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Plasma Diffusion in a Magnetic Field*

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The equations governing the diffusion of a fully ionized plasma across a magnetic field are derived. It is assumed that macroscopic quantities vary slowly across an ion radius of gyration, and that the interparticle collision frequency is much less than the gyration frequency. The relevant transport coefficients—electrical resistivity, thermal conductivity, and thermoelectric coefficient—are derived. Some similarity solutions of the equations are found.

I. INTRODUCTION

A FULLY ionized plasma, confined by a magnetic field, diffuses across the field by means of interparticle collisions,[1,2] when a transverse density gradient exists. In a recent paper,[3] Longmire and Rosenbluth investigated the effects of like-particle and unlike-particle collisions. They found that the effects of the latter predominate if the relative change in density over an ion radius of gyration is small.

In this paper the theory of diffusion is extended to include the effects of a transverse temperature gradient, in the same direction as the density gradient. A closed set of equations is derived, by an expansion in two small parameters: $\alpha \sim a|\nabla n|/n$, the ratio of the radius of gyration, a, to a characteristic macroscopic distance; and $\gamma \sim (\omega\tau)^{-1}$, the ratio of the collision frequency τ^{-1} to the gyration frequency $\omega \equiv eB/mc$.[4]

It is found that to second order in α, the ions and electrons diffuse at the same velocity $\mathbf{u}_d$; to higher order in α, the diffusion velocities are different,[3] and charge separation may occur. Such effects will not be considered in this paper. Hence, for our purposes, the densities of ions (assumed to be singly-charged, and all of the same mass) and electrons are equal. A consideration of relaxation times[5] shows that both components arrive at local Maxwell velocity distributions in a time short compared to the time for the plasma to diffuse across a distance equal to an ion radius of gyration; and that the two components come to the same temperature in a time roughly equal to the latter. Thus, to zero order in α, we shall take the ions and electrons as being locally Maxwellian, with equal densities and temperature.

The velocity distribution function $f(\mathbf{r},\mathbf{v},t)$ for each component is expanded as a power series in α:[6]

$$f=f_0+f_1+f_2+\cdots, \tag{1}$$

where

$$f_0^{i,e}=\frac{n}{2}\left(\frac{m_{i,e}}{2\pi kT}\right)^{\frac{3}{2}}\exp[-\tfrac{1}{2}m_{i,e}v^2/kT]. \tag{2}$$

The moments of f that we shall use are

$$\tfrac{1}{2}n\mathbf{u}^{i,e}\equiv\int d^3v\,\mathbf{v}f^{i,e}, \tag{3}$$

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[1] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956).

[2] Reference 1, Sec. 3.2b.

[3] C. Longmire and M. Rosenbluth, Phys. Rev. **103**, 507 (1956). See also A. Simon, Phys. Rev. **100**, 1557 (1955).

[4] Because each component of the plasma has different values of a, ω, and τ, we shall require that all the appropriate α and γ are small.

[5] Reference 1, Sec. 5.3.

[6] Similar expansions have been used by S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, New York, 1953); L. Spitzer, Astrophys. J. **116**, 299 (1952); Chew, Goldberger, and Low, Proc. Roy. Soc.

$$\mathbf{u}\equiv\frac{1}{m_i+m_e}(m_i\mathbf{u}^i+m_e\mathbf{u}^e), \tag{4}$$

$$\mathbf{j}\equiv\tfrac{1}{2}ne(\mathbf{u}^i-\mathbf{u}^e), \tag{5}$$

$$\mathbf{Q}\equiv\sum_{i,e}\int d^3v\,\tfrac{1}{2}m(\mathbf{v}-\mathbf{u})^2(\mathbf{v}-\mathbf{u})f. \tag{6}$$

The first-order correction f_1 contributes to these moments. It will be found that for no collisions ($\gamma=0$), there are component flows[7]

$$\tfrac{1}{2}n\mathbf{u}^{i,e}=\pm\frac{c}{e}\frac{\mathbf{B}\times\nabla(p/2)}{B^2}, \tag{7}$$

where p is the total zero-order pressure:

$$p=nkT. \tag{8}$$

From Eqs. (5) and (7), we obtain the first-order equation of equilibrium:

$$(1/c)\mathbf{j}\times\mathbf{B}=\nabla p. \tag{9}$$

To first order in γ, we shall find equal flows in the direction of the gradients:

$$\mathbf{u}_d=-\frac{\eta_\perp c^2}{B^2}\nabla p+\frac{3}{4}\frac{\eta_\perp c^2 n}{B^2}\nabla(kT)+c\frac{\mathbf{E}\times\mathbf{B}}{B^2}, \tag{10}$$

where[8]

$$\eta_\perp\equiv(8/3)(\pi/2)^{\frac{1}{2}}e^2m_e^{\frac{1}{2}}(kT)^{-\frac{3}{2}}\ln\Lambda. \tag{11}$$

The first two terms in (10) arise from momentum transfer between the electrons and ions, whose flows (7) oppose each other. In a magnetic field, gain of momentum produces guiding-center drift, as shown by Alfvén.[9] The appearance of the second term is a consequence of the fact that for Coulomb interactions, the collision frequency is velocity-dependent. The third term is the well-known $\mathbf{E}\times\mathbf{B}$ drift; the electric field $\mathbf{E}$ arises by induction from the motion of the diamagnetic plasma.

If $\mathbf{E}$ is measured in the frame moving with velocity $\mathbf{u}_d$:

$$\mathbf{E}'\equiv\mathbf{E}+(1/c)\mathbf{u}_d\times\mathbf{B}, \tag{12}$$

Equation (10) can be written in such a way as to resemble Ohm's law, with a thermoelectric coefficient λ:

$$\mathbf{j}=\eta_\perp^{-1}\mathbf{E}'-\lambda\nabla(kT)\times\mathbf{B}, \tag{13}$$

$$\lambda\equiv\tfrac{3}{4}(nc/B^2). \tag{14}$$

(London) **236**, 112 (1956); R. Landshoff, Phys. Rev. **76**, 904 (1949).

[7] Reference 1, Sec. 2.4.

[8] See reference 1, p. 73.

[9] H. Alfvén, *Cosmical Electrodynamics* (Oxford University Press, New York, 1950), Sec. 2.2.

Equation (9) remains valid to all orders in γ, but Eq. (10) would be modified by higher-order corrections. This paper will restrict itself to the lowest-order effects.

To first order in α and γ, the heat flow in the direction of the gradients, and relative to the frame $\mathbf{u}_d$ will be found to be

$$\mathbf{Q}=-K\nabla(kT)+\lambda kT\mathbf{E}'\times\mathbf{B}, \tag{15}$$

where

$$K=\frac{1}{4}\left(\frac{2m_i}{m_e}\right)^{\frac{1}{2}}\frac{n^2c^2kT}{B^2}\eta_\perp \quad \text{if} \quad m_i\gg m_e. \tag{16}$$

The dominant contribution to the thermal conductivity K is from ion-ion collisions, which do not contribute at all to the diffusion velocity $\mathbf{u}_d$. The appearance of the large numerical factor in K causes heat flow to be relatively much faster than mass flow [compare Eq. (10)]. Thus, the diffusion process will tend to occur at uniform temperature, in which case the thermoelectric effect becomes unimportant. Note that the same λ appears in (15) as in (13); this is required by the Onsager reciprocity relations.[10]

To second order in α, we shall find the moment equations expressing conservation of mass and energy:

$$\partial n/\partial t=-\nabla\cdot(n\mathbf{u}_d), \tag{17}$$

$$\partial p/\partial t+\mathbf{u}_d\cdot\nabla p=-(5/3)p\nabla\cdot\mathbf{u}_d+(2/3)(\mathbf{E}'\cdot\mathbf{j}-\nabla\cdot\mathbf{Q}). \tag{18}$$

In the equation of state (18), the three terms on the right represent, respectively, adiabatic heating (for a Maxwell gas), ohmic heating, and heating by heat conduction. The factor $\frac{2}{3}$ appears because the thermal energy density is $\frac{3}{2}$ the pressure.

The equations above are completed by the Maxwell equations

$$\nabla\times\mathbf{B}=(4\pi/c)\mathbf{j}, \tag{19}$$

$$\nabla\times\mathbf{E}=-(1/c)\partial\mathbf{B}/\partial t. \tag{20}$$

We have thus obtained a closed set of equations: (8), (9), (11)–(20). Each equation is valid to lowest order in the quantities that appear in it.

If an electric field exists along the magnetic field, the only change that need be made is in Eq. (13), which then becomes

$$\mathbf{j}=\boldsymbol{\sigma}\cdot\mathbf{E}'-\lambda\nabla(kT)\times\mathbf{B}, \tag{21}$$

where

$$\boldsymbol{\sigma}=\eta_\perp^{-1}(\mathbf{1}-\mathbf{nn})+\eta_{\parallel}^{-1}\mathbf{nn}, \tag{22}$$

and $\mathbf{n}$ is the unit vector along $\mathbf{B}$. The calculation[11] of $\eta_{\parallel}$ is independent of the presence of $\mathbf{B}$. The two resistivities are related[12]:

$$\eta_\perp=1.98\eta_{\parallel}. \tag{23}$$

[10] See, for example, S. R. DeGroot, *Thermodynamics of Irreversible Processes* (Interscience Publishers, Inc., New York, 1951), Sec. 61.

[11] L. Spitzer and R. Härm, Phys. Rev. **89**, 977 (1953).

[12] Reference 1, Sec. 5.4.

The next section contains the derivation of the equations above. In the last section some solutions are discussed.

II. DERIVATION OF EQUATIONS

The Boltzmann equation for ions is

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla+\frac{e}{m_i}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v\right]\times f^i(\mathbf{r},\mathbf{v},t)=C^{ii}+C^{ie},\quad(24)$$

where C^{ii} and C^{ie} represent the contributions of ion-ion and ion-electron collisions to $(\partial/\partial t)f^i$. An analogous equation applies to the electron function f^e.

We shall use the Boltzmann collision integral for the C's; for example,

$$C^{ie}(\mathbf{r},\mathbf{v},t)=-\int d^3uw\int d\Omega\sigma^{ie}\times[f^i(\mathbf{r},\mathbf{v},t)f^e(\mathbf{r},\mathbf{u},t)-f^i(\mathbf{r},\mathbf{v}',t)f^e(\mathbf{r},\mathbf{u}',t)],\quad(25)$$

where

$$\mathbf{w}\equiv\mathbf{v}-\mathbf{u},\quad(26)$$

$$\sigma^{ie}=\tfrac{1}{4}e^4\mu_{ie}^{-2}w^{-4}\csc^4(\tfrac{1}{2}\Theta),\quad(27)$$

$$\mu_{ie}^{-1}\equiv m_i^{-1}+m_e^{-1}.\quad(28)$$

The integration over angle will be cut off at small angles in the usual way.[8] Alternatively, one may use the Fokker-Planck formalism, and the same results are obtained.

To zero order in α, we take only the term in $\mathbf{B}$ on the left side of (24):

$$(\mathbf{v}\times\boldsymbol{\omega}_i)\cdot\nabla_v f_0^i(\mathbf{r},\mathbf{v},t)=C_0^{ii}+C_0^{ie},\quad(29)$$

where f_0^i and f_0^e are used in C_0, and

$$\boldsymbol{\omega}_{i,e}\equiv\pm e\mathbf{B}/m_{i,e}c.\quad(30)$$

Let the z axis be in the local direction of $\mathbf{B}$. Then, with

$$v_z=v\cos\theta,$$
$$v_x=v\cos\theta\cos\phi,$$
$$v_y=v\cos\theta\sin\phi,\quad(31)$$

the left side of (29) becomes $-\omega_i(\partial/\partial\phi)f_0^i$. Equations (29) for ions and electrons are satisfied by the local Maxwell distributions (2), where n and T are functions of position and time. (An arbitrary mass motion along $\mathbf{B}$ is set equal to zero.)

In order not to complicate the problem unduly, we assume that ∇n and ∇T are in the same direction (transverse to $\mathbf{B}$), which we take for the x axis, and that the induced field $\mathbf{E}$ is in the y direction. The first-order (in α) Boltzmann equation is

$$\mathbf{v}\cdot\nabla f_0^i+\frac{e}{m_i}\mathbf{E}\cdot\nabla_v f_0^i-\omega_i\frac{\partial}{\partial\phi}f_1^i=C_1^{ii}+C_1^{ie}.\quad(32)$$

(We treat ∇ as first-order in α; *a posteriori* it is found that $\partial/\partial t$ should be treated as second-order in α, and $\mathbf{E}$ as first-order in α and in γ.)

Multiply (32) by $m_i\mathbf{v}$ and integrate over velocity space. We find

$$\tfrac{1}{2}\nabla p-\tfrac{1}{2}ne\left(\mathbf{E}+\frac{1}{c}\mathbf{u}^i\times\mathbf{B}\right)=\mathbf{P}_1^{ie},\quad(33)$$

where $\mathbf{P}_1^{ie}$ is the rate of momentum transfer (to order α) from electrons to ions. The analogous electron equation is

$$\tfrac{1}{2}\nabla p+\tfrac{1}{2}ne\left(\mathbf{E}+\frac{1}{c}\mathbf{u}^e\times\mathbf{B}\right)=\mathbf{P}_1^{ei}.\quad(34)$$

Because momentum is conserved in collisions, $\mathbf{P}^{ie}+\mathbf{P}^{ei}=0$. Thus the sum of (33) and (34) is Eq. (9). Acceleration terms are higher order in α, and so do not appear in (9).

From the y-components of (33) and (34), we see that, to order α, the flows in the x direction are equal:

$$\mathbf{u}_x^i=\mathbf{u}_x^e\equiv\mathbf{u}_d.\quad(35)$$

Thus $j_x=0$, as required by Eq. (9); there is no charge separation.

To solve the integral equation (32) for f_1, we utilize the expansion in γ:

$$f_1=f_{10}+f_{11}+\cdots.\quad(36)$$

The zero-order (in γ) equation is

$$\mathbf{v}\cdot\nabla f_0^i-\omega_i(\partial/\partial\phi)f_{10}^i=0.\quad(37)$$

Its solution is easily found to be

$$f_{10}^i=\frac{v_y}{\omega_i}\left[\frac{1}{n}\frac{\partial n}{\partial x}-\frac{3}{2T}\frac{\partial T}{\partial x}\left(1-\frac{m_iv^2}{3kT}\right)\right]f_0^i.\quad(38)$$

The v_y moment of (38) yields Eq. (7).

To order γ, Eq. (32) becomes

$$\frac{e}{m_i}\mathbf{E}\cdot\nabla_v f_0^i-\omega_i\frac{\partial}{\partial\phi}f_{11}^i=C_{10}^{ii}+C_{10}^{ie},\quad(39)$$

where, with the substitution

$$f_{10}^{i,e}=f_0^{i,e}\Phi^{i,e},\quad(40)$$

the collision integral C_{10}^{ie} (for example) becomes

$$-f_0^i(v)\int d^3uwf_0^e(u)\int d\Omega\sigma^{ie}[\Phi^i(\mathbf{v})+\Phi^e(\mathbf{u})-\Phi^i(\mathbf{v}')-\Phi^e(\mathbf{u}')].\quad(41)$$

The quantities Φ are given by Eq. (38), and its analog for electrons. The solution is again easily found to be

$$f_{11}^i=\frac{m_iv_xc}{kT}\frac{E_y}{B}f_0^i+\frac{v_x}{v^2\omega_i^2}(F_2^{ii}+F_1^{ie}+F_2^{ie})f_0^i,\quad(42)$$

where

$$F_1^{ie} = -m_i^{-1}\left[\frac{\partial}{\partial x}\ln(nT^{-\frac{3}{2}})\right]\mathbf{v}\cdot\int d^3u w f_0^e(u)\int d\Omega \times\sigma^{ie}\Delta(m_i\mathbf{v}-m_e\mathbf{u}),$$

$$F_2^{ie} = \tfrac{1}{2}m_i^{-1}\left[\frac{\partial}{\partial x}(T^{-1})\right]\mathbf{v}\cdot\int d^3u w f_0^e(u)\int d\Omega \times\sigma^{ie}\Delta(m_i^2v^2\mathbf{v}-m_e^2u^2\mathbf{u}), \quad (43)$$

and

$$\Delta A = A - A', \quad (44)$$

the change in a quantity A in a collision. F_2^{ii} is analogous to F_2^{ie}, with $m_e^2\mathbf{u}$ replaced by $-m_i^2\mathbf{u}$; F_1^{ii} vanishes from momentum conservation.

Although f_{11} is a complicated function of v, its moments are simple. They are most easily evaluated by transforming the variables $\mathbf{u}$, $\mathbf{v}$ to the center-of-mass and relative velocities. The evaluation of the double integrals is then lengthy, but quite straightforward, and one obtains (10) and (15).

To second order in α, the Boltzmann equation (for ions) is

$$\frac{\partial}{\partial t}f_0^i+\mathbf{v}\cdot\nabla f_1^i+\frac{e}{m_i}\mathbf{E}\cdot\nabla_v f_1^i-\omega_i\frac{\partial}{\partial\phi}f_2^i = C_2^{ii}+C_2^{ie}. \quad (45)$$

Its zeroth moment yields the continuity equation (17). Its second moment involves energy transfer between electrons and ions. To eliminate this, we add the ion and electron energy moment equations, and, after a little manipulation, obtain the equation of state (18).

III. SOLUTIONS

We shall consider solutions of the diffusion equations for the case that $\mathbf{B}$ is everywhere in the z direction, and the gradients are everywhere in the x-direction.

Our first solution is for a plasma of finite extent in the x direction, and whose pressure is small compared to the field pressure $B^2/8\pi$. In such a case, we may neglect the induced electric field $\mathbf{E}$ (but not $\mathbf{E}'$), and treat the quantity B^2, wherever it appears, as a constant Our equations are then

$$p = nkT, \quad (8)$$

$$j = \frac{c}{B}\frac{\partial p}{\partial x}, \quad (9)$$

$$u_d = -\frac{\eta_\perp c^2}{B^2}\left(\frac{\partial p}{\partial x}-n\frac{\partial kT}{\partial x}\right), \quad (10')$$

$$E' = -\frac{1}{c}u_d B, \quad (12')$$

$$Q_d = -K\frac{\partial kT}{\partial x}+\lambda kTE'B, \quad (15)$$

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial x}(nu_d), \quad (17)$$

$$\frac{\partial p}{\partial t}+u_d\frac{\partial p}{\partial x} = -\frac{5}{3}p\frac{\partial u_d}{\partial x}+\frac{2}{3}\left(E'j-\frac{\partial Q_d}{\partial x}\right). \quad (18)$$

By the use of Eqs. (9), (12′), and (15), Eq. (18) becomes

$$\frac{\partial p}{\partial t} = -\frac{7}{6}\frac{\partial}{\partial x}(pu_d)+\frac{2}{3}\frac{\partial}{\partial x}\left(K\frac{\partial kT}{\partial x}\right). \quad (46)$$

We search for similarity solutions of the set (8), (10′), (17), (46), where

$$\eta_\perp = \eta_0(T/T_0)^{-\frac{3}{2}}, \quad (47)$$

and K is given by (16). We assume solutions of the form

$$n(x,t) = (t/t_0)^{-m_1}n_0 g(\xi^2), \quad (48)$$

$$p(x,t) = (t/t_0)^{-m_2}p_0[g(\xi^2)]^{m_3}, \quad (49)$$

$$\xi = \frac{x}{x_0}\left(\frac{t}{t_0}\right)^{-m_4}, \quad (50)$$

$$g(1) = 0, \quad (51)$$

$$g(0) = 1, \quad (52)$$

$$p_0 = n_0 kT_0. \quad (53)$$

Conservation of particles requires that $m_1 = m_4$. Substitution of Eqs. (48)–(53) into Eqs. (8), (10′), (17), and (46) yields the solution:

$$m_1 = m_2 = m_4 = \tfrac{1}{3},$$

$$m_3 = 1-(m_e/2m_i)^{\frac{1}{2}},$$

$$g(\xi^2) = (1-\xi^2)^{1-\frac{3}{2}(m_e/2m_i)^{\frac{1}{2}}},$$

$$x_0^2 = t_0\frac{\eta_0 c^2}{B^2}p_0\frac{3}{2}\left(\frac{1+21(m_e/2m_i)^{\frac{1}{2}}}{1+6(m_e/2m_i)^{\frac{1}{2}}}\right), \quad (54)$$

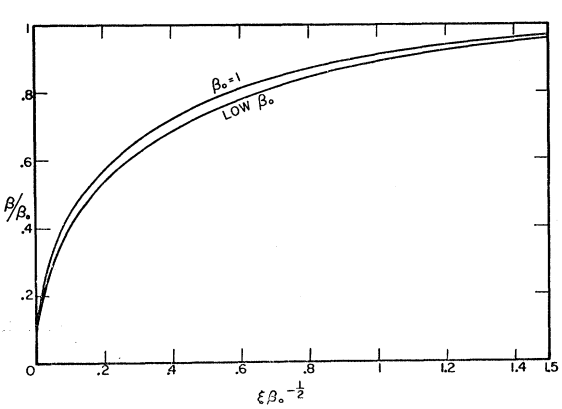

FIG. 1. Solution of the diffusion equations for a semi-infinite uniform plasma.

where, except in the last expression, we have expanded the solution in powers of $(m_e/m_i)^{\frac{1}{2}}$.

We see that the plasma is nearly isothermal:

$$T = T_0(1-\xi^2)^{-\frac{3}{4}(m_e/2m_i)^{\frac{1}{2}}}. \tag{55}$$

Only for ξ^2 nearly equal to 1, does the temperature differ appreciably from T_0. The distributions of density and pressure are nearly parabolic, but as $\xi^2 \to 1$, their gradients become infinite, as does the temperature. (Of course, in the neighborhood of $\xi^2=1$, the basic assumption of small α is not valid). As $(m_e/m_i) \to 0$, the solution becomes the isothermal parabolic solution found by Holladay.[13]

Another problem of interest is defined at $t=0$ by a surface of discontinuity at $x=0$ to the left of which there exists a vacuum with uniform B_z and to the right of which we have a plasma at uniform density, temperature and magnetic field in the z direction. The magnetic field at $x=-\infty$ is independent of time. In this case the appropriate similarity variable is $X/\sqrt{t}$, where X is the mass coordinate, i.e., the initial position of the plasma element. The resulting nonlinear equations were integrated on the IBM 650 at General Atomic by Miss Gwendolyn Roy. (The results are shown on Fig. 1.)

[13] J. Holladay, Los Alamos Scientific Laboratory Report LA-1962 (unpublished).

Here β is the ratio of material pressure to the vacuum magnetic pressure, and β_0 is the initial value of β. The similarity variable ξ is defined as

$$\xi^2 = 4\pi X^2/\eta_{\perp 0} t, \tag{56}$$

$\eta_{\perp 0}$ is the initial resistivity. Two cases are shown: $\beta_0=1$, i.e., no field initially in the plasma, and $\beta_0 \ll 1$.

It can be seen that the time for the front to diffuse a given distance is very nearly inversely proportional to β_0. This is reasonable since the diffusion proceeds by binary collisions. Because of the high thermal conductivity the wave is again almost isothermal, though with a very narrow spike at $\xi=0$. Thus at the point $\beta/\beta_0=0.5$ the temperature in both cases is only 3% higher than the initial temperature.

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