}$ goes over into

$$U_{J-1,\,J-1}{}^{Js}F_{J-1}{}^{Jm_Js}+U_{J-1,\,J+1}{}^{Js}F_{J+1}{}^{Jm_Js}, \quad (42)$$

and the unperturbed solution $j_{J+1}F_{J+1}{}^{Jm_Js}$ into

$$U_{J+1,\,J-1}{}^{Js}F_{J-1}{}^{Jm_Js}+U_{J+1,\,J+1}{}^{Js}F_{J+1}{}^{Jm_Js}.$$

We call the solutions for the uncoupled triplet and singlet states $U_{JJ}{}^{Js}$, $s=1$ and $s=0$, respectively. Following this notation, the partial wave expansion which we introduce for ψ is (suppressing the subscript p on ψ)

$$\psi_k(r)=\sum_{m_s'=-1}^{1}\psi_k{}^{s,m_sm_s'}(r)\chi_s{}^{m_s'}$$
$$=\sum_{J=0}^{\infty}\sum_{l=J-1}^{J+1}(2l+1)i^l\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}\sum_{l'=J-1}^{J+1}U_{ll'}{}^{Js}(r)$$
$$\times F_{l'}{}^{Jm_ss}(\hat{k},\hat{r})C(Jm_s;l0;sm_s). \quad (43)$$

The notation on the left-hand side has the following meaning: m_s is not a constant of the motion, so that if the unperturbed $\varphi_k(r)$ in Eq. (30) is given by Eqs. (38) or (41), the solution $\psi_k(r)$ of Eq. (30) will have parts corresponding to $m_s'=-1, 0, 1$; we have to think of an entrance channel m_s and an exit channel m_s' (spin flips).

We also need the angular momentum expansion of the Green's function, which is

$$G(r,r')=\sum(2l+1)G_l(r,r')Y_l^0(\hat{r},\hat{r}')[4\pi/(2l+1)]^{\frac{1}{2}},$$
$$G_l(r,r')=\int_0^{\infty}k''^2dk''\frac{j_l(k''r)j_l(k''r')f(p,k'')}{\Sigma-E_1(p_+,\Sigma)-E_1(p_-,\Sigma)}. \quad (44)$$

Finally, we insert the expansions Eqs. (41), (43), and (44) for $\varphi_k(r)$, $\psi_k(r)$, and G into Eq. (30), and drop the sums over J since J is a constant of the motion. The result is

$$\sum_{ll'}(2l+1)i^l\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}U_{ll'}{}^{Js}(r)F_{l'}{}^{Jm_ss}(\hat{k},\hat{r})C(Jm_s;l0;sm_s)$$
$$=\sum_l(2l+1)i^l\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}j_l(kr)F_l{}^{Jm_ss}(\hat{k},\hat{r})C(Jm_s;l0;sm_s)$$
$$+\sum_l\int dr'(2l+1)\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}G_l(r,r')Y_l^0(\hat{r},\hat{r}')V(r')$$
$$\times\sum_{l'l''}(2l'+1)i^{l'}[4\pi/(2l'+1)]^{\frac{1}{2}}U_{l'l''}{}^{Js}(r')$$
$$\times F_{l''}{}^{Jm_ss}(\hat{k},\hat{r}')C(Jm_s;l'0;sm_s). \quad (45)$$

In the last term, the integration over the angles of r' gives

$$(2l+1)\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}\int d\hat{r}'Y_l^0(\hat{r},\hat{r}')V(r')F_{l''}{}^{Jm_ss}(\hat{k},\hat{r}')$$
$$=4\pi F_l{}^{Jm_ss}(\hat{k},\hat{r})V_{ll''}{}^{Js}(r'). \quad (46)$$

When $\hat{k}\cdot\hat{r}=1$, Eq. (46) follows from the definition of $V_{ll''}{}^{Js}$ [see Eq. (62)]. We have used the addition

[26] The most satisfactory developments of scattering theory including the effects of tensor forces are by J. Ashkin and Ta-You Wu, Phys. Rev. **73**, 973 (1948), and J. Blatt and L. J. Biedenharn, Phys. Rev. **86**, 399 (1952).

theorem for Y_l^0

$$Y_l^0(\hat{r},\hat{r}')=\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}\sum_{m=-1}^{1}Y_l^m(\hat{k},\hat{r}')\times Y_l^m(\hat{k},\hat{r}),$$

and Eqs. (39) and (40) in deriving Eq. (46). For a tensor potential, $S_{12}V_T(r')$, where S_{12} is the tensor operator

$$V_{ll''}{}^{Js}(r')=S_{ll''}{}^{Js}V_T(r'), \tag{47}$$

where the S's are numbers given explicitly by Ashkin and Wu,[26] Eq. (26), for $s=1$. For $s=0$, a tensor potential gives $V_{ll''}{}^{Js}=0$, that is, a tensor potential gives no effect in singlet states. For a central potential $V_c(r')$, $V_{ll''}{}^{Js}(r')=\delta_{ll''}V_c(r')$, of course.

To complete the derivation, we put Eq. (46) into Eq. (45) and take the scalar product of the result with $F_{L'}{}^{Jm_ss}(\hat{k},\hat{r})$, integrate over the angles of r, and find (changing the dummy index l to l' on the left-hand side of the equation)

$$\sum_{l'}(2l'+1)i^{l'}\left(\frac{4\pi}{2l'+1}\right)^{\frac{1}{2}}U_{l'L'}{}^{Js}(r)C(Jm_s;l'0;sm_s)$$

$$=(2L'+1)i^{L'}\left(\frac{4\pi}{2L'+1}\right)^{\frac{1}{2}}j_{L'}(kr)C(Jm_s;L'0;sm_s)$$

$$+4\pi\sum_{l'l''}\int r'^2dr'G_{L'}(r,r')V_{L'l''}(2l'+1)i^{l''}$$

$$\times\left(\frac{4\pi}{2l'+1}\right)^{\frac{1}{2}}U_{l'l''}{}^{Js}(r')C(Jm_s;l'0;sm_s). \tag{48}$$

Multiplying through by $C(Jm_s;L0;sm_s)$, summing over m_s, and using

$$\sum_{m_s=-1}^{1}C(Jm_s;L0;sm_s)C(Jm_s;l'0;sm_s)=\delta_{Ll'}\frac{2J+1}{2l'+1} \tag{49}$$

gives (replacing L by l and L' by l')

$$U_{ll'}{}^{Js}=j_l(kr)\delta_{ll'}$$

$$+4\pi\sum_{l''}\int r'^2dr'G_{l'}(r,r')V_{ll''}{}^{Js}(r')U_{ll''}{}^{Js}(r'). \tag{50}$$

For $s=1$, it is easy to put Eq. (50) into matrix form similar to that used in Eq. (64) later. G and j as diagonal matrices, and j, V, U appear as they do in Eq. (64). For $s=0$, only $l=l'=J$ is possible and Eq. (50) has a much simpler appearance.

It is necessary to express the K matrices Eq. (32) in terms of the U's. This is done in Sec. III-D. Section IV is devoted to expressing the energies given by Eq. (11) or Eq. (14) in terms of this partial wave expansion.

Equation (50) [modified to take account of the hard core; see next section Eq. (60)] was used directly in the numerical calculations. Further details of the numerical work appear in Sec. V.

C. Removal of Difficulties Associated with the Presence of a Hard Core in the Potential

When a hard core is present in the potential, Eq. (50) presents a difficulty since the integral on the right-hand side runs over the core region also, and in the core region $V(r')=+\infty$, so that the integral is finite only because the wave function $U(r')$ vanishes in the core region. The value of the integral appears to be indeterminate. In this section we show how this difficulty can be overcome by altering the form of Eq. (50). The procedure is similar to the procedure used in avoiding the same difficulty in the usual two-body scattering problem.

The procedure is the same for all angular momentum states, so that for simplicity we consider an equation similar to Eq. (50) except that all indices are suppressed; namely,

$$U(r)=U_0(r)+4\pi\int_0^\infty G(r,r')V(r')U(r')r'^2dr'. \tag{51}$$

We next separate the integral on the right-hand side of this Eq. (51) into two parts

$$\int_0^\infty=\int_0^{r_c}+\int_{r_c}^\infty, \tag{52}$$

where r_c is the radius of the hard core. The second integral on the right-hand side of Eq. (52) causes no difficulty since the potential is finite for $r'>r_c$. In the first integral on the right-hand side $V(r')=+\infty$ and $U(r')=0$ for hard cores, but the product $V(r')U(r')$ is indeterminate. In the usual scattering problem it is exact (as we discuss below) to make the replacement

$$V(r')U(r')=\lambda\delta(r'-r_c),\quad r'\leqslant r_c, \tag{53}$$

where λ is determined by the condition that $U(r_c)=0$. Equation (53) says that the entire value of the first integral on the right-hand side of Eq. (52) comes from the core edge. This is easily seen to be correct in the scattering case by considering the behavior of VU for a square well repulsion. In this case the wave function for $r<r_c$ is $U=C\sinh\alpha r$ where C is a constant and α increases as the square root of the depth of the repulsive square well. For α large, U will increase very rapidly as r approaches r_c, and the product VU will also increase very rapidly, since V is constant (square well). Therefore, the product VU must behave very much as indicated in Eq. (53). Putting Eq. (53) into Eq. (51), we find

$$U(r)=U_0(r)+4\pi r_c^2G(r,r_c)\lambda$$

$$+4\pi\int_{r_c}^\infty G(r,r')V(r')U(r')r'^2dr'. \tag{54}$$

The condition $U(r_{\text{core}})=0$ then fixes λ, which is

$$\lambda=-\left[U_0(r_c)+4\pi\int_{r_c}^\infty G(r_c,r')V(r')U(r')r'^2dr'\right]\Big/$$

$$4\pi r_c^2G(r_c,r_c). \tag{55}$$

Inserting this value of λ into Eq. (54), we find

$$U(r)=S(r)+4\pi\int_{r_c}^{\infty}F(r,r')V(r')U(r')r'^2dr', \quad (56)$$

where

$$S(r)=U_0(r)-U_0(r_c)\frac{G(r,r_c)}{G(r_c,r_c)},$$
$$F(r,r')=G(r,r')-\frac{G(r,r_c)G(r_c,r')}{G(r_c,r_c)}. \quad (57)$$

The right-hand side of Eq. (56) should vanish for $r<r_c$. In the usual scattering problem, for which

$$U_0(r)=j_l(kr),$$
$$G(r,r')=-(Mk/4\pi)j_l(kr_<)n_l(kr_>), \quad (58)$$

where $r_<$ and $r_>$ are the lesser and greater of r and r', respectively, it follows immediately that the right-hand side of Eq. (56) vanishes for $r<r_c$. In the usual scattering problem

$$(\nabla^2+k^2)U_0(r)=0,$$
$$(\nabla^2+k^2)G(r,r')=\delta(r-r'), \quad (59)$$

and it follows from these that solutions of Eqs. (56) and (57) are solutions of the differential Schrödinger equation, and so solutions of Eq. (51). These facts prove the correctness of Eq. (53) in the usual two-body scattering problem.

In our problem, the Green's function does not have the simple property expressed in the second of Eqs. (58), and we cannot show that the right-hand side of Eq. (56) vanishes for $r<r_c$; similarly, we cannot show that solutions of Eqs. (56) and (57) are solutions of Eq. (51). However, Bethe and Goldstone[27] show that replacing Eq. (51) by Eqs. (56) and (57) for $r>r_c$ and setting $U(r)=0$ for $r<r_c$ leads to negligible error in the present problem. This point is also discussed in Appendix B.

The above procedure may be applied with trivial alterations to the coupled triplet states. To summarize, in Eq. (50), we replace j_l and G_l by

$$s_l(kr)=j_l(kr)-j_l(kr_0)\frac{G_l(r,r_c)}{G_l(r_c,r_c)},$$
$$F_l(r,r')=G_l(r,r')-\frac{G_l(r,r_c)G_l(r_c,r')}{G_l(r_c,r_c)}, \quad (60)$$

respectively.

D. Determination of the *K*-Matrix Elements

We next need to determine from the solutions of the integral equations the necessary K-matrix elements. We find the diagonal elements of K by substituting Eqs. (43) and (41) into Eq. (32), and making use of the fact that V is diagonal in J:

$$(k,s,m_s|K|k,s,m_s)=\sum_J\int d\mathbf{r}\sum_{l=J-1}^{J+1}(2l+1)(-i)^l$$
$$\times\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}j_l(kr)F_l{}^{Jm_ss}(\hat{k},\hat{r})C(Jm_s;l0;sm_s)$$
$$\times V(r)\sum_{l'=J-1}^{J+1}(2l'+1)i^{l'}\left(\frac{4\pi}{2l'+1}\right)^{\frac{1}{2}}$$
$$\times\sum_{l''=J-1}^{J+1}U_{l'l''}{}^{Js}F_{l''}{}^{Jm_ss}(\hat{k},\hat{r})C(Jm_s;l'0;sm_s). \quad (61)$$

Now

$$\int d\hat{r}F_l{}^{Jm_ss}(\hat{k},\hat{r})V(r)F_{l''}{}^{Jm_ss}(\hat{k},\hat{r})=V_{ll''}{}^{Js}, \quad (62)$$

independently of m_s, so that the matrix elements of K depend on m_s only through the Clebsch-Gordan coefficients. Summing over m_s, since it is this sum which appears in the energy expressions (see next section, Sec. IV), and using the theorem Eq. (49), we obtain

$$\sum_{m_s}(k,s,m_s|K|k,s,m_s)=4\pi\sum_J\sum_{l=J-1}^{J+1}(2J+1)$$
$$\times\int_0^{\infty}r^2drj_l(kr)\sum_{l'=J-1}^{J+1}V_{ll'}{}^{Js}U_{ll'}{}^{Js}. \quad (63)$$

It is interesting to note that the right-hand side of this equation can be written as the trace of a matrix product for $s=1$; that is,

$$\sum_{m_s}(k,1,m_s|K|k,1,m_s)=4\pi\sum_J(2J+1)\int_0^{\infty}r^2dr\ \mathrm{Trace}\begin{bmatrix}j_{J-1}&0&0\\0&j_J&0\\0&0&j_{J+1}\end{bmatrix}$$
$$\times\begin{bmatrix}V_{J-1,J-1}&0&V_{J-1,J+1}\\0&V_{J,J}&0\\V_{J+1,J-1}&0&V_{J+1,J+1}\end{bmatrix}\begin{bmatrix}U_{J-1,J-1}&0&U_{J-1,J+1}\\0&U_{JJ}&0\\U_{J+1,J-1}&0&U_{J+1,J+1}\end{bmatrix}. \quad (64)$$

[27] H. Bethe and J. Goldstone, Proc. Roy. Soc. (London) **A238**, 551 (1956).

Superscripts J, $s=1$ on V and U have been suppressed. For the singlet states, ($s=0$), Eq. (63) or Eq. (69) simplifies considerably, of course, since only $l=l'=J$ is possible.

E. Effect of the Hard Core on the K Matrices

The integral in Eq. (63) or Eq. (64) cannot be performed over the core region without using Eqs. (53) and (55). Proceeding just as we did in paragraph III-C (the core potential is diagonal in l and l'), we find Eq. (63) replaced by

$$\sum_{m_s}(k,s,m_s|K|k,s,m_s)=4\pi\sum_J(2J+1)\sum_{l=J-1}^{J+1}\left\{\frac{-j_l^2(kr_c)}{G_l(r_c,r_c)}+\int_{r_c}^{\infty}r^2drs_l(kr)\sum_{l'=J-1}^{J+1}V_{ll'}{}^{Js}U_{ll'}{}^{Js}\right\},\quad(65)$$

where s_l is given by Eq. (60).

IV. DETERMINATION OF THE SINGLE-PARTICLE ENERGIES

The procedure of the last section (Sec. III) determines the diagonal elements of the K matrix. From these, we can now evaluate the single-particle energies as given by Eq. (11) or Eq. (14).

In Eq. (11), the particle whose momentum is m_0 is either a proton (+) or neutron (−), and has its spin either up (α) or down (β). The sum over n_0 runs over particles in plane wave states whose spins are up and down and over both neutrons and protons. Thus, the energy of a (for example) proton whose spin is up is

$$\begin{aligned}E_0(m_0,+,\alpha)=&\frac{m_0^2}{2M}+\sum_{n_0}(m_0,+,\alpha;\,n_0,+,\alpha|K|m_0,+,\alpha;\,n_0,+,\alpha)\\&+\sum_{n_0}(m_0,+,\alpha;\,n_0,+,\beta|K|m_0,+,\alpha;\,n_0,+,\beta)\\&+\sum_{n_0}(m_0,+,\alpha;\,n_0,-,\alpha|K|m_0,+,\alpha;\,n_0,-,\alpha)\\&+\sum_{n_0}(m_0,+,\alpha;\,n_0,-,\beta|K|m_0,+,\alpha;\,n_0,-,\beta)\\&-\text{exchange}.\end{aligned}\quad(66)$$

It is conceivable that the upper limits of the four sums in Eq. (66) differ. For example, the proton density might not equal the neutron density. If they are not equal, we have to think of four Fermi momenta,

$$p_F(+,\alpha),\quad p_F(+,\beta),\quad p_F(-,\alpha),\quad p_F(-,\beta).\quad(67)$$

The first of these, for example, is the Fermi momentum for protons whose spin is up.

We have not kept this complication in the problem. One can imagine the complications which would come about because of the occurrence of four $f(p,k')$ referring to different Fermi momenta in the Green's function, Eq. (44). Also the E_1's in Eq. (44) would depend on the z component of the spin and isotopic spin of the particle referred to. However, the complication is not great in the present Sec. IV, and we retain it to facilitate a discussion of the symmetry energy later on.

We must change from spin functions like $\alpha\beta$, which are not eigenfunctions of the total spin, to spin functions $\chi_s^{m_s}$ used previously which are eigenfunctions of the total spin. As is well known,

$$\begin{aligned}\alpha\alpha&=\chi_1^1,\\\alpha\beta&=2^{-\frac{1}{2}}(\chi_1^0+\chi_0^0),\\\beta\alpha&=2^{-\frac{1}{2}}(\chi_1^0-\chi_0^0),\\\beta\beta&=\chi_1^{-1}.\end{aligned}\quad(68)$$

There are no elements of K which connect states of different total spin and no diagonal (forward or backward scattering) elements of K which connect states of different z component of the total spin.

There are equations just like Eqs. (68) for the isotopic spin; for example,

$$++=T_1^1.\quad(69)$$

Because spin and parity are conserved in the two-body interaction, the total isotopic spin is also conserved. Therefore K has no elements which connect states of different total isotopic spin or different z components of the total isotopic spin.

In this way, we find for Eq. (66)

$$\begin{aligned}E_0(m_0,+,\alpha)=&\frac{m_0^2}{2M}\\&+\sum_{n_0}(m_0;n_0|K(s=1,m_s=1,T=1,T_z=1)|m_0;n_0)\\&+\tfrac{1}{2}\sum_{n_0}[(m_0;n_0|K(s=1,m_s=0,T=1,T_z=1)|m_0;n_0)\\&+(m_0;n_0|K(s=0,m_s=0,T=1,T_z=1)|m_0;n_0)]\\&+\tfrac{1}{2}\sum_{n_0}[(m_0;n_0|K(s=1,m_s=1,T=1,T_z=0)|m_0;n_0)\\&+(m_0;n_0|K(s=1,m_s=1,T=0,T_z=0)|m_0;n_0)]\\&+\tfrac{1}{4}\sum_{n_0}[(m_0;n_0|K(s=1,m_s=0,T=1,T_z=0)|m_0;n_0)\\&+(m_0;n_0|K(s=0,m_s=0,T=1,T_z=0)|m_0;n_0)\\&+(m_0;n_0|K(s=1,m_s=0,T=0,T_z=0)|m_0;n_0)\\&+(m_0;n_0|K(s=0,m_s=0,T=0,T_z=0)|m_0;n_0)]\\&-\text{exchange}.\end{aligned}\quad(70)$$

We recall that the K's depend on m_s only through certain Clebsch-Gordan coefficients [see Eq. (61)]. This assures that the K's depend only on the magnitude of m_s, not its sign. The exchange terms will just give a factor 2 everywhere because of Pauli's principle:

$$\begin{aligned}&(\text{spin exchange})(\text{space exchange})\\&\qquad\times(\text{isotopic spin exchange})=-1.\end{aligned}\quad(71)$$

We now ignore the complication of four Fermi momenta and say they are all the same, and assume that the potentials are charge-symmetric so that they are independent of T_z and consequently the K's are independent of T_z. Changing the sign of some m_s in Eq. (70), we then find [dropping the $+$, α in $E_0(m_0,+,\alpha)$ because E_0 no longer depends on them]

$$E_0(m_0)=\frac{m_0^2}{2M}+\sum_{n_0}\sum_{T=0}^{1}\sum_{s=0}^{1}\frac{(2T+1)(2s+1)}{2}\times\sum_{m_s=-1}^{1}(m_0,n_0|K(s,m_s,T)|m_0,n_0). \quad (72)$$

There is no need to repeat Eq. (65) for $\sum_{m_s}$. The isotopic spin T only comes in through the parity dependence of the potentials.

Once we have the single-particle energy spectrum, it is a simple matter to compute the average binding energy per particle, E_b. If we define $V_0(m_0)$ by

$$E_0(m_0)=(m_0^2/2M)+V_0(m_0), \quad (73)$$

so that V_0 is the potential energy, E_b is given by

$$E_b=\int_0^{p_F}m_0^2dm_0\left[\frac{m_0^2}{2M}+\tfrac{1}{2}V_0(m_0)\right]\Big/\int_0^{p_F}m_0^2dm_0. \quad (74)$$

The factor $\frac{1}{2}$ in front of V_0 arises from the fact that V_0 arises from interactions of pairs of particles, and we must now count the potential energy of a pair twice.

V. COMPUTATIONAL PROCEDURE

The problem is to find a self-consistent solution[28] of Eqs. (11)–(15). The self-consistency problem arises because the energies appearing in the Green's functions (energy denominators) of the integral Eqs. (12) and (15) have an important effect on the K matrices, and the diagonal elements of the K matrices in turn determine the energies according to Eqs. (11)–(14). It was found that a self-consistent solution could be reached by iteration starting from any reasonable guess for $E_0(m_0)$ and $E_1(m_1,\Sigma)$, the simplest guess being

$$E_0(m_0)=m_0^2/2M, \quad E_1(m_1,\Sigma)=m_1^2/2M. \quad (75)$$

$E_0(m_0)$ and $E_1(m_1,\Sigma)$ were kept in the form of tables. M_0 took on the values $m_0=0.1p_F(0.2p_F)0.9p_F$. m_1 took on the values $m_1=1.1p_F(0.5p_F)2.1p_F$. For $m_1\geq 2.6p_F$, it was assumed that $E_1(m_1,\Sigma)=m_1^2/2M$ for all Σ, an assumption which was checked by calculation. Σ took on the values

$$\Sigma=2E_0(0)+f[2E_0(p_F)-2E_0(p_F)]$$
$$f=0.05(0.225)0.95. \quad (76)$$

Thus a table of $E_0(m_0)$ and $E_1(m_1,\Sigma)$ (an energy table) has the form shown in Table I.

[28] For a detailed discussion of the self-consistency problem see references 5 and 6.

TABLE I. Typical energy table used in computation.

On energy shell		Off energy shell					
m_0/p_F	$E_0(m_0)$	m_1/p_F	$f=0.05$	$f=0.225$	$f=0.500$	$f=0.725$	$f=0.900$
0.1	x	1.1	x	x	x	x	x
0.3	x	1.6	x	x	x	x	x
0.5	x	2.1	x	x	x	x	x
0.7	x	2.6					
0.9	x	↓		$E_1(m_1,\Sigma)=m_1^2/2M$			

An iteration consists of producing a new energy table from an old energy table; we know the old energy table either from a guess like Eq. (75) or as a result of the previous iteration.

The new energies are given by Eqs. (11) and (14), which are not used directly but replaced by equations like Eq. (72). The sum over m_s which appears in Eq. (72) is given by Eq. (65); this sum over m_s depends on

$$k=\tfrac{1}{2}|m_0-n_0|, \quad \text{and} \quad p=|m_0+n_0|, \quad (77)$$

or on the magnitude of n_0 and the polar angle θ of n_0 referred to m_0 as polar axis. The sum over n_0 which appears in Eq. (72) is replaced by an integral over the Fermi sea; the K matrices have to be calculated for enough values of n_0 to make it possible to perform this integral numerically. The mesh of n_0, θ_0 (or n_1, θ_1) used was

$$n_0=0.1p_F(0.2p_F)0.9p_F,$$
$$\cos\theta_0=-0.8(0.4)0.8. \quad (78)$$

That this mesh suffices was checked by refining it; a refined mesh leads to the same new energies. Actually, it was found that a much cruder mesh suffices; this is discussed below.

The sum over m_s which appears in Eq. (72) also depends on s and T, and a sum over J appears in Eq. (65). That is, the new energies are also sums of diagonal elements of K matrices for several different states of the two-nucleon system. The states we consider are

$$^3S_1+{}^3D_1, \quad ^3D_2, \quad ^3D_3, \quad ^1S_0, \quad ^1D_1. \quad (79)$$

That is, we consider all even parity states for $l\leq 2$. The odd-parity states are ignored for reasons given in the discussion (Sec. VI). $l=4$ was found not to affect the self-consistent solutions.

The new energies are given in terms of wave functions through Eqs. (65) and (72). The wave functions $U_{ll'}{}^{Js}$ which appear in Eq. (65) depend on m_0 (or m_1), n_0, θ_0, (or n_1,θ_1) and the state of the two nucleon system. The wave function depends on Σ, which is given by Eq. (13) if we are calculating a new $E_0(m_0)$ or is one of the five values corresponding to the five f values in Eq. (76) if we are calculating $E_1(m_1,\Sigma)$.

The wave functions are solutions of integral equations like Eq. (50). These equations were solved by replacing them by sets of simultaneous linear equations

in a well-known way. The mesh used was

$$r,r'=0.6(0.1)1.3(0.2)2.5(0.5)6.0\times10^{-13}\text{ cm}, \quad (80)$$

for $r_{\text{core}}=0.5\times10^{-13}$ cm and a similar mesh for $r_{\text{core}}=0.2\times10^{-13}$ cm. That this mesh suffices was checked by refining it. The really difficult part of solving the integral equations was in constructing the Green's functions according to Eq. (44). This was done by numerical integration; the k'' mesh used was

$$k''=0(0.5)(10^{-13}\text{ cm})^{-1}, \quad (81)$$

with analytical corrections applied for $k^{-1}>(10^{-13}$ cm$)^{-1}$ (see below). That this mesh suffices was checked by calculation with a finer mesh.

The old energy tables were used to obtain the E_1's which appear in the denominator of the integrand in Eq. (44). The old energy tables were interpolated quadratically in m_1 [and m_0 also to get $\Sigma=E_0(m_0)+E_0(n_0)$ for on-the-energy-shell propagation] and linearly in Σ to obtain $E_1(m_1,\Sigma)$. (It is at this point the self-consistency problem occurs, of course.)

At this point some idea of the enormous magnitude of the calculation emerges. For each entry in the energy table (20 in all) we must construct a new entry which is a sum of two-body interactions. The sum is replaced by an integral over the Fermi sea; the mesh of n_0 consists of 5 magnitudes of n_0 and 5 angles of n_0 relative to m_0. We must construct the Green's function for every value of r and r' (441 in all) for $l=0$ and 2, and for every entry in the table and every magnitude and angle of r_0. This is $441\times20\times5\times5\times2=441\,000$ Green's functions in all, each of which is a numerical integral as shown in Eq. (44).

And yet an iteration took only four hours. There is no point in describing the calculation in more detail; we have given all the meshes we used and the techniques were all straightforward.

We did effect a saving in calculating time by using a cruder mesh of n_0 and $\cos\theta_0$. This mesh was based on the following considerations. The K matrices depend mainly on the relative momentum

$$\mathbf{k}_0=\tfrac{1}{2}(\mathbf{m}_0-\mathbf{n}_0), \quad (82)$$

and have a polynomial expansion like Eq. (35),

$$K(k_0)=A+Bk_0^2+Ck_0^4+\text{etc.} \quad (83)$$

Also, we do not expect these expansions to require many terms. If only the terms through the coefficient C are retained, K depends on $\cos\theta_0$ only quadratically. It is obvious that

$$\int_{-1}^{1}d\mu f(\mu)=f(+\sqrt{\tfrac{1}{3}})+f(-\sqrt{\tfrac{1}{3}}), \quad (84)$$

if $f(\mu)$ is of the form $a+b\mu+c\mu^2+d\mu^3$. We found that the integration rule Eq. (84) works very well when applied to integrating the K matrices over the angles of n_0, and so replaced the mesh

$$\cos\theta_0=-0.8(0.4)0.8, \quad (85)$$

by

$$\cos\theta_0=-\sqrt{\tfrac{1}{3}},+\sqrt{\tfrac{1}{3}}, \quad (86)$$

reducing the time requirement by a factor 0.4.

If Eq. (83) is as accurate as this result suggests, it ought also be possible to carry out the integral over the magnitude of n_0 in the same way. It is easily seen that

$$\int d\hat{n}_0K(k)=f(n_0^2), \quad (87)$$

that is, the integral is a function of n_0^2 only. It is easily checked that

$$\int_0^{p_F}n_0^2dn_0f(n_0^2)=f[(0.5814968p_F)^2]+f[(0.92836494p_F)^2], \quad (88)$$

if $f(n_0^2)$ is of the form $a+bn_0^2+cn_0^4$. We found the integration rule Eq. (88) works very well also. Thus we replaced the mesh

$$n_0=0.1p_F(0.2p_F)0.9p_F, \quad (89)$$

by

$$n_0=0.5814968p_F,\quad 0.92836494p_F, \quad (90)$$

reducing the time requirements by another factor 0.4.

The two reductions might have been expected to cut the time from four hours to thirty-eight minutes; actually forty-five minutes were required per iteration.

Five iterations starting from the guess Eq. (75) sufficed to make the new energy table agree with the old to one part in 10^5.

The analytical correction to the Green's function integral referred to above is the following. For $k''>10$ $(10^{-13}$ cm$)^{-1}$, $E_1(p_+,\Sigma)+E_1(p_-,\Sigma)=k''^2/M$ and Σ is negligible compared to $E_1(p_+,\Sigma)+E_1(p_-,\Sigma)$. The correction is

$$M\int_K^{\infty}dk''j_l(k''r)j_l(k''r'), \quad (91)$$

for which it is easy to obtain an analytical expression for the asymptotic form (valid as $K\to\infty$). For $r=r'$, for example, the correction is

$$\frac{M}{r}\int_{Kr}^{\infty}\frac{\sin^2x}{x^2}dx\approx\frac{M}{2Kr^2}+\frac{\sin2Kr}{4K^2r^3}+\cdots. \quad (92)$$

VI. RESULTS

Note added in proof.—In the computations of this section the contribution of the $^1D_2$ state was inadvertently computed using the triplet central even-state potential. This error has been corrected; the new results are given in Table II together with a summary of the previous results. The considerable shift in the properties shows the sensitivity of the results to the detailed potential form.

TABLE II. Predicted nuclear properties. Column A summarizes the results of this paper; column B gives the results corrected for the singlet D contribution; column C gives the new results obtained from the Gammel-Thaler potentials.

	A	B	C
Binding energy (Mev)	14.6	18.5	15.2
Equilibrium spacing (10^{-13} cm)	1.00	0.95	1.02
Compressibility (Mev)	187	167	172
Effective mass at Fermi surface	$0.66M$	$0.72M$	$0.73M$
Symmetry energy (ratio to ideal Fermi gas)	2.69	2.47	1.69

The calculations of this section have also been repeated using the Gammel-Thaler potentials[29,30] which include spin-orbit forces, and which predict correctly all scattering results in the low and intermediate energy region (0–150 Mev). The computed nuclear properties are given in Table II. The actual potentials used have been slightly modified from those of references 29 and 30 to give correctly the low energy scattering parameters and deuteron properties; the potential parameters are given in Table III.

TABLE III. Constants of the Gammel-Thaler potentials. The potentials have the Yukawa form and repulsive cores with $r_c=0.40\times10^{-13}$ cm.

State	Strength (Mev)	Inverse range $(10^{-13}\text{ cm})^{-1}$
Triplet central even	−877.4	2.091
Triplet tensor even	−159.4	1.045
Singlet even	−434	1.450
$L\cdot S$ even	−5000	3.70
Triplet central odd	−14.0	1.00
Triplet tensor odd	22.0	0.80
Singlet odd	130.0	1.00
$L\cdot S$ odd	−7315	3.70

The substantial change of the symmetry energy from the prediction of the Gammel-Christian-Thaler potentials is caused by the effect of the odd-state central forces where both the repulsive singlet and attractive triplet contributions act to lower the symmetry energy [see Eq. (107)].

In our calculations we have used the Gammel-Christian-Thaler two-body potential.[31] This potential has hard repulsive cores

$$V(r)=+\infty, \quad r<r_c \tag{93}$$

outside of which are central and tensor potentials of Yukawa shape

$$V(r)=V_c\frac{\exp(-\mu_c r)}{\mu_c r}+S_{12}V_T\frac{\exp(-\mu_t r)}{\mu_t r}, \quad r>r_c. \tag{94}$$

The core size is independent of spin and parity. V_c, μ_c, V_t, and μ_t are spin- and parity-dependent ($^3V_c^+$, for example, is the depth of the central potential for triplet even-parity states). The values of the core size, depths, and inverse ranges are

$$r_c=0.5\times10^{-13}\text{ cm},$$

State	$-V_c$ Mev	μ_c $(10^{-13}\text{ cm})^{-1}$	$-V_t$ Mev	μ_t $(10^{-13}\text{ cm})^{-1}$
triplet even	6395	2.936	45	0.73421
singlet even	905.6	1.7	···	···
tripled odd	150	1.5	−57.5	1.12
singlet odd	−113	1.0	···	···

(95)

Gammel, Christian, and Thaler found this potential by fitting the low-energy nucleon-nucleon data (properties of the deuteron and triplet and singlet scattering lengths and effective ranges) and some of the high-energy (up to 310 Mev) nucleon-nucleon data. The p-p polarization data at 100–310 Mev is not fitted by this potential. This fact casts considerable doubt on the triplet odd-parity potentials. In the calculations, we have set the odd-parity potentials, both triplet and singlet, equal to zero. This procedure has the following justification.

Investigations[30] carried out after the completion of this work show that the p-p polarization can be fitted by including a very short-ranged spin-orbit term in the triplet odd-parity potential. In these investigations, it was found that the central term in the triplet odd-parity potential is zero. The tensor and spin-orbit terms in the triplet odd-parity potential give no binding in first order (in which the K-matrix elements are replaced by V-matrix elements; that is, in Born approximation) and the slight binding effect they produce in second order is compensated by the singlet odd-parity repulsion, which though strong, occurs only in isotopic spin $T=0$ states and so has little statistical weight. Investigation of this point using the latest Gammel-Thaler potentials indeed shows that the odd-state contribution is very nearly zero, at least in the phase-shift approximation to the K matrices. [Still more recent work has shown the presence of a short-ranged spin-orbit term in the triplet even-parity potential.[31] These short-ranged terms are not effective at low energies (they begin to be important at 100 Mev), and the properties of nuclear matter depend on nucleon-nucleon scattering in the region 0–100 Mev.] Thus, were we to redo these calculations, we would and could use potentials which give very good fits to all the nucleon-nucleon data in the energy range 0–310 Mev, putting the calculation on a firmer basis.

We return now to a description of the results we obtained with the even parity potentials given in Eq. (95). The results for the binding energy per particle (E_b) are shown in Fig. 1 as a function of r_0 which is defined in terms of the volume per particle (Ω/N) as follows:

$$\Omega/N=\tfrac{4}{3}\pi r_0^3. \tag{96}$$

Two curves are given corresponding to two different values of the average excitation parameter Δ defined in Eq. (5). The lower curve is for $\Delta=0$, the upper for $\Delta=E_0(p_f)-E_0(0)$. Near the energy minimum, the

[29] J. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 (1957).
[30] J. Gammel and R. M. Thaler, Phys. Rev. **107**, 1337 (1957).
[31] Gammel, Christian, and Thaler, Phys. Rev. **105**, 311 (1957).

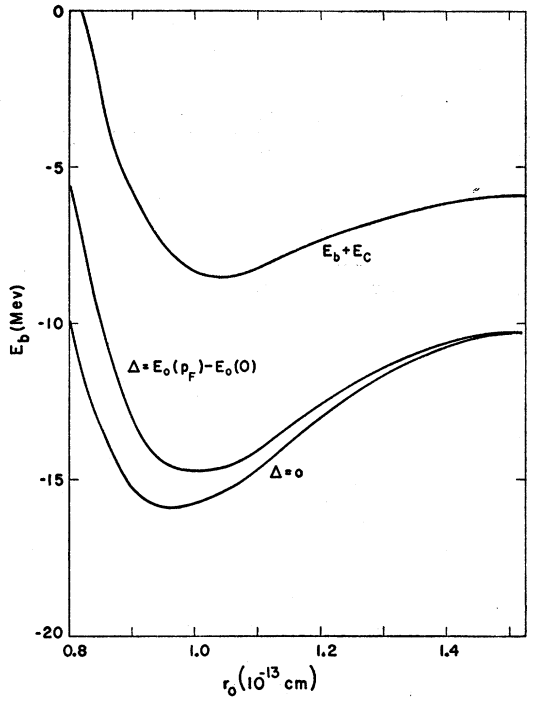

FIG. 1. Binding energy per particle as a function of r_0 [defined by Eq. (103)]. The two lower curves are for $\Delta=0$ (the lowest curve) and $\Delta=E_0(p_F)-E_0(0)$. The uppermost curve is obtained from the curve with $\Delta=E_0(p_F)-E_0(0)$ by adding the Coulomb energy of $Z=82$ protons, taking the nuclear radius to be $R=(2Z)^{\frac{1}{3}}r_0$. (It is assumed that $N=Z$.)

latter value of Δ corresponds to an excitation energy of about 70 Mev. We believe that this value is reasonable although probably somewhat too large, so that the correct solution probably lies close to but below the upper curve. The small effect of the value of Δ on E_b justifies the approximations made in Sec. II, and shows that these approximations do not introduce an error of more than a few tenths of a Mev into E_b. This error is the same order of magnitude as those introduced by the cluster corrections.

The energy minimum in Fig. 1 occurs [for $\Delta=E_0(p_f)-E_0(0)$] at $r_0=1.00\times10^{-13}$ cm and $E_b=-14.6$ Mev/particle. These are to be compared with the value $r_0=1.07\times10^{-13}$ cm deduced from high-energy electron-nucleus scattering and the semiempirical value $E_b=-15.5$ Mev/particle. We have also investigated the effect on the equilibrium density of the Coulomb repulsion, taking as an example $N=Z=82$. The energy curve with the Coulomb energy added is shown in Fig. 1. The position of the energy minimum now occurs close to the empirical value $r_0=1.07\times10^{-13}$ cm, corresponding to a drop of 15–20% in density from the case where Coulomb forces are neglected. Thus the Coulomb repulsion causes an appreciable "blowing up" of the nucleus. It must be emphasized that the position of the energy minimum may be similarly affected by the repulsion arising from the surface energy; also a shift in the position of the energy minimum may occur when the neutron to proton ratio is shifted from unity. We did not estimate these effects from our present results, although we expect them to be no larger than the Coulomb effect.

We have also determined the nuclear compressibility from these results. This was done by fitting a second degree polynomial to the computed energy points at $r_0=0.98$, 1.07, and 1.12×10^{-13} cm. The compressibility is

$$K=r_0^2(d^2E_b/dr_0^2), \tag{97}$$

evaluated for r_0 at the energy minimum. We find $K=187$ Mev, which is a considerably larger value than that determined from a study of the isotope shifts.[32] It is not clear that this difference is meaningful, particularly since our value presumably refers for finite nuclei to the central regions of high density.

We have also determined E_b in the approximation in which the K-matrix elements are those calculated for free scattering. The appropriate limit is[33,34]

$$(k|K_l|k)\rightarrow-(4\pi/Mk)\delta_l(k). \tag{98}$$

The energy *versus* density curve in this approximation is given in Fig. 2; there is no sign of saturation in the vicinity of normal density and the binding energy at normal density is 52.0 Mev. The large discrepancy between this result and that obtained with the exclusion effect and the self-consistency requirements is a measure of their importance.

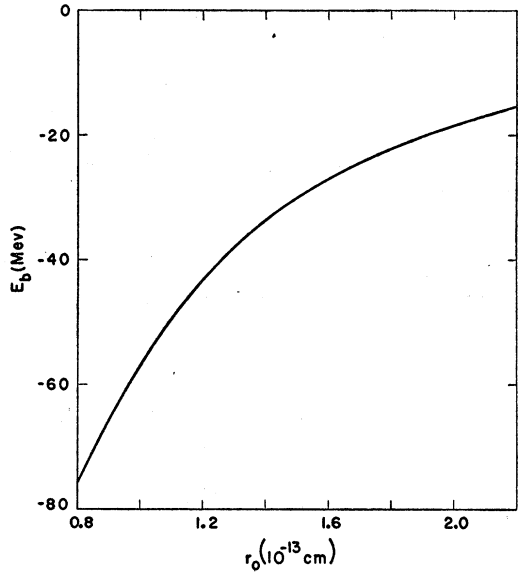

FIG. 2. Binding energy per particle computed directly from free-scattering phase shifts for even states alone. The phase shifts are computed from the same potentials used in the main computations.

[32] K. W. Ford and D. L. Hill, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Stanford, 1955), p. 46.
[33] N. Fukuda and R. Newton, Phys. Rev. **103**, 1558 (1956).
[34] B. de Witt, Phys. Rev. **103**, 1565 (1956).

Another approximation which may be compared with our more exact procedure is that which includes the exclusion principle but neglects self-consistency requirements. This approximation results from setting $E_0(p)=E_1(p,\Sigma)=p^2/2M$ in calculating the Green's functions for the integral equations. For $r_0=0.98\times10^{-13}$ cm this approximation gives $E_b=-34.4$ Mev/particle, whereas our more exact procedure gives $E_b=-14.6$ Mev/particle. This large difference would be absent for potentials which could be treated by a rapidly convergent perturbation expansions such as Yukawa or exponential potentials without repulsive cores.[35] The large effect for the actual potentials considered is a measure of the inadequacy of a perturbation evaluation of the energy.

The energy *versus* density curve has also been determined for a smaller value of the repulsive core radius, $r_c=0.20\times10^{-13}$ cm. These potentials still give a fit to the low-energy data and are assumed to act in even-parity states only. These potentials fail to predict correctly many of the features of high-energy scattering; thus the determination of E_b *vs* r_0 for these potentials is of interest only in showing the sensitivity of the energy to the core size. The results are shown in Fig. 3 as a function of density. There is no saturation near normal density. We conclude that the saturation phenomenon is very sensitive to the radius of the repulsive core, as of course is not unexpected.

A further point of interest is the contribution of the tensor force to the binding energy. This contribution vanishes in first order (Born approximation). The spin average of the tensor operator is zero, so that the magnitude of the tensor contribution is a measure of the importance of the higher order terms in the K matrices. The tensor contribution has been evaluated at $r_0=0.98\times10^{-13}$ cm where it is found to be 6 Mev/particle, or 15% of the potential energy.

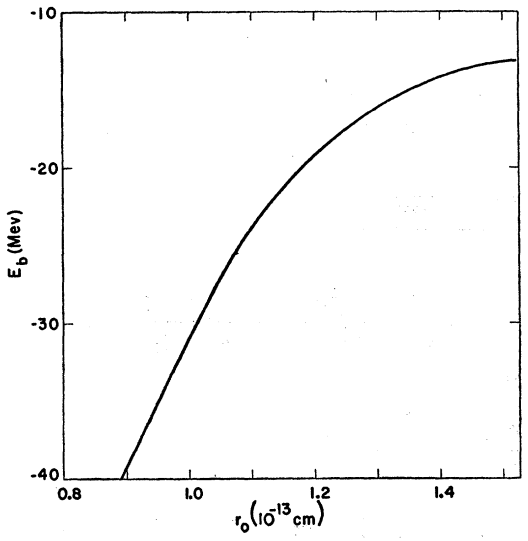

FIG. 3. Binding energy per particle computed from even-parity potentials with repulsive cores of $r_c=0.2\times10^{-13}$ cm. The potentials agree with the low-energy data.

[35] P. Martin (private communication) has found that this large self-consistency effect seems to be absent for certain special classes of nonlocal potentials.

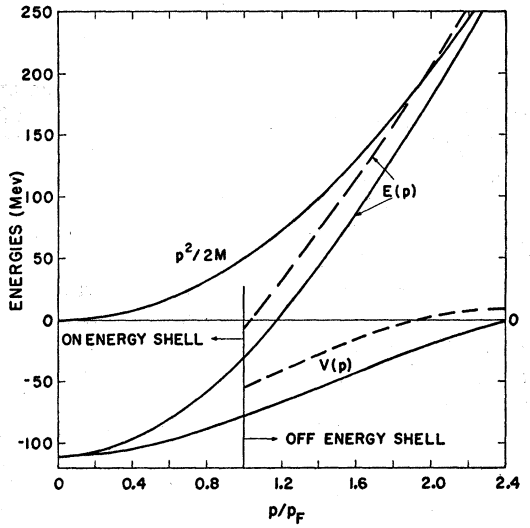

FIG. 4. Self-consistent $E_0(p)$ and $E_1(p,\Sigma)$ for $r_0=0.98\times10^{-13}$ cm. The self-consistent potential energy $V(p)$ and the kinetic energy $p^2/2M$ are also given. The solid curves for $V(p)$ and $E(p)$ are for Σ corresponding to $f=0.95$ [see Eq. (76)] and the dashed curves for $f=0.05$.

The self-consistent single-particle energy spectrum is given in Fig. 4 for $\Delta=0$. We give $E_0(p)$ and $E_1(p,\Sigma)$, both labeled $E(p)$ in Fig. 4. If we subtract the kinetic energy $p^2/2M$ from the single-particle energies $E(p)$, we find the single-particle potential energies $V(p)$. These are also shown in Fig. 4. For $p<p_f$, $E(p)$ is $E_0(p)$, the energies of unexcited particles. For $p>p_f$, $E(p)$ is $E_1(p,\Sigma)$, the energies of excited particles. When a particle is excited, its energy depends on the states from which it and the particle with which it interacted were excited through the parameter Σ defined by Eq. (13). The dashed curve gives the energy of a particle excited together with another particle both coming from near the bottom of the Fermi sea. This particle is far off the energy shell even when its momentum is close to p_F. The lower curve corresponds to the energy of a particle excited together with another particle both coming from near the top of the Fermi sea. Such a particle is almost on the energy shell, and hence the lower curve is almost continuous with the energy spectrum below the Fermi sea.

The alteration of the energy spectrum from the energy spectrum for free particles (which is $p^2/2M$ in Fig. 4) is very pronounced; the energy required to excite a particle from $p=0$ to $p=1.5p_F$ is about 70% greater than the kinetic energy change. This corresponds to an effective mass for particle motion in nuclear matter which is considerably less than the normal mass. We may define an effective mass as

follows:

$$M^*(p)=[(1/p)(dE/dp)]^{-1}. \quad (99)$$

This effective mass will be a slowly varying function of p. $M^*(p)$ averaged over the Fermi distribution is 0.62 with M^*/M changing from 0.56 at $p=0$ to 0.66 at $p=p_F$. These values are close to values previously obtained[5,6] and also close to that deduced by Johnson and Teller[36] from their theory of nuclear matter.

Another property of interest, closely related to the shape of the single-particle energy spectrum, is the nuclear symmetry energy, E_τ. Our estimate of E_τ is based on the following considerations. In Eq. (70) we ignore the dependence of the K matrices on T_z which would result from an inequality in the number of neutrons and protons. This approximation has the same basis as many other approximations we have made; namely, the leading terms in the K's (the first Born approximation) are just matrix elements of the potential V, which we take to be charge-symmetric and hence independent of T_z. Nearly all of the higher order terms in the K's would be treated correctly by an approximation in which the four (or two) Fermi momenta in Eq. (67) would be replaced by an average Fermi momentum and the four (or two) single-particle energy spectra would be replaced by some average single-particle energy spectrum in calculating the Green's functions which appear in the integral equations for the K matrices. Such an approximation would make the K's independent of T_z.

In this approximation the dependence of $E_0(m_0,+,\alpha)$ on the symbol $+$ comes entirely from the fact that the limit on the first two sums in Eq. (70) $[p_F(+,\alpha)=p_F(+,\beta)=p_F(+)]$ is not the same as the limits on the last two sums, $[p_F(-,\alpha)=p_F(-,\beta)=p_F(-)]$.

If we wrote Eq. (66) and subsequently Eq. (70) for $E_0(m_0,-,\alpha)$, we would find exactly the same result except the upper limit of the first two sums would now be $p_F(-)$ and the upper limit of the last two sums would be $p_F(+)$. [$T_z=-1$ instead of $T_z=+1$ would appear in the equation corresponding to Eq. (70), but this does not matter if the K's are independent of T_z.] It is easy to see that

$$E_0(m_0,+,\alpha)-E_0(m_0,-,\alpha)=\Sigma+\tfrac{1}{2}\Sigma-\tfrac{1}{2}\Sigma-\tfrac{1}{4}\Sigma-\text{exchange}, \quad (100)$$

where the sums all run from $p_F(-)$ to $p_F(+)$ and are exactly the same four sums which appear in Eq. (70). Finally, much as we found Eq. (72), we find dropping α in $E_0(m_0,+,\alpha)$ and $E_0(m_0,-,\alpha)$ since neither of these E_0's depend on α.

$$E_0(m_0,+)-E_0(m_0,-)=\sum_{n_0}\sum_{T=0}^{1}\sum_{s=0}^{1}(T-\tfrac{1}{2})\times\sum_{m_s=-1}^{1}(m_0,n_0|K(s,m_s,T)|m_0,n_0). \quad (101)$$

The first sum runs from $p_F(-)$ to $p_F(+)$. When $p_F(-)-p_F(+)$ is small, we may do the sum over n_0 approximately as follows:

$$E_0(m_0,-)-E_0(m_0,+)=[p_F^3(-)-p_F^3(+)]\times[A/(2p_F^3)]\sum_{T=0}^{1}\sum_{s=0}^{1}(T-\tfrac{1}{2})\times\sum_{m_s=-1}^{1}(m_0,n_0|K(s,m_s,T)|m_0,n_0)_F. \quad (102)$$

The subscript F on the K matrices means they are to be evaluated for the magnitude of n_0 equal to Fermi momentum p_F [see Eq. (108) below] and averaged over the angles of n_0.

We need an estimate of $p_F(-)-p_F(+)$. It is known that nuclei contain more neutrons than protons, and that the neutron and proton density distributions have the same radii. Therefore the neutrons are more dense than the protons, and $p_F(-)>p_F(+)$. Treating the neutrons and protons separately as degenerate Fermi gases, we find

$$p_F(-)=(2N/A)^{\frac{1}{3}}p_F,\qquad p_F(+)=(2Z/A)^{\frac{1}{3}}p_F. \quad (103)$$

The average binding energy per proton $E_b(+)$ is given by an equation just like Eq. (74):

$$E_b(+)=\frac{3}{p_F^3(+)}\int_0^{p_F(+)}\left[\frac{m_0^2}{2M}+\tfrac{1}{2}V(m_0,+)\right]\times m_0^2dm_0, \quad (104)$$

where

$$E_0(m_0,+)=(m_0^2/2M)+V_0(m_0,+). \quad (105)$$

There are corresponding expressions for $E_b(-)$, of course. Expanding everything in powers of $(N-Z)/A$, we find

$$\frac{ZE_b(+)+NE_b(-)}{A}-E_b(0)=\left(\frac{N-Z}{N+Z}\right)^2\left(\frac{1}{6}\frac{p_F^2}{m}+\frac{1}{6}p_F\frac{dV(p_F)}{dp_F}+S\right), \quad (106)$$

where

$$S=\frac{A}{8}\sum_{T=0}^{1}\sum_{s=0}^{1}(T-\tfrac{1}{2})\times\sum_{m_s=-1}^{1}(m_0n_0|K(s,m_s,T)|m_0n_0)_{Av}. \quad (107)$$

The average indicated in the last term is defined as

$$K_{Av}=\left[\int\frac{d\hat{n}_0}{4\pi}(m_0n_0|K|m_0n_0)\right]_{m_0=n_0=p_F}. \quad (108)$$

[36] M. H. Johnson and E. Teller, Phys. Rev. **93**, 357 (1954).

The coefficient of $[(N-Z)/(N+Z)]^2$ is the symmetry energy E_τ. The first term in E_τ,

$$E_\tau(F)=\tfrac{1}{6}p_F^2/M, \tag{109}$$

is the symmetry energy of the Fermi gas. The second term arises from the velocity dependence of the potential and combines with the first, using Eq. (97) to replace M by the effective mass M^*. The last term arises from the parity dependence of the two nucleon potential.

Inserting the numerical values given by our results and dropping as before the odd-state contributions, we find

$$E_\tau/E_\tau(F)=[(M/M^*)+1.19], \tag{110}$$

where $E_\tau(F)=p_F^2/6M$ is the symmetry energy of a Fermi gas. Since at the Fermi surface $M/M^*=1.50$, the effect from the spin dependence of the interaction is nearly as large as that from the velocity dependence, with the combined effect 2.69 times as large as that of a Fermi gas. Teller and Johnson estimated for heavy nuclei that the symmetry energy should be $(0.51)^{-1}=1.96$ times that of a Fermi gas, a value which is somewhat lower than that we have obtained.[37] We have estimated the effects of the odd-state interaction on the symmetry energy, taking the scattering phases of Gammel and Thaler.[29,30] The effects of the attractive triplet odd and repulsive singlet odd interactions both lower the symmetry energy. We have estimated that this effect reduces $E_\tau/E_\tau(F)$ from 2.69 to 2.31 which still is above the phenomenologically deduced Teller-Johnson value. This discrepancy can be traced to the considerably more rapid drop with energy of the singlet even than the triplet even interactions, suggesting that the repulsive core in the latter should perhaps be slightly larger than in the former case. We have not at the present investigated this discrepancy further.

We have also determined the effect of the two-body potential on the wave function describing the interaction of two particles. This is of particular interest in the theory of the shell model since it shows to what extent the particle motion departs strongly from independent particle motion. We have taken for simplicity the case of the singlet even-parity interaction and a relative momentum $K=\frac{1}{2}|p_i-p_j|=0.128\times(10^{-13}\text{ cm})^{-1}$ corresponding to two particles moving near the bottom of the Fermi sea. In Fig. 5 we give the wave function obtained by solution of the integral equation together with the function [see Eq. (60)]

$$S(r)=\frac{\sin Kr}{Kr}-\frac{\sin Kr_c}{Kr_c}\frac{G(r,r_c)}{G(r_c,r_c)}, \tag{111}$$

which is the correct solution if the influence of the attractive part of the potential on the wave function is neglected. For comparison, we also give the unperturbed wave function $(\sin Kr)/Kr$ and the form taken by $S(r)$ if we replace the Green's function by the Green's function appropriate to free scattering:

$$S_{\text{free}}(r)=\frac{\sin Kr}{Kr}-\frac{\tan Kr_c}{Kr_c}\cos Kr. \tag{112}$$

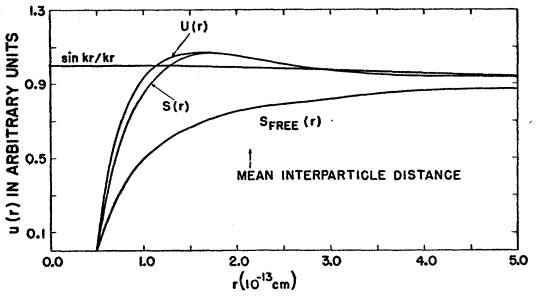

FIG. 5. Wave functions for ^1S state, and relative momentum $k_0=0.128\times(10^{-13}\text{ cm})^{-1}$.

A variety of interesting features are exhibited by the results given in Fig. 5. Most striking is the difference between $S(r)$ or $U(r)$ and $S_{\text{free}}(r)$. These differ primarily because the effect of the exclusion principle is to change the $1/r$ asymptotic falloff of $G(r,r_c)$ to an exponential falloff with falloff distance of the order of the Fermi wavelength $\lambda_F=\hbar/p_F$. This strong damping effect of the exclusion principle on $S(r)$ and $U(r)$ leads to a rapid approach to the free solution $(\sin Kr)/Kr$ at a distance about equal to the mean particle spacing. It is also interesting to note the small difference between $S(r)$ and $U(r)$, which amounts in this case to 15 to 20% at most. It must not be inferred from this result, however, that Born's approximation can be used in treating the attraction. The reason is clear when it is realized that the actual effect of the interaction appears as the difference between the large repulsion arising from the core and the somewhat larger effect arising from the attractive part of the potential. Consequently, a relatively small error in the wave function leads to large errors in the binding energy; numerical investigation of this question shows that all of the binding effect is lost if $U(r)$ is replaced by $S(r)$.

APPENDIX A

The infinite ladder of equations begins with Eqs. (1) and (2) in the text, which we renumber (1_0) and (2_0), the subscript 0 reminding us that these equations are the first step in the ladder. Equations (3) and (4) in the text become Eqs. (1_1) and (2_1), the second step in the ladder.

We note that $E_1(s_1,m_1;m_0,n_0)$ is given by a set of equations exactly similar to Eqs. (1_1) and (2_1) except that the letters m and s are everywhere interchanged

[37] A. E. S. Green [Phys. Rev. 99, 1410 (1955)] gives a value for the symmetry energy of 25 Mev at a nuclear radius of approximately $1.16A^{\frac{1}{3}}\times10^{-13}$ cm. This corresponds to $E_\tau/E_\tau(F)=2.10$.

(except m_0 remains m_0). A typical step in the ladder is

$$E_i(m_i,D_i)=(m_i^2/2M)+\sum_{ni}(m_in_i|K_i(m_i,D_i)|m_in_i)$$

D_i, which specifies the state of excitation of the medium, is labelled by the array

$$D_i=\begin{pmatrix} m_i & s_1s_2s_3 & \cdots s_i \\ m_{i-1}s_1s_2s_3 & \cdots n_{i-1} \\ \vdots & \vdots\vdots & \vdots \\ m_3 & s_1s_2s_3 & \cdots n_{i-1} \\ m_2 & s_1s_2n_2 & \cdots n_{i-1} \\ m_1 & s_1n_1n_2 & \cdots n_{i-1} \\ m_0 & n_0n_1n_2 & \cdots n_{i-1} \end{pmatrix}. \qquad \text{(A-1}_i\text{)}$$

$E_i(m_i,D_i)$ is the energy of a particle (P) whose momentum is m_i as a consequence of collisions of P with other particles: in the initial state P has momentum m_0 and the other particles have momenta $n_0\cdots n_{i-1}$. The subsequent steps in the sequence of collisions is shown in detail in D_i. For example, in the first step P collides with the particle (Q) whose momentum initially is n_0; the momentum of P becomes m_1 and the momentum of Q becomes s_1 and the momenta of the other particles remain unchanged. One thinks of a Feynman diagram, of course, and the letter D stands for diagram.

$K_i(m_i,D_i)$ is a solution of

$$(m'n'|K_i(m_i,D_i)|m_in_i)=(m'n'|V|m_in_i)+\sum_{m_{i+1}s_{i+1}}\times\frac{(m'n'|V|m_{i+1}s_{i+1})(m_{i+1}s_{i+1}|K_i(m_i,D_i)|m_in_i)}{D},$$

$$D=E_0(m_0)+\sum_{j=0}^{i}E_0(n_j)-\sum_{j=0}^{i+1}E_j(s_j,D_j)-E_{i+1}(m_{i+1},D_{i+1}). \qquad \text{(A-2}_i\text{)}$$

The meaning of the notation should be clear; for example, $E_i(s_i,D_i)$ is the energy of the particle whose momentum is s_i as a consequence of the sequence of collisions described by D_i; $E_i(s_i,D_i)$ is the energy in the final state of the last particle with which P interacts. It is understood that D_{i-1} is a "subgraph" of D_i, D_{i-2} is a "subgraph" of D_{i-1}, etc.

We note that $E_0(m_0)$ depends only on m_0. However, $E_1(m_1,s_1;m_0,n_0)$ depends on m_1, and, in addition $E_0(m_0)+E_0(n_0)-E_1(s_1,m_1;m_0,n_0)$, because, in Eqs. (3–4) in the text there are sums over the variables n_1, s_2, and m_2 so that $E_1(m_1,s_1;m_0,n_0)$ cannot depend on these, and even though m_0, n_0, and s_1 are not summed over, they appear only in the combination $E_0(m_0)+E_0(n_0)-E_1(s_1,m_1;m_0,n_0)$. Examination of Eqs. (A-1$_i$) and (A-2$_i$) shows that $E_i(m_i,D_i)$ depends on m_i, and, in addition

$$H(m_i,D_i)=E_0(m_0)+\sum_{j=0}^{i-1}E_0(n_j)-\sum_{j=1}^{i}E_j(s_j,D_j). \qquad \text{(A-3)}$$

We may rewrite Eqs. (A-1$_i$) and (A-2$_i$) as follows:

$$E_i[m_i,H(m_i,D_i)]=\frac{m_i^2}{2M}+\sum_{ni}(m_in_i|K_i[H(m_i,D_i)]|m_in_i), \qquad \text{(A-4)}$$

where

$$(m'n'|K[H(m_i,D_i)]|m_in_i)=(m'n'|V|m_in_i)+\sum_{m_{i+1}s_{i+1}}\times\frac{(m'n'|V|m_{i+1}s_{i+1})(m_{i+1}s_{i+1}|K[H(m_i,D_i)]|m_in_i)}{D}, \qquad \text{(A-5)}$$

$$D=H(m_i,D_i)+E_0(n_i)-E_{i+1}(s_{i+1},H(s_{i+1},D_{i+1}))-E_{i+1}(m_{i+1},H(m_{i+1},D_{i+1})). \qquad \text{(A-5')}$$

By the definition Eq. (A-3) of H,

$$\begin{aligned}H(s_{i+1},D_{i+1})&=H(m_i,D_i)+E_0(n_i)-E_{i+1}(m_{i+1},D_{i+1}) \\ &=H(m_i,D_i)+E_0(n_i)-E_{i+1}(m_{i+1},H(m_{i+1},D_{i+1})),\end{aligned} \qquad \text{(A-6)}$$

and similarly

$$H(m_{i+1},D_{i+1})=H(m_i,D_i)+E_0(n_i)-E_{i+1}(s_{i+1},H(s_{i+1},D_{i+1})). \qquad \text{(A-7)}$$

All of the steps in the ladder have this same structure. If we allow the function H to be independent of its arguments and take on a continuum of values as a parameter in the equations, we may drop the subscript i and replace the ladder of equations by a finite set of equations. Instead of Eq. (4) we have

$$E(m,H)=\frac{m^2}{2M}+\sum_{n}(mn|K(H)|mn), \qquad \text{(A-8)}$$

where, instead of Eqs. (A-5) and (A-5'), we have

$$(m'n'|K(H)|mn)=(m'n'|V|mn)+\sum_{m''n''}\frac{(m'n'|V|m''n'')(m''n''|K(H)|mn)}{D}, \qquad \text{(A-8')}$$

$$D=H+E_0(n)-E(m'',H(m'',n''))-E(n'',H(n'',m'')),$$

where, according to Eqs. (A-6) and (A-7),

$$\begin{aligned}H(m'',n'')&=H+E_0(n)-E(n'',H(n'',m'')), \\ H(n'',m'')&=H+E_0(n)-E(m'',H(m'',n'')).\end{aligned} \qquad \text{(A-8'')}$$

It is necessary to solve Eqs. (A-8'') simultaneously to determine $H(m'',n'')$ and $H(n'',m'')$. This is a purely numerical or algebraic problem. The dependence of $E(m,H)$ or H might be such that either Eqs. (A-8'') have no solution or not unique solutions. However, we believe that $E(m,H)$ is a weakly varying, monotonically

decreasing function of H such that Eqs. (A-8″) always have solutions.

The Eqs. (11)–(15) which we deal with in the main text are very similar to Eqs. (A-8)–(A-8″). The quantity Σ in the text is analogous to the quantity H except that Σ takes on finite values only (from $2E_0$ to $2E_F$) while H takes all values from 0 to $+\infty$ (perhaps not quite all). Also, in the text, we do not have the problem of solving Eqs. (A-8″) simultaneously. However, the problems of solving Eqs. (A-8)–(A-8″) are not sufficiently greater than the problem of solving Eqs. (11)–(15) in the main text to make us not wish that we had solved Eqs. (A-8)–(A-8″) without the additional approximations given in the main text.

APPENDIX B

We consider a simplified problem for which it is easy to determine the correction to our treatment of the core. We consider the S state since this is affected most by the core; we also disregard the attraction. We arrive at the simplified model by approximating to the correct Green's function as follows: the action of the exclusion principle in allowing only transitions from the Fermi gas to states above the Fermi surface is approximated by adding a mean excitation energy,

$$\frac{\alpha^2}{2M}=\frac{p_F^2}{2M}-\frac{3}{5}\frac{p_F^2}{2M}=\frac{2}{5}\frac{p_F^2}{2M}, \tag{B-1}$$

to the energy denominator; that is, we take

$$G(r,r')=-\frac{4\pi^M}{(2\pi)^5}\int_0^\infty \frac{\sin kr}{kr}\frac{\sin kr'}{kr'}\frac{k^2dk}{k^2+\alpha^2}$$

$$=-\frac{M}{8\pi\alpha rr'}\{\exp(-\alpha|r-r'|)-\exp[-\alpha(r+r')]\}. \tag{B-2}$$

This Green's function, together with the integral equation

$$U(r)=(\sin k_0r/k_0r)+4\pi\int r'^2dr'G(r,r')V(r')U(r') \tag{B-3}$$

defines the simplified problem. The Green's function for the simplified problem satisfies

$$(1/M)(-\alpha^2+\nabla^2)G(r,r')=(1/4\pi r^2)\delta(r-r'). \tag{B-4}$$

To solve the integral equation, we make the replacement

$$V(r')U(r')=\lambda\delta(r'-r_c)+f(r'), \tag{B-5}$$

where $f(r')(r'<r_c)$ will give rise to a small correction to the K matrix and must be determined so that $U(r)$ vanishes for $r<a$. We find, eliminating λ as before,

$$U(r)=\frac{\sin k_0r}{k_0r}-\frac{\sin k_0r_c}{k_0r_c}\frac{G(r,r_c)}{G(r_c,r_c)}$$

$$+4\pi\int G(r,r')-\left[\frac{G(r,r_c)G(r',r_c)}{G(r_c,r_c)}\right]f(r')r'^2dr'. \tag{B-6}$$

To determine $f(r)$, we operate on Eq. (B-5) with $(-\alpha^2+\nabla^2)/M$ in the region for which $U(r)=0$, namely, for $r<r_c$. The result is

$$f(r)=\left(\frac{k_0^2+\alpha^2}{M}\right)\frac{\sin k_0r}{k_0r}. \tag{B-7}$$

The K matrix is also given directly in terms of λ and $f(r)$; that is,

$$K=4\pi\int\frac{\sin k_0r}{k_0r}V(r)U(r)r^2dr=-\left(\sin k_0r_c\Big/\frac{k_0r_c}{G(r_c,r_c)}\right)^2$$

$$+4\pi\int r^2dr\left[\frac{\sin k_0r}{k_0r}-\frac{\sin k_0r_c}{k_0r_c}\frac{G(r,r_c)}{G(r_c,r_c)}\right]f(r). \tag{B-8}$$

We are interested in k_0r_c and αr_c both less than one; thus we make the approximation

$$\frac{G(r,r_c)}{G(r_c,r_c)}=1-(r_c^2-r^2)\frac{\alpha^2}{3!}. \tag{B-9}$$

Then, also setting $k_0=0$ and replacing in the leading term $G(r_c,r_c)$ by its lowest approximation, we find

$$K=\frac{4\pi r_c}{M}+\frac{4\pi\alpha^2}{3!}\int_0^{r_c}r^2dr(r_c^2-r^2)\frac{\alpha^2}{M}$$

$$=\frac{4\pi r_c}{M}\left[1+\frac{4}{1125}(p_Fr_c)^4\right].$$

In our problem, at the energy minimum, $p_Fr_c=0.71$, so that the correction term is about 0.1% which is quite negligible.

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