Wave Equation for a Massless Particle with Arbitrary Spin*

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A wave equation for a noninteracting particle with zero mass and arbitrary spin s is given in this paper. The Hamiltonian is proportional to the inner product of the momentum and spin operators so that the wave function has $2s+1$ components. As an auxiliary condition, solutions with spins not parallel or antiparallel to the momentum are discarded. With this condition the theory is Lorentz-covariant. The energy, momentum, and angular momentum are defined in terms of expected values of the usual type of displacement operator. The specialization $s=\frac{1}{2}$ is the two-component neutrino theory and $s=1$ gives Maxwell's equations for the photon.

I. INTRODUCTION

RECENTLY Lee and Yang,[1] in their researches on parity nonconservation and beta decay, have emphasized the importance of Weyl's[2] two-component neutrino theory. Also it has been shown that Maxwell's equations for the electromagnetic field in vacuum can be written in the form of a particle type of wave equation.[3] In each of these two theories the wave equation is

$$H\phi=i\hbar\partial\phi/\partial t, \tag{1}$$

where

$$H=(c/s)\mathbf{p}\cdot\mathbf{s}, \tag{2}$$

$\mathbf{p}$ is the momentum operator $-i\hbar\boldsymbol{\nabla}$, and $\mathbf{s}$ are the angular momentum matrices for spin s equal to $\frac{1}{2}$ or 1. Furthermore the reflection properties of the two theories are also parallel. These two facts suggest the existence of a general theory for massless particles of arbitrary spin $s=\frac{1}{2}, 1, \frac{3}{2}\cdots$ with the above Hamiltonian. The purpose of this paper is to give the details of this theory.

The treatment of the photon differs from that of the two-component neutrino in two respects which are carried over into the general theory: 1. As an auxiliary condition, solutions of the wave equation with spins not parallel or antiparallel to the momentum are discarded. 2. A distinction is made between the spinor components ψ of the field and the wave function components ϕ which are used to form the energy, momentum, and angular momentum from the displacement operators of the field. For arbitrary spin the relation is

$$\psi=|H/c|^{s-\frac{1}{2}}\phi. \tag{3}$$

Both of these differences are necessary to make the theory Lorentz-covariant. The first is consistent with the fact that only polarizations with or against the momentum are permissible for a relativistic particle with zero mass.[4]

The possibility of constructing such a theory can be seen from the general Dirac-Pauli-Fierz discussion[5] of field theories. Their equations are

$$\partial_{\dot{\lambda}\mu}\varphi_{\dot{\sigma}\cdots}{}^{\mu\rho\cdots}=i\kappa\chi_{\dot{\lambda}\dot{\sigma}\cdots}{}^{\rho\cdots}, \tag{4}$$

$$\partial^{\nu\dot{\mu}}\chi_{\dot{\mu}\dot{\sigma}\cdots}{}^{\rho\cdots}=i\kappa\varphi_{\dot{\sigma}\cdots}{}^{\nu\rho\cdots}, \tag{5}$$

where φ, χ are spinors symmetric in the dotted and undotted indices (ranging from 1 to 2) and κ is proportional to the mass. If the mass is zero the system permits of only two independent states[5] and the second equation uncouples from the first. Then Eqs. (5) are a set of first-order equations which apply to a particle with spin s, with a $(2s+1)$-component wave function χ, and with two independent states. However Eqs. (5) cannot be identified with Eq. (1) because they contain auxiliary equations. The relationship is illustrated by the spin one specialization in which $i\kappa\chi_{\dot{\lambda}\dot{\sigma}}$ corresponds to the electromagnetic field and $\varphi_{\dot{\sigma}}{}^{\nu}$ to the four-potential. Equation (5) gives all four of the Maxwell field equations[6]; the curl equations are Eq. (1) and the divergence equations are the auxiliary condition.

II. BASIC EQUATIONS

In this section the plane wave solutions of Eq. (1) which fulfill the auxiliary condition are given and some of their properties are listed.

The substitution

$$\phi=u\exp[i\hbar^{-1}(\mathbf{p}\cdot\mathbf{x}-Wt)] \tag{6}$$

reduces Eq. (1) to the matrix eigenvalue problem

$$(c/s)\mathbf{p}\cdot\mathbf{s}u=Wu. \tag{7}$$

By specializing to an axis in the direction of $\mathbf{p}$, one sees that the auxiliary condition requires

$$W=\pm cp. \tag{8}$$

The usual representation of the angular momentum

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[1] T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1671 (1957).
[2] H. Weyl, Z. Physik **56**, 330 (1929).
[3] R. H. Good, Jr., Phys. Rev. **105**, 1914 (1957).
[4] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **34**, 211 (1948).
[5] See, for example, H. Umezawa, *Quantum Field Theory* (Interscience Publishers, Inc., New York, 1956), Chap. IV, Sec. 3.
[6] O. Laporte and G. E. Uhlenbeck, Phys. Rev. **37**, 1380 (1931).

matrices (with the factor $\hbar$ not included) is[7]

$$(s_1)_{m+1,m}=(s_1)_{m,m+1}=\tfrac{1}{2}[(s-m)(s+m+1)]^{\frac{1}{2}}, \tag{9a}$$

$$(s_2)_{m+1,m}=-(s_2)_{m,m+1}=-\tfrac{1}{2}i[(s-m)(s+m+1)]^{\frac{1}{2}}, \tag{9b}$$

$$(s_3)_{m,m}=m, \tag{9c}$$

where the elements not listed are zero and the subscript m, which ranges from $-s$ to $+s$, refers to the row or column which has m on the diagonal in s_3. In this representation the solutions of Eqs. (7) and (8) are

$$(u_\pm)_m=\left[\frac{p_1^2+p_2^2}{4p^2}\right]^{\frac{1}{2}s}\left[\frac{(2s)!}{(s+m)!(s-m)!}\right]^{\frac{1}{2}}\left[\frac{\pm p+p_3}{p_1+ip_2}\right]^m. \tag{10}$$

Some of the properties of these solutions are[8]:

$$u_\pm^H u_\pm=1, \tag{11a}$$

$$u_\pm^H u_\mp=0, \tag{11b}$$

$$u_\pm^H s_i u_\pm=\pm s p_i/p, \tag{11c}$$

$$u_\pm^H s_i u_\mp=0, \tag{11d}$$

$$u_\pm^H \partial u_\pm/\partial p_1=\pm i s p_2 p_3/[p(p_1^2+p_2^2)], \tag{11e}$$

$$u_\pm^H \partial u_\pm/\partial p_2=\mp i s p_1 p_3/[p(p_1^2+p_2^2)], \tag{11f}$$

$$u_\pm^H \partial u_\pm/\partial p_3=0, \tag{11g}$$

$$u_\pm^H \partial u_\mp/\partial p_i=0. \tag{11h}$$

These are required in the arguments below.

III. TRANSFORMATION PROPERTIES

The transformation properties of the spinor components ψ in the general case can be inferred from the properties for spin $\frac{1}{2}$, in which case ψ is a spinor of first rank. With respect to the proper Lorentz transformation,

$$x_\alpha'=a_{\alpha\beta}x_\beta, \tag{12}$$

the spin $\frac{1}{2}$ transformation is

$$\psi'(x')=\Lambda\psi(x), \tag{13}$$

where

$$a_{\alpha\beta}\sigma_\beta=\Lambda^H\sigma_\alpha\Lambda. \tag{14}$$

Here σ_4 is i and σ_j ($j=1, 2, 3$) are the Pauli matrices. The three complex parameters β_i, defined by

$$\Lambda=\exp(i\tfrac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}), \tag{15}$$

where β^2 is $\beta_i\beta_i$, are convenient for discussing the transformation. It is easily seen that this parametrization exists for every Λ and that the β_i have the following properties (proofs are given in the Appendix):

1. When the β_i are all real, the transformation is an angular displacement in the right-hand sense through angle β.
2. When the β_i are all pure imaginary,

$$\boldsymbol{\beta}=i(\mathbf{v}/v)\ \mathrm{arc\ tanh}(v/c), \tag{16}$$

the transformation is a pure Lorentz transformation with relative velocity $\mathbf{v}$.

3. In general, $\boldsymbol{\beta}$ is a complex three-vector with respect to simultaneous rotations of primed and unprimed axes.

For the general spin, with respect to the Lorentz transformation of Eq. (12), the transformation

$$\psi'(x')=\exp(i\boldsymbol{\beta}\cdot\mathbf{s})\psi(x) \tag{17}$$

can be assigned. It is clear that this assignment is consistent when products of transformations are made because the calculation of the transformation matrix $\exp(i\boldsymbol{\beta}_C\cdot\mathbf{s})$ for transformations A and B successively applied,

$$\begin{aligned}\exp(i\boldsymbol{\beta}_C\cdot\mathbf{s})&=\exp(i\boldsymbol{\beta}_B\cdot\mathbf{s})\exp(i\boldsymbol{\beta}_A\cdot\mathbf{s})\\&=\exp\{i\boldsymbol{\beta}_B\cdot\mathbf{s}+i\boldsymbol{\beta}_A\cdot\mathbf{s}+\tfrac{1}{2}[i\boldsymbol{\beta}_B\cdot\mathbf{s}, i\boldsymbol{\beta}_A\cdot\mathbf{s}]\\&\quad+(1/12)[i\boldsymbol{\beta}_B\cdot\mathbf{s},[i\boldsymbol{\beta}_B\cdot\mathbf{s}, i\boldsymbol{\beta}_A\cdot\mathbf{s}]]\\&\quad+(1/12)[[i\boldsymbol{\beta}_B\cdot\mathbf{s}, i\boldsymbol{\beta}_A\cdot\mathbf{s}], i\boldsymbol{\beta}_A\cdot\mathbf{s}]+\cdots\},\end{aligned} \tag{18}$$

depends only on the commutation rules for $\mathbf{s}$.[9] Since the commutation rules are the same for all spins, the consistency for all spin follows from that for spin $\frac{1}{2}$. The components of ψ form a spinor since they transform linearly under Lorentz transformations.

The next step is to show that a function $\psi(x)$, that satisfies Eq. (1) and the auxiliary condition in the unprimed system, transforms into a function that satisfies Eq. (1) and the auxiliary condition in the primed coordinate system. In other words, the wave equation and the auxiliary condition together form a covariant statement. [Since ψ and ϕ are related by Eq. (3), if ψ satisfies Eq. (1) and the auxiliary condition, so also does ϕ.] To make the proof, let the solution of Eq. (1) and the auxiliary condition in the unprimed system be

$$\begin{aligned}\psi(x)=(2\pi\hbar)^{-\frac{3}{2}}\int d\mathbf{p}K_+(\mathbf{p})p^{s-1}u_+(\mathbf{p})\exp[i\hbar^{-1}(\mathbf{p}\cdot\mathbf{x}-cpt)]\\+(2\pi\hbar)^{-\frac{3}{2}}\int d\mathbf{p}K_-(\mathbf{p})p^{s-1}u_-(\mathbf{p})\\\times\exp[i\hbar^{-1}(\mathbf{p}\cdot\mathbf{x}+cpt)],\end{aligned} \tag{19}$$

where $K_\pm p^{s-1}$ are the coefficients for expanding $\psi(x)$ into plane waves. [The factor p^{s-1} is included because $K_\pm$ turns out to be a scalar.] To show the covariance it is sufficient to consider a pure Lorentz transformation in the 3-direction and a pure rotation about the 3-axis.

[7] See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), second edition, Sec. 24.

[8] The superscripts ^{C} and ^{H} denote the complex conjugate and Hermitian conjugate matrices. Unless otherwise specified, Latin indices run from one to three, Greek from one to four (x_4 is ict) and a sum is understood to be made on indices repeated within a term.

[9] F. Hausdorff, Leipzig, Ber. Ges. Wiss., math-phys. Kl. **58**, 19 (1906).

The special Lorentz transformation can be written as

$$x_1'=x_1,\quad x_2'=x_2, \tag{20}$$

$$x_3'-ix_4'=e^{i\beta}(x_3-ix_4), \tag{21}$$

$$\psi_m'=e^{i\beta m}\psi_m. \tag{22}$$

In the usual way, the integrals in Eq. (19) can be put into a covariant form in terms of $K(p_\mu)$, $u(p_\mu)$ defined by

$$\begin{aligned}K(\mathbf{p},p_4)&=K_+(\mathbf{p})\quad\text{when}\quad -ip_4>0\\&=K_-(\mathbf{p})\quad\text{when}\quad -ip_4<0,\end{aligned} \tag{23}$$

and similarly for u. One finds

$$\begin{aligned}\psi(x)&=-i(2\pi\hbar)^{-\frac{3}{2}}\int d^4pK(p)p^{s-1}u(p)\exp[i\hbar^{-1}p_\mu x_\mu]\\&\qquad\times[\delta(p+ip_4)+\delta(p-ip_4)]\\&=-2i(2\pi\hbar)^{-\frac{3}{2}}\int d^4pK(p)p^su(p)\\&\qquad\times\exp[i\hbar^{-1}p_\mu x_\mu]\delta(p_\rho p_\rho).\end{aligned} \tag{24}$$

To reproduce a plane wave expansion in the primed system, one makes a change of integration variable from p_μ to p_μ' parallel to the coordinate transformation:

$$p_1'=p_1,\quad p_2'=p_2, \tag{25}$$

$$p_3'\pm p'=e^{i\beta}(p_3\pm p). \tag{26}$$

When this is combined with Eq. (10) it is seen that

$$e^{i\beta m}p^su_m(p)=p'^su_m(p'). \tag{27}$$

Therefore the transformation of Eqs. (20) to (22) gives

$$\psi'(x')=-2i(2\pi\hbar)^{-\frac{3}{2}}\int d^4p'K(p)p'^su(p')\times\exp[i\hbar^{-1}p_\mu'x_\mu']\delta(p_\rho'p_\rho').$$

Since this result is of the same form as Eq. (24), $\psi'(x')$ fulfills Eq. (1) and the auxiliary condition in the primed coordinate system. Also $K(p)$ is to be identified with $K'(p')$,

$$K'(p')=K(p). \tag{28}$$

As well as K, K_+, and K_- are separately scalars. A similar proof applies for the pure rotation

$$x_1'+ix_2'=e^{-i\beta}(x_1+ix_2), \tag{29}$$

$$x_3'=x_3,\quad x_4'=x_4, \tag{30}$$

$$\psi_m'=e^{i\beta m}\psi_m, \tag{31}$$

and again K is found to be a scalar.

As well as the continuous transformations, reflections also must be considered. With respect to the space reflection

$$x_i'=-x_i,\quad x_4'=x_4,$$

and the time reflection

$$x_i'=x_i,\quad x_4'=-x_4,$$

the transformation of the spinor components is[8]

$$\psi'(x')=[C\psi(x)]^C, \tag{32}$$

where in this representation, Eqs. (9), C is defined by

$$C_{m,n}=(i)^{2m}\delta_{m,-n}. \tag{33}$$

The matrix C is Hermitian and unitary. It anticommutes with s_1, s_3 which are real and it commutes with s_2 which is pure imaginary. Therefore one can write

$$Cs_i=-s_i^CC, \tag{34}$$

and in consequence the covariance of Eq. (1) follows immediately. The transformation properties of K are

$$K_\mp'(\mathbf{p})=K_\pm^C(\mathbf{p}), \tag{35}$$

$$K_\pm'(\mathbf{p})=K_\pm^C(-\mathbf{p}), \tag{36}$$

for the space and time reflections respectively. These are easily found by substituting the plane wave expansion on the right in Eq. (32) and using the results

$$\begin{aligned}[Cu_\pm(\mathbf{p})]^C&=u_\mp(\mathbf{p})\\&=u_\pm(-\mathbf{p})\end{aligned} \tag{37}$$

to express $\psi'(x')$ also as a plane wave expansion.

IV. EXPECTED VALUES FOR PHYSICAL QUANTITIES

Just as in the special case of the photon,[10] for every transformation

$$x_\mu'=x_\mu'(x),\quad \phi'=\phi'(\phi),$$

that leaves the form of Eq. (1) unchanged, there is a conserved quantity whose density is $\phi^H(H/|H|)\mathcal{O}\phi$ and whose flux is $(c/s)\phi^H\mathbf{s}(H/|H|)\mathcal{O}\phi$, where the operator $\mathcal{O}$ is defined by

$$\phi'(x)=\mathcal{O}\phi(x). \tag{38}$$

To find $\mathcal{O}$ for the transformations discussed in the previous section, one first determines $\mathcal{O}_\psi$ in the usual way such that

$$\psi'(x)=\mathcal{O}_\psi\psi(x)$$

and then, according to Eq. (3),

$$\mathcal{O}=|H|^{-s+\frac{1}{2}}\mathcal{O}_\psi|H|^{s-\frac{1}{2}}. \tag{39}$$

For transformations to new origins of space and time, the wave function is assumed to be a scalar and one finds the operators p_i, iH/c. For rotations of the space axes and pure Lorentz transformations the operators $\mathcal{O}$ are

$$\mathbf{J}=\mathbf{x}\times\mathbf{p}+\hbar\mathbf{s}, \tag{40}$$

$$\mathbf{G}=|H|^{-s+\frac{1}{2}}[\mathbf{x}H-c^2t\mathbf{p}-ic\hbar\mathbf{s}]|H|^{s-\frac{1}{2}}. \tag{41}$$

[10] Reference 3, p. 1918.

The corresponding conserved quantities are

$$P_i=\int d\mathbf{x}\phi^H(H/|H|)p_i\phi, \tag{42}$$

$$P_4=(i/c)\int d\mathbf{x}\phi^H(H/|H|)H\phi, \tag{43}$$

$$\Theta_{ij}=\int d\mathbf{x}\phi^H(H/|H|)\epsilon_{ijk}J_k\phi, \tag{44}$$

$$\Theta_{i4}=-\Theta_{4i}=(i/c)\int d\mathbf{x}\phi^H(H/|H|)G_i\phi. \tag{45}$$

The first three are the momentum, energy, and angular momentum and the last gives the center of energy theorem.

The justification for these definitions and for the original choice of wave function in Eq. (3) is that P_μ is a Lorentz four-vector and that $\Theta_{\mu\nu}$ is an antisymmetric tensor (Θ_{44} is defined to be zero). The tensor properties for the special transformations of Eqs. (20) to (22) and (29) to (31) can be demonstrated by using the plane wave expansions of Eq. (19) as well as Eqs. (11) to rewrite P, Θ in terms of integrations in momentum space:

$$P_\mu=-2\int d^4p\delta(p_\rho p_\rho)K^C(p_4/|p_4|)p_\mu K, \tag{46}$$

$$\Theta_{\mu\nu}=-2\int d^4p\delta(p_\rho p_\rho)K^C(p_4/|p_4|)\times(p_\nu x_\mu-p_\mu x_\nu+\hbar T_{\mu\nu})K, \tag{47}$$

where x_μ is $i\hbar\partial/\partial p_\mu$. The quantity $T_{\mu\nu}$ is defined by

$$(T)_{\mu\nu}=\frac{is}{p_1^2+p_2^2}\begin{bmatrix} 0 & 0 & p_2p_4 & -p_2p_3 \\ 0 & 0 & -p_1p_4 & p_1p_3 \\ -p_2p_4 & p_1p_4 & 0 & 0 \\ p_2p_3 & -p_1p_3 & 0 & 0 \end{bmatrix}, \tag{48}$$

and it has the property that

$$T_{\mu\nu}(p')=a_{\mu\rho}a_{\nu\sigma}T_{\rho\sigma}(p), \tag{49}$$

when the momenta are transformed according to Eqs. (20) to (22) or (29) to (31). The fact that P, Θ are tensors with respect to the continuous Lorentz group follows then from the transformation property of K, Eq. (28). Also, using Eqs. (35) and (36) and expressing P and Θ explicitly in terms of K_+ and K_-, one sees that they are regular tensors by space reflection and pseudotensors by time reflection, as required.

The quantity

$$N=\int d\mathbf{x}\phi^H\phi = -2i\int d^4p\delta(p_\rho p_\rho)K^CK, \tag{50}$$

which is to be interpreted as the number of particles, is constant in time and is a scalar for both the continuous transformations and the reflections. For a Fermi particle N must be normalized to unity.

V. DISCUSSION

This gives a c-number theory for a single particle of zero mass and spin s. For bosons one expects that particles can be accumulated into a single state until the wave function becomes observable, as for the Maxwell field; then Eq. (1) becomes the equation for the observable field and N is the total number of particles.

In the special case $s=\frac{1}{2}$ the spinor components ψ are identical with the wave function ϕ and the theory reduces to that of the two-component neutrino.

In the special case $s=1$ the theory applies to the photon. The presentation above differs from the previous work[3] in the choice of representation of the spin one matrices and in a constant factor in ψ. In a different representation,

$$\bar{s}_i=Ss_iS^{-1}, \tag{51}$$

the C matrix with the properties of Eqs. (34) and (37) is found from

$$\bar{C}=S^CCS^{-1}, \tag{52}$$

and in the representation used earlier C is the identity. According to Eq. (32), the electric field $\mathbf{E}$ transforms as an axial vector and the magnetic field $\mathbf{B}$ as a polar vector, opposite to the earlier assignment.[11] This is the transformation rule recently suggested by Wigner[12] and Landau.[13]

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APPENDIX

The purpose of this Appendix is to outline proofs of the assertions made in Sec. III concerning the β parameters.

In the special case of real β_i, rewriting Eq. (15) as

$$\Lambda=\cos\tfrac{1}{2}\beta+i(\boldsymbol{\beta}/\beta)\cdot\boldsymbol{\sigma}\sin\tfrac{1}{2}\beta,$$

[11] Reference 3, Eq. (21e).

[12] E. P. Wigner, Bull. Am. Phys. Soc. Ser. II, 2, 36 (1957); Revs. Modern Phys. 29, 255 (1957).

[13] L. Landau, Nuclear Phys. 3, 127 (1957).

and substituting into Eq. (14), one obtains

$$a_{ij}=\delta_{ij}\cos\beta+\epsilon_{ijk}(\beta_k/\beta)\sin\beta+(\beta_i\beta_j/\beta^2)(1-\cos\beta),$$
$$a_{i4}=a_{4i}=0,\quad a_{44}=1.$$

It is seen that β_i is the angular displacement vector because it is the eigenvector of a_{ij} and a_{ii} is $(1+2\cos\beta)$; the sense can be verified by considering the transformation for small $\boldsymbol{\beta}$,

$$a_{ij}=\delta_{ij}+\epsilon_{ijk}\beta_k.$$

When the β_i are pure imaginary, they can be written as in Eq. (16) and a similar calculation leads to the pure Lorentz transformation in standard form.[14]

In general, $\boldsymbol{\beta}$ is a complex three-vector with respect to simultaneous rotations of primed and unprimed axes. To see this, let the spinor transformation matrix Λ^- correspond to a rotation of axes so that it is unitary and satisfies

$$\bar{a}_{ij}\sigma_j=(\Lambda^-)^H\sigma_i\Lambda^-,$$

where $\bar{a}_{ij}$ are the coordinate transformation coefficients.

[14] C. Møller, *The Theory of Relativity* (Oxford University Press, New York, 1952), p. 41, Eq. (25).

If this rotation is performed in both the primed and unprimed axes, related by

$$\psi'=\exp(i\tfrac{1}{2}\beta_j\sigma_j)\psi, \tag{53}$$

the new wave functions are

$$\bar{\psi}'=\Lambda^-\psi',\quad \bar{\psi}=\Lambda^-\psi.$$

Then, by operating from the left with Λ^- in Eq. (53) one finds that

$$\bar{\psi}'=\exp(i\tfrac{1}{2}\bar{\beta}_j\sigma_j)\bar{\psi},$$

where

$$\bar{\beta}_i=\bar{a}_{ij}\beta_j,$$

so that $\boldsymbol{\beta}$ is a vector in this special sense.

There only remains to show that any transformation matrix Λ can be written in the form $\exp(i\frac{1}{2}\beta_j\sigma_j)$. Any such matrix can be written as a product of pure rotations and pure Lorentz transformations which, as argued above, are of this form. As is seen from Eq. (18), the exponential form is preserved when products of exponentials are taken since the commutation rules for $\boldsymbol{\sigma}$ can be used to reduce every term on the right in Eq. (18) until it is linear and homogeneous in σ_i.

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Nonlinear Coupling in Low-Energy Meson Theory*

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It is shown that a very large class of meson-nucleon interactions leads to the same first-order scattering equations as would be obtained by adding simple nonlinear terms to the familiar linear pseudovector coupling in the Chew-Low-Wick formalism. The class of interactions under consideration is large enough to include what one would expect to be a reasonable form for the nonrelativistic limit of the charge-symmetric pseudoscalar interaction.

In the gauge-invariant extension of the theory to photoproduction, the P-wave coupling constant is not exhibited explicitly in the energy-independent S-wave part of the inhomogeneous terms of the integral equation for the transition amplitude. It is shown directly, however, that the contribution of the higher order terms in the limit of zero total energy is exactly as required for the satisfaction of the Kroll-Ruderman theorem.

I. INTRODUCTION

CHEW and Low[1] have shown that the assumption of a linear pseudovector meson-nucleon coupling in a fixed extended-source meson theory leads to integral equations for the scattering matrices that have provided a successful qualitative description of low-energy meson-nucleon scattering. An extension of this approach[2] to low-energy meson photoproduction met with similar success.

Application of a Foldy[3]-type transformation to the relativistic pseudoscalar theory, for example, suggests that the linear pseudovector coupling is not of sufficient generality to be considered as an approximately equivalent nonrelativistic coupling even for low-energy processes, except in the case of weak coupling. It is the purpose of this note to demonstrate that a more general choice of equivalent nonrelativistic coupling leads to exactly the same scattering equations as obtained by

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