Influence of Shell Structure on the Level Density of a Highly Excited Nucleus*

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(Received June 17, 1957)

Some features of the influence of shell structure on the level density of a highly excited nucleus ($\simeq$10 Mev) are studied on the basis of an idealized independent-particle model for which accurate formulas can be obtained in closed form. The theory is especially useful for estimating the ratio of the level densities of two nuclei in which the same degenerate (shell model) levels are being filled in the ground state, but which differ by a few units in the number of particles which occupy these levels. Observable effects persist to high energies ($\simeq$10 Mev) and they become very large in the neighborhood of the magic numbers. Thus, it is possible to determine experimentally whether or not the shell model of the nucleus retains any validity at high excitations.

I. INTRODUCTION

MANY years ago Bethe[1] calculated the density of energy levels of a heavy nucleus which is excited to any energy of about 10 Mev, on the simplifying assumption that the nucleons move as independent particles in a suitable average potential (Hartree approximation). Then the energy levels of the nucleus are given simply by the sums of the energies of the individual particles. Given a set of independent-particle levels, the problem of calculating the average level density of the excited nucleus is reduced to counting the number of levels of the combined system of particles which lie within a suitable interval of energy. In the counting process due regard must, of course, be paid to the exclusion principle.

Counting, while simple in principle, may be very difficult in practice, and so it is with the problem outlined—except for very low values of the excitation energy. Bethe's derivation of the level density is therefore based on the "statistical method of counting" in which an essential step is the evaluation of certain sums over discrete states which, generally speaking, can only be carried out by some approximate method. Bethe used the well-known "continuous approximation" in which the sums are replaced by integrals. The subsequent work of Van Lier and Uhlenbeck[2] showed, in a rather general way, that in this approximation only one parameter (one for each kind of particle) pertaining to the arrangement of the independent-particle levels is of importance for the level density of the whole system, namely, the number of quantum states per unit energy near the Fermi level,[3] which we shall denote by ρ. By adopting a reasonable dependence of ρ on the mass number A, Bethe explained two gross features of the observed level spacings: For a given nucleus the level spacing decreases rapidly with excitation energy, and for a fixed value of the excitation energy, the level spacing decreases rapidly with increasing A.

During recent years there have been significant developments, both theoretical and experimental which make a further study of the level-density problem worthwhile:

1. A good deal of experimental information on nuclear energy levels has been accumulated and more will become available in the near future. The data, especially those obtained from slow-neutron resonance experiments, have been analyzed by various authors[4] in terms of Bethe's theory and other very similar theories[5] which are based on the continuous approximation. It seems fair to say that these theories are in over-all agreement with experiment, as noted above, but are incapable of reproducing the observed irregular variations by factors of 2 or more as the particle numbers change by a few units.[6] Moreover, the theories fail altogether in the regions of the magic numbers where anomalously large level spacings occur.

2. During the last decade the independent-particle model of the nucleus has been revived with notable successes in the form given to it by Mayer and by Jensen, Haxel, and Suess.[7] More recently it has even been stated, as a result of the work initiated by Brueckner and collaborators,[8] that a theoretical foundation of the shell model, including a clear delineation of its region of validity, has been found.

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] H. A. Bethe, Phys. Rev. **50**, 332 (1936).

[2] C. Van Lier and G. E. Uhlenbeck, Physica **4**, 531 (1937).

[3] This is certainly the case for the energy range under discussion ($\sim$10 Mev); for much higher energies see N. Rosenzweig, Phys. Rev. **105**, 950 (1957).

[4] See especially T. D. Newton, Can. J. Phys. **34**, 804 (1956); also Harvey, Hughes, Carter, and Pilcher, Phys. Rev. **99**, 10 (1955); G. Brown and H. Muirhead, Phil. Mag. **2**, 473 (1957).

[5] J. M. B. Lang and K. J. LeCouteur, Proc. Phys. Soc. (London) A**47**, 585 (1954); J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), p. 371.

[6] To make this point clear, consider a set of isobars excited to the same energy. According to Bethe's theory all the isobars would have the same level density.

[7] M. Goeppert Mayer, Phys. Rev. **75**, 1969 (1949); Haxel, Jensen, and Suess, Phys. Rev. **75**, 1766 (1949) and Z. Physik **128**, 295 (1950). A comprehensive account of the scope of the shell model is given in M. G. Mayer and D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley and Sons, Inc., New York, 1955). We shall refer to the scheme of independent-particle levels given on p. 58 as the Mayer-Jensen scheme.

[8] Brueckner, Eden, and Francis, Phys. Rev. **98**, 1445 (1955). For a more complete set of references see H. A. Bethe, Phys. Rev. **103**, 1353 (1956).

One of the main purposes of this paper is to show that, contrary to some widely-held opinions, a refinement of the statistical method of counting is capable of yielding valuable information about nuclear structure when confronted with observed level spacings at high energies of excitation ($\sim$10 Mev). For example, our work shows rather convincingly that it should be possible to deduce from such data whether or not the nuclear shell model with a strong spin-orbit force retains its validity at high energies of excitation ($\sim$10 Mev). This possibility arises mainly from the fact that, in the regions of some of the magic numbers, the Fermi levels (i.e., the incomplete subshells) have an especially high degeneracy. Our calculations show that a high degeneracy of the Fermi levels leads to pronounced differences in the level density of nuclei for which the same independent-particle (Fermi) levels are being filled in the ground state, but which differ in the *number* of particles which occupy these levels in the ground state. If the shell model is valid, one may hope to obtain (eventually) detailed information about such quantities as the distance between the shells[9] and the number of quantum states per shell.

The theory based on the continuous approximation has fairly clear limitations in connection with these questions: by means of it one may hope, at best, to deduce the *ratio* of the average number of states per shell to the average distance between the shells. Moreover, the continuous theory makes no distinction between the level densities of two nuclei which have the same arrangement of levels and which differ only in the number of particles which occupy the Fermi level.

Actually, the double problem of considering the effects of shell structure on the level density and of refining Bethe's theory was clearly recognized by Margenau[10] long ago, in the era before the popularity of the shell model waned in the light of the success of Bohr's theory of the compound nucleus. Margenau avoided the continuous approximation by carrying out some numerical computations[11] which, in spite of their very limited scope, clearly suggested that shell effects may be important even at high excitations. More recently, the problem of improving on the continuous approximation has also been considered by Bloch.[12] His method is partially analytical, but since he attempts to treat a rather elaborate model he has to make many approximations. Bloch considers two interesting physical questions, viz., the distribution of angular momenta and the effects of the interaction between the particles, and he furnishes some evidence that the method of statistical counting is adequate for the purpose at hand.

In this paper we have attempted to shed some further light on the effects of shell structure by steering a middle course between the extremes of direct numerical computation and elaborate analytical approximations. We shall do this by treating an idealized nuclear model which has at least some of the essential features of shell structure, and yet is sufficiently simple to permit an accurate solution of the mathematical problem in a closed form which can easily be surveyed. In fact, our final formulas bear a close resemblance to the familiar results of the continuous approximation, with which they become identical when the energy of excitation is sufficiently high. The precise criteria in terms of the parameters of the shell structure are quite simple, and they show that the shell effects for our model nucleus are still very significant at excitation energies which correspond roughly to an energy of 10 Mev for a *real* nucleus. We shall place considerable emphasis on the rather convincing evidence, which we have obtained, that the statistical method of counting is sufficiently accurate for describing the (considerable) shell effects.

Our simple model consists of a mixture of two kinds of Fermi particles (neutrons and protons) each of which occupies a set of uniformly spaced energy levels. Furthermore, the degeneracy of each independent-particle level is the same, although the spacings and the degeneracies of the neutron and proton systems may be different. This model enables us to evaluate the crucial sums over states very accurately by means of the Euler-Maclaurin summation formula. For the sake of both clarity and brevity we shall treat the case of one kind of particle in Sec. II and the more pertinent case of two kinds in Sec. III.

Although we have succeeded in obtaining accurate results for our fictitious model, the application of these results to the more realistic level schemes leads only to a rough, though useful, estimate of the shell effects in nuclei (Sec. IV). With this qualification we may summarize the conclusions of physical interest as follows: A potential well having a depth of about 40 Mev and a range equal to the nuclear radius leads to the correct order of magnitude (on an absolute basis) for the level density for the entire range of mass numbers. An *average* degeneracy of 5 to 6 particles per level leads to an (irregular) variation in the level spacing (for a constant energy of excitation) within a small range of mass numbers. The magnitude of this variation is compatible with experiment. The value of 5 to 6 for the average degeneracy is consistent with the Mayer-Jensen scheme of levels. The anomalously large variations in the regions of the magic numbers can be accounted for by the especially high degeneracies and the varying occupations of the Fermi levels in the ground state.

[9] Knowledge of this type would enable one to deduce the magnitude of the effective mass of a nucleon inside nuclear matter; see, e.g., V. F. Weisskopf, Revs. Modern Phys. **29**, 174 (1957). However, we shall not deal with that problem in this paper.

[10] H. Margenau, Phys. Rev. **59**, 627 (1941).

[11] A numerical solution of the saddle-point problem may still be the best approach for elaborate nuclear models. With modern computing machinery, a comprehensive study along these lines is entirely feasible.

[12] C. Bloch, Phys. Rev. **93**, 1094 (1954).

II. (APPROXIMATE) SOLUTION OF THE COMBINATORIAL PROBLEM FOR ONE KIND OF PARTICLE

In this section we shall assume that all the Fermi particles are identical. This is the simplest case both from the physical and mathematical point of view. Nonetheless, the method of our treatment as well as almost all the features of the shell effects are revealed in this simple case. Therefore, after presenting a fairly detailed exposition of the simple case in this section, a much briefer account will suffice for the system which consists of two kinds of Fermi particles (Sec. III).

More specifically, the model consists of a number N of noninteracting Fermi particles which occupy a set of uniformly spaced energy levels, each of which is g-fold degenerate. The constant difference between the adjacent levels will serve as the unit of energy. The numerical value of the energy of the lowest level which a particle can occupy will be set equal to zero.

Let E denote the total energy of the system, including the zero point energy E_0. The energy of excitation Q is given by

$$Q = E - E_0. \tag{1}$$

Clearly, all the values of E and Q which are accessible to the system are integers.

We are interested in calculating the number of distinct ways in which the N particles can occupy the levels so as to give a total energy E. We shall denote that number by $C_g(N,E)$. In applying quantities of the type C_g to nuclei we shall have to make the usual assumption[2] that C_g represents the average density of states in a small interval of energy containing E (see Sec. V). For convenience, we shall even now refer to C_g as the level density.

Throughout this work we shall restrict our considerations to degenerate Fermi systems. By this we mean the following: given a number of particles N, we shall assume that the energy of excitation Q is not greater than the amount required to create one hole in the lowest independent-particle level of the system. Or, equivalently, given a value of Q, we shall assume that there are enough particles present so that Q will again not be larger than the amount required to create a hole in the zero level. A rough formulation of the above condition which is sufficiently close for our purpose is

$$gQ \lesssim N. \tag{2}$$

A little reflection shows that for a degenerate system, $C_g(N,E)$ is determined entirely by g, Q, and the number of particles n which occupy the Fermi level in the ground state of the system. We shall therefore employ the convenient, though mathematically unsound, notation

$$C_g(N,E) = C_g(n,Q). \tag{3}$$

1. Exact Solution for Low Values of the Excitation Energy

For low values of Q it is possible to obtain the exact solution to the combinatorial problem either by direct enumeration or from the generating function,

$$\sum_{N',E'} C_g(N',E') x^{N'} y^{E'} = \prod_{m=0}^{\infty} (1+xy^m)^g, \tag{4}$$

the sums and the product being over all the positive integers and zero. We shall write down the exact solution for $Q=0$, 1, 2 and 3. To save space we shall introduce the auxiliary function

$$F_{g,n}(\sigma) = \binom{g}{n-\sigma} + \binom{g}{n+\sigma}, \tag{4a}$$

which we shall further abbreviate by $F(\sigma)$. Then

$$\begin{aligned} C_g(n,0) &= \tfrac{1}{2}F(0), \\ C_g(n,1) &= gF(1), \\ C_g(n,2) &= g^2F(0) + gF(1) + \binom{g}{2}F(2), \\ C_g(n,3) &= g^2F(0) + g\left[1+\binom{g}{2}\right]F(1) \\ &\quad + g^2F(2) + \binom{g}{3}F(3). \end{aligned} \tag{5}$$

The factors F arise from the various occupations of the Fermi level. We shall return to a further consideration of this fact in Sec. IV.

In the above form the result is not easily surveyed. However, there is one general feature which emerges: n occurs only in the factors $F_{n,g}$. Since $F_{n,g}$ is invariant under the replacement

$$n \to g - n, \tag{6}$$

it is rigorously true for all values of Q (for a degenerate system) that

$$C_g(n,Q) = C_g(g-n, Q). \tag{7}$$

This means that (for our model) a system with n particles has the same level density as a system with n holes.

We have used the formulas (5) to obtain some exact numerical values for $C_g(n,Q)$ which are listed in column 4 of Table I. It will be noted that for a given value of g and Q the level density is greatest for a system which has a half-filled shell and is least for a system which has a completed shell in the ground state. If g is an even integer, then convenient measure of this shell effect is the ratio

$$R_g(n,Q) \equiv C_g(\tfrac{1}{2}g,Q)/C_g(n,Q). \tag{8}$$

TABLE I. Data illustrating that the asymptotic formula $C_g'(n,Q)$ provides an accurate description of the shell effect—measured by $R_g'(n,Q)$—even for low values of Q.

1 g	2 Q	3 n	4 Exact $C_g(n,Q)$	5 Asympt $C_g'(n,Q)$	6 C_g'/C_g	7 Exact $R_g(n,Q)$	8 Asympt $R_g'(n,Q)$
2	0	1	2	3.85	1.93	1.00	1.00
		2	1	a	a	2.00	a
	1	1	4	6.22	1.56	1.00	1.00
		2	4	5.08	1.27	1.00	1.22
	2	1	12	13.89	1.16	1.00	1.00
		2	9	11.43	1.27	1.33	1.22
	3	1	24	28.99	1.21	1.00	1.00
		2	20	24.27	1.21	1.20	1.19
6	0	3	20	24.54	1.23	1.00	1.00
		6	1	a	a	20.00	a
	1	3	180	211.5	1.17	1.00	1.00
		6	36	44.42	1.23	5.00	4.76
	2	3	1080	1191	1.10	1.00	1.00
		6	297	335.8	1.13	3.64	3.55
	3	3	4852	5254	1.08	1.00	1.00
		6	1588	1759	1.11	3.06	2.99
12	0	6	924	1043	1.13	1.00	1.00
		9	220	259.6	1.18	4.20	4.02
		12	1	a	a	924.00	a
	1	6	19 008	20 690	1.09	1.00	1.00
		9	6732	7377	1.10	2.82	2.80
		12	144	154.6	1.07	132.00	133.80
	2	6	217 404	232 400	1.07	1.00	1.00
		9	91 476	98 310	1.07	2.38	2.36
		12	4644	5124	1.10	46.81	45.3
	3	6	1 779 008	1 885 000	1.06	1.00	1.00
		9	833 680	886 100	1.06	2.13	2.13
		12	67 840	72 970	1.08	26.22	25.83

[a] At these values the asymptotic formula C' breaks down.

Some numerical values of $R_g(n,Q)$ are tabulated in column 7 of Table I. As noted above, the maximum ratio is given by $R_g(g,Q)$. It should also be noted that $R_g(g,Q)$ increases with increasing values of g.

Evidently, the shell effect is very important for low values of Q, as would be expected. However, we shall soon see that the effect persists to very high excitation energies, and that it probably plays a significant and sometimes a dominant role in the understanding of the level density of a highly excited nucleus.

2. Derivation of the Asymptotic Formula

We shall now derive an asymptotic formula for $C_g(n,Q)$ for large values of Q by means of the Darwin-Fowler method which was first used in a similar connection by Van Lier and Uhlenbeck.[2] From (4) it follows that $C_g(N,E)$ is given exactly by the contour integral

$$C_g(N,E)=\oint\oint\frac{\prod_m(1+xy^m)^g}{x^{N+1}y^{E+1}}dxdy. \tag{9}$$

The integrand has one and only one saddle point on the positive real axes. It is convenient to make the exponential transformation:

$$x=e^{\alpha},\quad y=e^{-\beta}. \tag{10}$$

Carrying out the saddle-point integration in the usual way, one obtains

$$C_g(N,E)\simeq\frac{e^{f(\alpha,\beta)}}{2\pi(\det A)^{\frac{1}{2}}}\equiv C_g'(N,E), \tag{11}$$

in which

$$f(\alpha,\beta)=E\beta-N\alpha+g\sum_m\ln(1+e^{\alpha-\beta m}), \tag{12}$$

the sum being over all integers; and A is the symmetric matrix of the second partial derivatives,

$$A=\begin{pmatrix}f_{\alpha\alpha} & f_{\alpha\beta}\\ f_{\beta\alpha} & f_{\beta\beta}\end{pmatrix}. \tag{13}$$

All the quantities must be evaluated at the saddle point (α,β) which is determined by the equations

$$f_\alpha=f_\beta=0. \tag{14}$$

At this point one must deal with the sums which occur in the equations. In the past, this has been done by such means as numerical computation,[10] the "continuous approximation,"[1,2,5] and more elaborate analytical approximations.[12] Our method differs from all of these. For our model a virtually exact and entirely tractable expression for $\sum\ln(1+e^{\alpha-\beta m})$ can be obtained by means of the Euler-Maclaurin summation formula provided that we retain the degeneracy assumption (2) and make the further assumption that the system is highly excited, i.e., that

$$Q\gg 1. \tag{15}$$

We now wish to outline the course of our mathematical argument which will lead to the solution of the saddle-point problem. We shall obtain an expression for the sum of logarithms in (12) which becomes *exact* as $\alpha\to\infty$ and $\beta\to 0$. We shall then verify that the solution obtained in this way indeed satisfies the assumptions (2) and (15). Since the solutions of Eqs. (14) are unique, the procedure is mathematically sound.

The Euler-Maclaurin formula applied to our sum reads

$$\sum_{m=0}^{\infty}\ln(1+e^{\alpha-\beta m})=\int_0^\infty\ln(1+e^{\alpha-\beta x})dx+\tfrac{1}{2}\ln(1+e^{\alpha})+\frac{\beta}{12(1+e^{-\alpha})}+\epsilon(\alpha,\beta). \tag{16}$$

The remainder $\epsilon(\alpha,\beta)$ is readily shown to be of the form

$$\epsilon(\alpha,\beta)=O(e^{-\alpha})+O(\beta^2). \tag{17}$$

If $\alpha>0$, the integral in (16) is given by the convergent expansion:

$$\int_0^\infty \ln(1+e^{\alpha-x})dx=\frac{\alpha^2}{2}+\frac{\pi^2}{6}+\sum_{k=1}^\infty \frac{e^{-k\alpha}}{k^2}. \tag{18}$$

Thus, as $\alpha\rightarrow\infty$ and $\beta\rightarrow 0$ the expression for $f(\alpha,\beta)$, on which subsequent calculations are based, assumes the relatively simple form:

$$f(\alpha,\beta)=E\beta-N\alpha+\frac{g}{\beta}\left(\frac{\alpha^2}{2}+\frac{\pi^2}{6}\right)+g\left(\frac{\alpha}{2}+\frac{\beta}{12}\right). \tag{19}$$

By using the above expression for f, the saddle-point problem can be solved *exactly* by elementary algebraic operations. By use of the identity

$$E-\frac{1}{2g}(N-\tfrac{1}{2}g)^2=Q-\frac{1}{2g}(n-\tfrac{1}{2}g)^2, \tag{20}$$

C_g' may be expressed in the form

$$C_g'(n,Q)=\frac{\exp[\pi(\frac{2}{3}gQ_s)^{\frac{1}{2}}]}{(48)^{\frac{1}{4}}Q_s}, \tag{21}$$

in which

$$Q_s=Q+\frac{g}{12}-\frac{1}{2g}(n-\tfrac{1}{2}g)^2. \tag{22}$$

To complete the argument we shall verify that the solution, which we have obtained, satisfies the assumptions (2) and (15). From Eqs. (14) one finds that

$$g\pi^2/(6\beta^2)=Q_s, \tag{23}$$

and

$$g\alpha/\beta=N-\tfrac{1}{2}g. \tag{24}$$

From (23) one sees that $\beta\rightarrow 0$ implies $Q\rightarrow\infty$. If β is determined by holding Q at some fixed (large) value, then Eq. (24) shows that $\alpha\rightarrow\infty$ implies $N\rightarrow\infty$. However, that is entirely consistent with the assumption (2) since, as we have already noted, the solution to the combinatorial problem, if the system is degenerate, depends only on n—not on N. Therefore, we may take N as large as we please or, equivalently, let $\alpha\rightarrow\infty$.

3. Remarks about the Asymptotic Formula

(a) *Adequacy of the Asymptotic Formula*

We have evaluated the asymptotic formula, Eq. (21), for very low values of Q, and have listed these in Table I, column 5. These values should be compared with the *exact* values given in column 4. It will be noted that even for low values of Q the relative error is small, as may be seen from the values of the ratio C_g'/C_g which are listed in column 6.

More significant for our purpose is the fact that the shell effect is adequately described by the asymptotic formula even for low values of Q. To see this we define

$$R_g'(n,Q)=C_g'(\tfrac{1}{2}g,Q)/C_g'(n,Q) \tag{25}$$

in analogy to (8). The values of R_g' are listed in column 8 of Table I, and they should be compared with the *exact* values given in column 7. It should be noted that the error in R_g' is small compared to the variations in R_g.

From the derivation we may expect that the asymptotic formula will be (relatively) even more accurate as Q increases. We therefore conclude that the asymptotic formula adequately describes the effects of shell structure for the model under discussion.

(b) *Symmetry between Particles and Holes*

The asymptotic solution has the symmetry property

$$C_g'(n,Q)=C_g'(g-n,\,Q), \tag{26}$$

which, as we have demonstrated, the exact solution also possesses.

(c) *Asymptotic Behavior of the Shell Effect*

As $Q\rightarrow\infty$, $R'\rightarrow 1$, and the shell effect disappears. However, the effect declines rather slowly with increasing energy. To obtain a convenient measure of this behavior, let us assume that $g/Q\ll 12$. Then the variation in the denominator of (21) may be neglected, and one obtains

$$\ln R_g'(n,Q)\simeq\pi\left(\frac{2g}{3Q}\right)^{\frac{1}{2}}\left[\frac{g}{16}-\frac{1}{4g}(n-\tfrac{1}{2}g)^2\right]; \tag{27}$$

and for the maximum effect (half-filled shell to closed shell) one obtains

$$\ln R_g'(g,Q)\simeq\tfrac{1}{16}\pi g(2g/3Q)^{\frac{1}{2}}\simeq\tfrac{1}{6}g(g/Q)^{\frac{1}{2}}. \tag{28}$$

Formulas (27) and (28) are in good qualitative agreement with the numerical results obtained by Margenau[10] on the basis of a more elaborate nuclear model (see Margenau's Fig. 4). However, Margenau's calculations are not extensive enough to exhibit the rather slow variation of (28) with energy.

Thus, for very *large* values of Q our formula (21) becomes independent of n, and we shall denote the resulting expression by $B_g(Q)$; i.e.,

$$C_g'(n,Q)\simeq\frac{\exp[\pi(\frac{2}{3}gQ)^{\frac{1}{2}}]}{(48)^{\frac{1}{4}}Q}\equiv B_g(Q). \tag{29}$$

Formula (29) is the familiar result which was first obtained by Bethe and also by Van Lier and Uhlenbeck on the basis of the "continuous approximation." It may be worth noting that formula (29) can be obtained formally from (21) for *all* values of Q by setting

$$n=g\left(\frac{1}{2}\pm\frac{1}{(6)^{\frac{1}{2}}}\right)\simeq 0.9g \quad\text{or}\quad 0.1g. \tag{30}$$

(*d*) *Comparison of Systems having the Same Number of Quantum States per Unit Energy*

In the above we emphasized the differences in the level density which arise from different occupations of the Fermi level for a fixed value of g. It is of some importance to note, however, that there may be considerable differences in the level densities of systems which have different values of g but which have the *same* number of (independent particle) quantum states per unit energy.

In order to show this we shall introduce explicitly the constant spacing between adjacent levels, to be denoted by γ. (Up to this point we had $\gamma=1$.) The number of quantum states per unit energy is given by g/γ. The level density becomes $C_g'(n,Q/\gamma)/\gamma$. Equations (25) to (28) hold with Q replaced by Q/γ. If $Q/\gamma \gg g^3/36$, then

$$C_g'(n,Q/\gamma)\rightarrow B_g(Q/\gamma). \tag{31}$$

In order to compare various systems which have the same value of g/γ, we have plotted in Fig. 1 the values of the dimensionless ratio $\log_{10}(C_g'/B_g)$ against the pure number $\log_{10}(gQ/\gamma)$ for various values of g and n. Clearly there are significant differences when $gQ/\gamma \lesssim g^4/36$; and in the same region the result $B_g(Q/\gamma)$ is obviously inadequate. For a fixed value of g, the curves in Fig. 1 illustrate once more the shell effects resulting from different occupations n of the Fermi level.

In Sec. IV we shall take the point of view that the level density of a nucleus which is excited to about 8 Mev should be described by the curves in those regions in which the shell effects are considerable.[13]

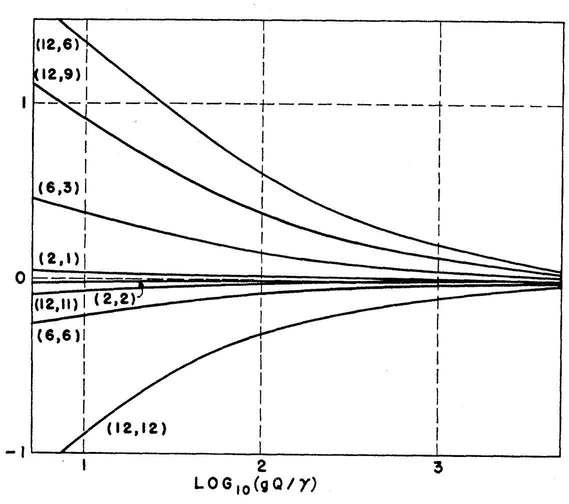

FIG. 1. Plot of $\log_{10}[C_g'(n,Q/\gamma)/B_g(Q/\gamma)]$ *vs* $\log_{10}(gQ/\gamma)$. The graph illustrates the various aspects of the shell effect discussed in the text; g and n are specified by (g,n).

[13] Actually, conclusions for nuclei must be based on the results for two kinds of particles [see Eq. (34)].

III. TWO KINDS OF PARTICLES

In this section we shall discuss the case which is more pertinent to the nuclear problem; we shall assume that the system consists of *two* kinds of Fermi particles, neutrons and protons. Qualitatively, the result will be much the same as in the previous section. However, quantitatively the differences are significant, and they matter even in the very rough evaluation of the nuclear shell effects which we shall undertake in Sec. IV.

We shall retain the assumptions that the spacing between the levels is uniform and that the degeneracy of each level is the same; however, we shall allow these quantities to be different for the neutron and proton systems. We shall use the following symbols. For the neutron system: N denotes the number of neutrons, n the number in the Fermi level, g the degeneracy of each level, and γ the spacing between adjacent levels. For the proton system: P denotes the number of protons, p the number in the Fermi level, d the degeneracy of each level, and δ the spacing between the levels. For the combined system: E denotes the total energy, Q the excitation energy, $C_{g,d}(N,P,\gamma,\delta,E)$ the level density, and $C_{g,d}'$ the asymptotic formula for $C_{g,d}$. As a matter of convenience we shall occasionally drop some or all of the variables.

We shall again assume that the system is degenerate and we may then write, in the same sense as in Sec. II,

$$C_{g,d}(N,P,\gamma,\delta,E)=C_{g,d}(n,p,\gamma,\delta,Q). \tag{31}$$

For low values of Q it is easy to obtain the exact value of $C_{g,d}$ by direct enumeration. We shall cite only one example which we shall use below for comparison with the asymptotic formula. For $\gamma=\delta=Q=1$, one obtains

$$C_{g,d}(n,\,p,\,Q=1)=g\binom{d}{p}\left[\binom{g}{n-1}+\binom{g}{n+1}\right]+d\binom{g}{n}\left[\binom{d}{p-1}+\binom{d}{p+1}\right]. \tag{32}$$

As before, the symmetry property between holes and particles hold rigorously for all values of Q. Thus

$$C_{g,d}(n,p)=C_{g,d}(g-n,\,p)=C_{g,d}(n,\,d-p). \tag{33}$$

The combinatorial problem can be solved asymptotically by the Darwin-Fowler method in a similar way and with the same degree of rigor as was done in Sec. II for one kind of particle. We shall therefore merely cite the result:

$$C_{g,d}'(n,p,\gamma,\delta,Q)=\frac{\exp\{\pi[\tfrac{2}{3}(g/\gamma+d/\delta)Q_s]^{\frac{1}{2}}\}}{4[216Q_s^5(g/\gamma)^2(d/\delta)^2/(g/\gamma+d/\delta)^3]^{\frac{1}{4}}}, \tag{34}$$

in which

$$Q_s = Q + \frac{g\gamma}{12} - \frac{\gamma}{2g}(n-\tfrac{1}{2}g)^2 + \frac{d\delta}{12} - \frac{\delta}{2d}(p-\tfrac{1}{2}d)^2.$$

It should be noted that C' satisfies the symmetry relations (34).

For comparison we have calculated some exact values for C using (32) and the corresponding values for C' using (34) and have listed the results in Table II. A convenient measure of the shell effect is the ratio

$$R_{g,d}(n,p,Q) = C_{g,d}(\tfrac{1}{2}g,\tfrac{1}{2}d,Q)/C_{g,d}(n,p,Q). \qquad (35)$$

The corresponding quantity obtained from the asymptotic formula is denoted by R'. Some numerical values of R and R' are listed in Table II. We see that the shell effect is very important for low values of Q, and that the asymptotic formula provides an accurate and simple description of it. Furthermore, the shell effect persists to high energies: Assuming that $Q \gg (1/12)(g\gamma + d\delta)$, one obtains in complete analogy to (28):

$$\ln R_{g,d}'(g,d,Q) \simeq \frac{g\gamma+d\delta}{6}\left[\left(\frac{g}{\gamma}+\frac{d}{\delta}\right)\frac{1}{Q}\right]^{\frac{1}{2}}. \qquad (36)$$

However, as $Q \rightarrow \infty$, the replacement of Q_s by Q in $C_{g,d}'$ Eq. (34), produces a negligible error, and the shell effects disappear. (One then obtains, incidentally, the result of the cruder continuous approximation.[2])

IV. APPLICATION TO NUCLEI

In this section we shall attempt an order-of-magnitude discussion of the degree to which the arrangement of the nucleons in well-defined independent-particle levels affects the level density of nuclei. For this purpose we shall assume that nuclei have their levels arranged according to the shell model with a strong spin-orbit force,[7] and we shall suppose that the energy levels of a nucleus are given by the sum of the independent-particle levels, paying due regard to the Fermi statistics as was done in Secs. II and III. In order to make some numerical estimates it will be necessary to extend the results of the preceding sections to the more complicated level scheme of Mayer and Jensen. In this paper we shall describe a simple *approximate* method of doing that, and we shall use the method to make some rough estimates of the shell effects. We shall be concerned primarily with the range of excitation energies which corresponds to the binding energy of the "last" neutron.

1. Nuclear Model

We shall adopt a literal interpretation of the nuclear shell model with a strong spin-orbit force, i.e., we shall assume that the energy levels of a nucleus are given by the eigenvalues of a Hamiltonian having the form:

$$H = \sum H_i,$$
$$H_i = -\frac{\hbar^2}{2m}\nabla_i^2 + V(r_i) + G(r_i)\mathbf{l}_i \cdot \mathbf{s}_i. \qquad (37)$$

In the above, the interactions between nucleons are taken into account insofar as they give rise to the effective potentials $V(r)$ and $G(r)$. We shall neglect all further perturbations.

It is well known that in terms of the above model many nuclear data pertaining to the ground state and to the low-lying excited states can be explained, especially in the regions of the magic numbers. There is much less evidence that the model remains valid at higher excitations.[14] Nonetheless, we shall adopt the assumption that the energy levels of a highly excited nucleus (5 to 10 Mev) are given by the sums of the eigenvalues of the independent-particle Hamiltonian (37).

Approximate eigenvalues for suitable wells have been obtained by various authors.[15] For our present considerations we shall only need some of the qualitative features of these solutions, the most important being the following: There are only a few (2 to 6) bound levels above the Fermi level. This statement holds both for the neutron and the proton systems. Let us consider a nucleus which is excited to about 5–10 Mev. Then it should be noted that, provided the mass number $A \gtrsim 20$, this energy is not sufficient to create a hole in the lowest independent-particle level of the nucleus, and the nuclear system therefore is degenerate in the sense described in Secs. II and III.

Thus, the situation described above is similar to the one of Sec. III in those respects which are responsible for the shell effect: (1) the existence of well-defined levels, each of which holds an appreciable number $(2j+1)$ of particles; (2) the existence of nuclei which have the same (or nearly the same) shell structure and which differ only in the number of particles which

TABLE II. Illustrates that the asymptotic formula C' provides an accurate description of the shell effect which is measured by R'. The parameters not listed have the values $\gamma=\delta=Q=1$, $d=6$, $p=3$, and $g=12$.

n	Exact C	Asympt C'	Exact R	Asympt R'
6	546 480	634 873	1.00	1.00
7	483 120	561 402	1.13	1.13
8	331 980	386 140	1.65	1.64
9	174 240	203 086	3.14	3.13
10	67 560	78 987	8.09	8.04
11	18 240	21 374	30.0	29.7
12	3060	3576	179	178

[14] E. Fermi, Nuovo cimento **11**, 407 (1954).
[15] K. Bleuler and C. Terreaux, Helv. Phys. Acta **28**, 245 (1955); A. E. S. Green, Phys. Rev. **104**, 1617 (1956).

TABLE III. Data showing that the nuclear shell model leads to the correct order of magnitude for the level spacing and indicates the maximum variation due to shell structure for "average" nuclei.

Q/γ	Approx. A	Level spacing in ev
1	50	500–15 000
2	75	50–1000
3	125	10–100
4	175	2–15
5	200	0.3–2.5
6	225	0.1–0.6

occupy the Fermi levels, and (3) the nuclei are degenerate for the range of energy under discussion. It therefore seems rather clear, in the light of Secs. II and III, that one may expect considerable shell effects in nuclei. We shall now consider the matter in greater detail.

2. Shell Effects in Regions Remote from the Magic Numbers; $Q=5$–10 Mev

(a) *Order of Magnitude of Level Spacing and Maximum Shell Effect*

The above nuclear model, when completely defined, leads to definite values of the level density. We shall show, in a rough way, that the adoption of reasonable values for the parameters which occur in the Hamiltonian (37) results in the correct order of magnitude for the level density for the entire range of mass numbers.

With Green[15] we shall assume that $V(r)$ has a depth of about 40 Mev and a range given by the nuclear radius. The strength of the spin-orbit coupling is assumed to have a magnitude which is consistent with the existence of the observed magic numbers. The eigenvalue problem so defined leads to a definite number of bound levels. In columns 1 and 2 of Table III we have listed the number of bound levels which lie above the Fermi level and the approximate mass number A which corresponds to it.

For the purpose of estimating the absolute magnitude of the average spacing by means of formula (34) of Sec. III, we shall use for g and d values which are obtained by forming *averages* over the Mayer-Jensen scheme of levels. We find the following:

Number of particles:	20	40	75	100	150
Average number of particles per level:	3	4	5	6	6

For our rough estimate we have adopted the values $g=6$, $d=4$. The parameters γ and δ are fixed as follows. In the first place we shall assume that $\gamma=\delta$. Next, let Q be the binding energy of the last neutron (5 to 10 Mev), then Q/γ represents the number of bound levels above the Fermi level, i.e., the numbers in column 2 of Table III. On this basis we have calculated the order of magnitude of the average level spacing in ev and have listed the results in column 3 of Table III. The indicated *range* in the level spacing represents the difference between half-filled and completely filled Fermi levels and represents the *maximum* shell effect for nonmagic nuclei. Both the order of magnitude and the maximum fluctuation are compatible with the observed level spacings.[16]

(b) *Minimum Shell Effect*

The simple model of Sec. III should also suffice for predicting the minimum effect which the arrangement of nucleons in shells produces on the level density. In almost any independent-particle model which has been proposed the degeneracy of each neutron level and of each proton level is at least 2. Let us use formula (35) for a rough evaluation of the minimum shell effect. Let us set $g=d=2$; for simplicity again let $\gamma=\delta$, and put $Q/\gamma=4$ in rough correspondence to the neutron binding energy of a moderately heavy nucleus. With these values of the parameters, one obtains

$$R_{g,d}(n=1, p=1)\simeq 2.$$

The above has the following implications: For excitation energies roughly equal to the neutron binding energy, the shell structure of nuclei causes the level density to vary by a factor of at least 2 as the mass numbers change by about 5 units.

3. Effects Near the Magic Numbers $Q=5$ to 10 Mev

As noted previously, the shell effect increases rapidly with increasing g and/or d. Now, it follows from the Mayer-Jensen scheme of levels that in the region of the magic numbers the values of g and/or d are especially *large* because of the high degeneracy which is associated with the high values of the orbital angular momenta of the levels $g_{9/2}$, $g_{7/2}$, $h_{9/2}$, $h_{11/2}$, $i_{13/2}$, etc., which separate the major shells. We shall see below that when we are dealing with models in which the degeneracy varies from level to level, as it does in the Mayer-Jensen scheme, the quantities which are decisive for the shell effect are still the same as in Secs. II and III, viz., the degeneracies of the Fermi levels and the occupations in the ground state of the system. That, together with the high degeneracies, leads immediately to the conclusion that the shell effect should be especially large for nuclei in the regions of the magic numbers.

[16] The levels observed in slow-neutron resonance experiments usually refer to only one or two values of the total angular momentum of the compound nucleus. In order to obtain the order of magnitude of the level spacing for all the levels, the observed level spacing must by multiplied by an appropriate factor, which is given, for example, by Bloch, reference 12, Eq. (17). We are primarily interested in estimating the variation in the level density resulting from the filling of an incomplete shell. For that purpose it is not necessary to reproduce accurately the absolute magnitudes of the level density, and no attempt has been made to do so in this paper.

(a) *Significance of the Fermi Level for Low Q*

We shall briefly discuss the dominant role which the occupation of the Fermi level plays in the shell effect. Let us focus our attention on two nuclei in which the same neutron level and the same proton level is being filled. Then the only difference between the two nuclei lies in the number of neutrons and in the number of protons which occupy the Fermi levels. Let both nuclei be excited to the same energy Q. We make the following assertion about the *ratio* of the level densities of these two nuclei: The ratio is given *approximately* by the ratio of the quantities $C_{g,d}'$ of Sec. III with the understanding that now g and d refer to the degeneracy of the *Fermi* levels, n and p are the numbers of neutrons and protons which occupy the *Fermi* levels in the ground state, and γ and δ are average values obtained from the immediate vicinity of the Fermi level.

The above statement has approximate validity provided the excitation energy is not too high and the degeneracies of the Fermi levels are appreciable. We shall not give a general analytical treatment of this important feature. However, an insight into the situation may be gained from the following simple example.

Let us suppose, for simplicity, that there is only one kind of particle, and let the spacing between adjacent levels again be unity. Let the degeneracy of the Fermi level be g and the degeneracies of all other levels be λ. Denote the level density by $D_\lambda(n,Q)$. Let us introduce the ratio

$$R_\lambda(n,Q)=D_\lambda(\tfrac{1}{2}g,Q)/D_\lambda(n,Q), \qquad (38)$$

in complete analogy to (8). For low values of Q, D is given exactly by the formulas (5) provided one replaces g by λ in the factors which multiply the functions F (only). Then it will be noted that

$$R_\lambda(n,0)=R_g(n,0), \quad R_\lambda(n,1)=R_g(n,1). \qquad (39)$$

Thus, our assertion regarding the ratios holds exactly for $Q=0$ and 1. For $Q=2$ and 3 the ratios are the same to within 8% and 15% respectively for values of λ having the magnitude which is characteristic of nuclear levels. From this example, and others as well, we conclude that the theory of Sec. III may be used for a rough estimate of the shell effect in nuclei for excitation energies which correspond to the neutron binding energy. This amounts to an extrapolation from $Q=3$ to $Q\cong 6$.

We shall now consider some particular nuclei. Nowhere in this section has any attempt been made to establish the best values of the parameters, we merely wish to assess the order of magnitude of the effect.

(b) *Isotopes of Tin*

The ground states of Sn isotopes are presumably formed by filling the $6h_{11/2}$ level in the Mayer-Jensen scheme. The proton $g_{9/2}$ level is completely filled, and a major proton shell is closed. According to our theory, the total variation in the level densities of the Sn isotopes is given by $C'(\tfrac{1}{2}g,d)/C'(g,d)$ with $g=12$, $d=10$. We shall set $\gamma=1$ and shall take the larger separation between major proton shells into account by putting $\delta=2$. We shall set $Q=5$ to correspond roughly to the neutron binding energy. With these numbers the level densities vary by a factor of 30.

TABLE IV. Possible values for the ratio of the level densities of Au^{197}/Pb^{208}. This illustrates the large shell effect which occurs in the regions of the magic numbers.

γ	δ	Q	Ratio
1	1	4	7.8×10^3
		5	3.5×10^3
		6	1.9×10^3
1	2	4	3.0×10^4
		5	1.2×10^4
		6	5.8×10^3
2	1	4	1.5×10^5
		5	5.1×10^4
		6	2.3×10^4

(c) *Ratio of the Level Densities of* Au^{197} *and* Pb^{208}

In the region of the periodic table which contains these two nuclei, there is a competition in the filling of several levels of the Mayer-Jensen scheme. This presumably means that the competing levels lie fairly close together, and for our rough estimate of the shell effect we shall treat these levels as coincident. Thus, the Fermi level of the proton system consists of the coincident $4d_{3/2}$ and $3s_{1/2}$ levels. The Fermi level of the neutron system consists of the coincident $7i_{13/2}$, $4p_{3/2}$ and $4p_{1/2}$ levels. Thus, we arrive at the following values of the parameters:

	g	d	n	p
Au^{197}	20	6	8	3
Pb^{208}	20	6	20	6

We have evaluated the *ratio* of the level densities for several values of γ, δ, and Q and have listed the results in Table IV. It is seen that the shell effect is enormous, and by a suitable choice of the parameters one can evidently account for the especially large level spacing of the nuclei surrounding Pb^{208} which has been observed in neutron resonance experiments. In this connection it should be noted that we have compared the level densities for the same value of the excitation energy, whereas the neutron binding energies may differ by as much as 30%. Within the framework of our theory we find that the difference in binding energies can account at the most for a factor of 10–50 in the ratio of the level densities, the remaining factor of about 100 is due to the shell effect.

4. Higher Excitation Energies

We shall briefly discuss, in a qualitative way, the shell effect for $Q\gtrsim 10$ Mev on the assumption that the nuclear model as outlined above remains valid. We shall be interested only in the density of levels in which *all* the particles are bound. If Q is larger than the binding energy of the last particle, then the theory of Secs. II and III does not apply, strictly speaking, because we assumed there that an infinite number of levels is available for occupation by the particles, whereas actually only a finite number—in fact only very few—are bound in the nuclear well. For that very reason, however, one may expect the shell effect to persist to even higher energies than is indicated by the results of Secs. II and III.

For $10 \lesssim Q \lesssim 40$, the nucleus remains a degenerate Fermi system in the sense described in Sec. II, and the shell effect is probably considerable. For Q of the order of several hundred Mev, the picture is rather complex; the shell effect is presumably diminished.

The picture becomes very simple again for extremely high energies of excitation (although the nuclear model presumably breaks down long before that point). As we have already noted, the number of levels in the nuclear well is finite, and therefore there exists an upper limit, $Q_{\max}$, to the excitation energy for which *all* the particles are bound. This $Q_{\max}$ is of the order of 1 Bev. In a small energy range below $Q_{\max}$ ($\sim$10 Mev) the situation is essentially the same as for the relatively low energies (5–10 Mev) which we have considered at length in this section; in fact, the only difference within the framework of the model is that the particles and holes have interchanged their roles. Thus, near $Q_{\max}$ the density of bound levels again becomes very small. The shell effect becomes large and is determined essentially by the degeneracies and occupations of the Fermi level of the system of holes.

V. MISCELLANEOUS REMARKS

1. Our mathematical treatment is based on a strict independent-particle model. This leads to the result that a certain value of the total energy E is realized in many ways (see, e.g., Table I) and that there is a relatively large (empty) interval of energy between the adjacent levels of the system. This does not correspond to what one finds in a real nucleus. It is well known, for example from neutron resonance experiments, that the levels of a nucleus are spread out, more or less uniformly, over the finite interval of energy in question. It is therefore necessary, at least in principle, to admit the existence of perturbations which will remove the degeneracy (presumably the only degeneracy which is ordinarily left in a nuclear level is the degeneracy with respect to the z component of the total spin). On the other hand, in order to retain the main features of the shell effect, it is probably necessary to assume that the shifts in energy, which are caused by the perturbations, are small compared to the spacing between the shells in the Mayer-Jensen scheme. The perturbations which will produce these effects must presumably be looked for in the residual part of the interaction between the particles which is not represented by the effective potential, Eq. (37). We have not considered these important questions at all, but have proceeded on the assumption that the residual interactions do not affect appreciably the average level density as obtained from our calculation.

2. In Sec. IV we presented an order-of-magnitude discussion of the shell effects which one may expect to find in nuclear level densities. A detailed analysis of the experimental data on level spacings derived from neutron resonance experiments is being carried out at the present time in collaboration with L. M. Bollinger. The analysis proceeds on the basis of considerations which are similar to those of Sec. IV, except that we also take into quantitative account the fact that only certain spin states are observed in slow neutron resonance scattering. If we should find that the main features of the shell effects, as described in this paper, are realized in nuclear level densities, we would infer that the shell model retains a considerable degree of validity at high energies of excitation ($\simeq$10 Mev).

ACKNOWLEDGMENTS

The author is grateful to Dr. L. M. Bollinger, Dr. J. Cook, Dr. D. Kurath, Dr. Kiuck Lee, and Professor M. Peshkin for several stimulating and useful discussions. The author wishes to thank Professor H. Bethe and Professor G. E. Uhlenbeck for an illuminating discussion. It is a pleasure to acknowledge the contribution to this work made by Miss M. A. Fisherkeller and Mr. W. Greenhow of the Applied Mathematics Division of this Laboratory, who carried out most of the numerical computations on the Argonne digital computer.