Distributions of Fission Neutron Numbers*

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(Received June 24, 1957)

It is shown, on reasonable assumptions as to the distribution of excitation energy among fission fragments, that the probabilities P_ν of observing ν neutrons from fission are given approximately, in cumulative form, by the "Gaussian" distribution,

$$\sum_{n=0}^{\nu} P_n = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{(\nu-\bar{\nu}+\frac{1}{2}+b)/\sigma} \exp(-t^2/2)dt.$$

In this equation $\bar{\nu}$ is the average number of neutrons, related to the average total excitation, b is a small adjustment ($b<10^{-2}$), and σ is the root-mean-square width of the distribution of total excitation in units of the average excitation energy change E_0 per emitted neutron. It is shown that all experimental data on neutron emission probabilities are reasonably well represented by this distribution with $\sigma \cong 1.08$, with the exception of Cf^{252}, which requires $\sigma = 1.21 \pm 0.01$. An estimate that $E_0 = 6.7 \pm 0.7$ Mev gives an excitation energy distribution and a rate of change of $\bar{\nu}$ with incident neutron energy ($d\bar{\nu}/dE_n \cong 1/E_0$) in reasonable accord with experiment. These conclusions should also hold approximately for fission induced by higher energy neutrons, in which case a few neutrons may be emitted before fission.

INTRODUCTION

THE probability P_ν of emission of any given integral number ν of prompt neutrons from fission can be calculated from the distribution of excitation energy among the fission fragments, if sufficiently accurate experimental information can be obtained. Leachman[1] has performed such calculations, based on neutron evaporation theory, the results of which are in good agreement with experiment. He obtained the distribution of total excitation energy from ionization chamber data on the kinetic energy of fission fragments, corrected for estimated experimental dispersion and ionization defect, and converted to excitation energy by means of extrapolated atomic masses. The excitations of the individual fragments were then obtained by assuming their distributions to be independent and identical. Other parameters used were the effective nuclear temperature and the binding energy of each neutron emitted. Calculations were performed for three mass ratios in each case, and the results were weighted together. Finally, the ionization defect was adjusted to give the experimental value of $\bar{\nu}$, the average number of neutrons emitted.

Because of the complexity of such calculations, and the number of parameters, it was considered desirable to correlate the various sets of experimental data on neutron emission probabilities by means of a simpler calculation, based on a minimum of parameters. The results of this comparison should give some information on the distribution of the excitation energy of fission fragments, as well as a simple means of predicting the emission probabilities P_ν for given $\bar{\nu}$.

* Work done under the auspices of the U. S. Atomic Energy Commission. A preliminary account was given recently [James Terrell, Bull. Am. Phys. Soc. Ser. II, 2, 105 (1957)].

[1] R. B. Leachman, Phys. Rev. **101**, 1005 (1956); R. B. Leachman and C. S. Kazek, Jr., Phys. Rev. **105**, 1511 (1957).

CALCULATION OF NEUTRON NUMBER DISTRIBUTION

It is assumed that neutrons will be emitted from the fission fragments whenever this is energetically possible. Two simplifying assumptions are made: (1) that the emission of any neutron from any fission fragment reduces its excitation by a value ΔE which is nearly equal to the average value $E_0 = \langle \Delta E \rangle_{Av}$, and (2) that the total excitation energy of the two primary fragments from binary fission has a Gaussian or normal distribution with rms deviation σE_0 from the average energy $\bar{E}$. The energy ΔE is the sum of the binding energy and kinetic energy of an emitted neutron and is of the order of 7 Mev, but of course is not a constant. However, there is presumably a tendency for the sum, ΔE, to be more nearly constant than either of its components, with the neutrons emitted first from a fragment having the lower binding energies and higher kinetic energies. Assumption (2) appears quite reasonable on the basis of experimental determinations[2–5] of the distribution of fission-fragment kinetic energy, inversely related to the distribution of excitation energy for a given pair of fission fragments. Furthermore, the basic idea of correlation between fragment excitation and neutron emission is borne out by data of Hicks *et al.*,[6] although it is in apparent conflict with data of Fraser and Milton.[3,7]

Three parameters, σ, E_0, and $\bar{E}$, have already been

[2] D. C. Brunton and G. C. Hanna, Can. J. Research **A28**, 190 (1950); D. C. Brunton and W. B. Thompson, Can. J. Research **A28**, 498 (1950).

[3] J. S. Fraser and J. C. D. Milton, Phys. Rev. **93**, 818 (1954).

[4] Smith, Friedman, and Fields, Phys. Rev. **102**, 813 (1956); Smith, Fields, and Friedman, Phys. Rev. **106**, 779 (1957).

[5] J. C. D. Milton and J. S. Fraser, Bull. Am. Phys. Soc. Ser. II, 2, 197 (1957), and private communication.

[6] Hicks, Ise, Pyle, Choppin, and Harvey, Phys. Rev. **105**, 1507 (1957).

[7] J. C. D. Milton, Chalk River Report CRP-642-A, 1956 (unpublished).

defined, and the way in which the total excitation energy is divided between the two fragments has not yet been specified. However, it will be shown that it is possible to eliminate one parameter from discussion, and that the neutron number distribution is relatively insensitive to the way in which the total excitation energy is shared. Consider first the mathematically simplest case, in which all of the excitation energy is given to one fragment. Since the excitation energy has a Gaussian distribution, and each emitted neutron reduces the excitation by $\Delta E \cong E_0$, the neutron emission probabilities P_ν are given approximately, in cumulative form, by

$$\sum_0^{\nu} P_n = \frac{1}{\sigma E_0(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{(\nu+1)E_0} \times \exp[-(E-\bar{E})^2/2\sigma^2E_0^2]dE. \quad (1)$$

Explicit mention of E_0 may be eliminated from this equation by changing to the variable $t=(E-\bar{E})/\sigma E_0$ and then defining $\bar{E}/E_0=\bar{\nu}+\frac{1}{2}-b$, giving

$$\sum_0^{\nu} P_n = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{(\nu-\bar{\nu}+\frac{1}{2}+b)/\sigma} \exp(-t^2/2)dt = \frac{1}{2}+\frac{1}{2}f[(\nu-\bar{\nu}+\frac{1}{2}+b)/\sigma], \quad (2)$$

in which $f(x)$ is a normal probability integral, tabulated in various publications[8] and given by the expression

$$f(x)=(2\pi)^{-\frac{1}{2}}\int_{-x}^{x}\exp(-t^2/2)dt. \quad (3)$$

The set of neutron emission probabilities, P_ν, given in cumulative form by (2), involves only two independent parameters, $\bar{\nu}$ and σ, for b is determined by the necessary condition $\sum_0^\infty \nu P_\nu=\bar{\nu}$. It may be shown that

$$b\cong\frac{1}{2}-\frac{1}{2}f[(\bar{\nu}+\frac{1}{2})/\sigma], \quad (4)$$

and that $b<10^{-2}$ for all experimental data to be discussed in this paper, so that it is almost completely negligible.

The cumulative distribution of neutron numbers given by (2) is shown as the set of points (A) in Fig. 1. The parameters $\bar{\nu}$ and σ have been taken as the representative experimental values 2.5 and 1.08, respectively (see Table I). The points may be connected by a straight line because the vertical scale used is a "probability scale" in which the vertical dimension represents the inverse function $f^{-1}(2y-1)$ rather than the indicated scale y, which in this case is $\sum_0^\nu P_n$. On such a graph any continuous Gaussian distribution appears as a straight line, and a discrete "Gaussian" distribution appears as points on a straight line.

[8] For instance, *Tables of Normal Probability Functions*, National Bureau of Standards, Applied Mathematics Series No. 23 (Superintendent of Documents, U. S. Government Printing Office, Washington, D. C., 1953).

Although the "Gaussian" distribution (A) is mathematically simple, it is not physically reasonable that all of the excitation should always be given to one of the two fission fragments; a more reasonable assumption is that the fragments share the excitation in an uncorrelated way. Although it is probable that the heavy and light fragments have excitation energy distributions depending on their respective energy level densities, it is simplest to assume, as Leachman does,[1] that the two distributions are identical, though independent. With this assumption, the neutron number distribution from each fragment will be given by expression (2), with $\bar{\nu}/2$ and $\sigma/\sqrt{2}$ substituted for $\bar{\nu}$ and σ. The two identical distributions $P_\nu(L)$ and $P_\nu(H)$ must then be folded together to yield $P_\nu=\sum_{n=0}^{\nu} P_n(L)P_{\nu-n}(H)$. The resultant distribution, for the same parameters as (A), is the set of points (B) in Fig. 1. The points fall very nearly on a straight line, and distributions (A) and (B) would probably be experimentally indistinguishable. It must be remembered, however, that neither distribution (B) nor any other distribution in which both fragments share the excitation will necessarily have $\bar{E}/E_0\cong\bar{\nu}+\frac{1}{2}$, as is true of ($A$); actually, ($B$) has $\bar{E}/E_0\cong\bar{\nu}+1$.

In order to see what would be the effect of correlation between the fragment excitation energies, it may be assumed that the two fragments always share the total excitation energy equally. In this case the distribution of neutrons from each fragment is of the form (2) with $\bar{\nu}/2$ and $\sigma/2$ replacing $\bar{\nu}$ and σ. The complete correlation assumed here results in the complete absence of odd

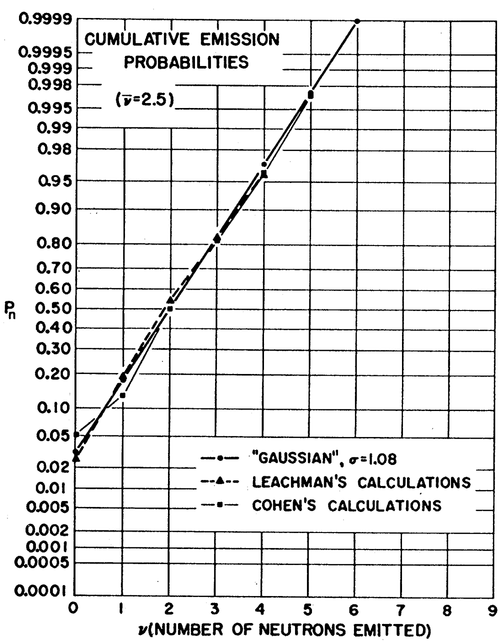

FIG. 1. Cumulative neutron emission probabilities for a Gaussian distribution of total excitation energy and various modes of distribution of the energy between two fission fragments. Neutrons are assumed to require a constant energy for emission.

neutron numbers with $P_\nu = P_{\nu/2}(L) = P_{\nu/2}(H)$. The points of this distribution, shown as (C) in Fig. 1, fall considerably above and below the "Gaussian" distribution (A), although the general trend is similar. If it is assumed that the correlation of excitation energies is almost perfect, but that there is a random difference averaging $E_0/4$ (about 2 Mev), the result is the distribution shown as (D) in Fig. 1. Even this small lack of correlation gives a distribution appreciably closer to (A), although even numbers of neutrons are still highly favored. However, even a small variation of the energy ΔE required per neutron would bring distributions (C) and (D) much closer to (A), so that much favoring of even numbers of neutrons is unlikely even with high correlation of excitation energies. Furthermore, high correlation is apparently inconsistent with Fong's statistical theory of fission.[9] Allowing ΔE to fluctuate in cases (A) and (B) would make essentially no difference in the calculated distributions if the fluctuations were small compared to E_0, although the width of the neutron number distribution would increase very slightly.

These considerations give some justification to the simple "Gaussian" distribution of Eq. (2), so that distributions of fission neutrons could be predicted on the basis of an estimate of σ, the ratio of excitation energy width to E_0. Although the approach taken in this paper will, instead, be that of determining σ from the neutron distributions, it is interesting to estimate the effective value of σ used implicitly by Leachman in his successful calculations.[1] He used neutron binding energies having an average value of about 5.0 Mev, based on an extrapolated semiempirical mass surface, and a nuclear temperature of 1.4 Mev. This would result in an average neutron energy of 2.8 Mev in the frame of reference of the fission fragment if the neutrons were not always limited in their maximum energy by the available excitation. As a result of this last factor, the kinetic energy per neutron is reduced to around 2 Mev for this temperature,[10] so that in Leachman's calculations $E_0 \cong 7$ Mev. Leachman also assumed a distribution of total excitation having a full width at half-height of about 17.8 Mev,[10] including all possible mass ratios, and corresponding to a root-mean-square deviation of 7.56 Mev for a Gaussian distribution. Thus $\sigma \cong 1.08$ in his calculations. Cohen[11] has also calculated neutron number distributions, by a method considerably simpler than Leachman's. He assumed a set of neutron binding energies having an average value of about 5.6 Mev,[12] based on an extrapolated mass calculation,[13] and a neutron kinetic energy of 1.5 Mev except for the last neutron. Essentially he was assuming $E_0 \cong 7.1$ Mev and an excitation full width of 19 Mev, giving an rms $\sigma \cong 1.13$. Because of the effective values

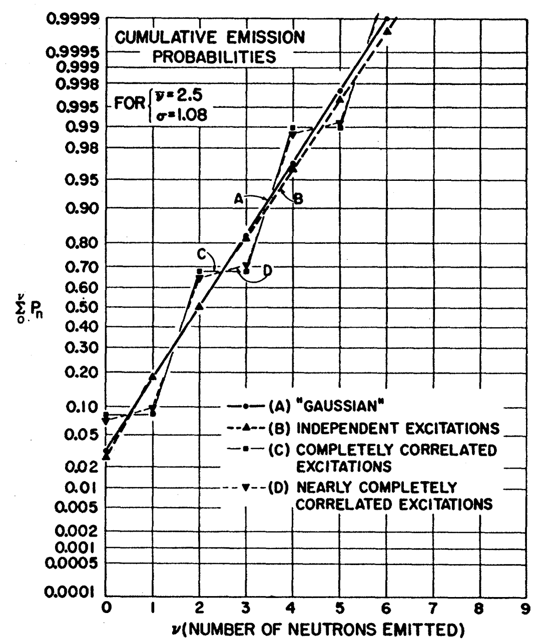

FIG. 2. Cumulative neutron emission probabilities as given by a "Gaussian" distribution and as calculated by Leachman and by Cohen.

of σ used in Leachman's and Cohen's calculations, it is not surprising that their results resemble "Gaussian" distributions for $\sigma = 1.08$ (this choice of σ will be justified in the next section). All three distributions are shown in Fig. 2 for the case $\bar{\nu} \cong 2.5$ (U^{235} thermal fission). Cohen's results for small numbers of neutrons are slightly different because of his assumption of fairly strong correlation of fragment excitation energies.

The conclusion which is indicated by the results of these various calculations is that if the excitation change ΔE per emitted neutron is more or less constant, and if the total excitation energy has a roughly Gaussian distribution, the resultant distribution of neutron numbers will be close to the "Gaussian" distribution of expression (2). This conclusion is essentially independent of the mode in which the two fragments share the excitation, and should also be true if a few neutrons are emitted before fission, with about the same value of ΔE. The two assumptions involved seem fairly reasonable on the basis of existing knowledge. It should be noted that the neutron number distribution of (2) does not include E_0 as a parameter, but depends only on the ratios of average excitation and excitation width to E_0.

COMPARISON WITH EXPERIMENT

It is of interest to see how well the experimentally observed neutron number distributions from fission fit the calculated "Gaussian" distribution. For this purpose all of the published data[14–16] on emission probabilities,

[9] P. Fong, Phys. Rev. **102**, 434 (1956).
[10] R. B. Leachman (private communication).
[11] Cohen, Cohen, and Coley, Phys. Rev. **104**, 1046 (1956).
[12] B. L. Cohen (private communication).
[13] N. Metropolis and G. Reitwiesner (unpublished).
[14] Diven, Martin, Taschek, and Terrell, Phys. Rev. **101**, 1012 (1956).
[15] Hicks, Ise, and Pyle, Phys. Rev. **101**, 1016 (1956).
[16] J. E. Hammel and J. F. Kephart, Phys. Rev. **100**, 190 (1955).

TABLE I. Widths of neutron number distributions from fission. For neutron-induced fission the energy of the neutrons is given; for U^{235} at 1.25 and 4.8 Mev it has not been completely proved that thermal or epithermal fissions do not contribute. The quantities σ, D, and Γ_2 are measures of the widths of the neutron number distributions; σ has been derived from either D or Γ_2 on the assumption of a "Gaussian" distribution. Standard deviations are given for all experimental quantities.

Fissioning nuclide	Reference	Experimental data $\bar{\nu}$	σ	$D=\langle\nu^2\rangle_{Av}-\bar{\nu}^2$	$\Gamma_2=(\langle\nu^2\rangle_{Av}-\bar{\nu})/\bar{\nu}^2$	Calculated for "Gaussian" distribution, $\sigma=1.08$ D	Γ_2
Pu^{242}	15	2.18±0.09	1.069±0.035	1.19±0.07	0.792±0.014	1.213	0.796
Pu^{240}	14–16	2.26±0.05	1.109±0.012	1.28±0.03	0.807±0.005	1.218	0.796
Pu^{236}	15	2.30±0.19	1.11 ±0.11	1.28±0.21	0.807±0.039	1.221	0.796
Pu^{238}	15	2.33±0.08	1.115±0.023	1.30±0.05	0.809±0.009	1.223	0.796
$U^{235}+n$ (80 kev)	14	2.47±0.03	1.072±0.021	1.22±0.04	0.795±0.007	1.231	0.797
$U^{233}+n$ (80 kev)	14	2.58±0.06	1.041±0.041	1.16±0.08	0.786±0.013	1.236	0.798
Cm^{242}	15	2.65±0.09	1.053±0.013	1.18±0.03	0.791±0.004	1.238	0.799
$U^{235}+n$ (1.25 Mev)	17	2.65±0.07	1.04 ±0.06	1.15±0.12	0.786±0.018	1.238	0.799
$U^{238}+n$ (1.5 Mev)	17	2.65±0.09	1.23 ±0.08	1.56±0.18	0.845±0.025	1.238	0.799
Cm^{244}	14,15	2.82±0.05	1.036±0.018	1.15±0.05	0.790±0.005	1.243	0.802
$Pu^{239}+n$ (80 kev)	14	3.05±0.08	1.14 ±0.07	1.38±0.14	0.821±0.015	1.246	0.806
$U^{235}+n$ (4.8 Mev)	17	3.20±0.08	1.20 ±0.06	1.51±0.13	0.835±0.013	1.247	0.810
Cf^{252}	14,15	3.86±0.07	1.207±0.012	1.54±0.04	0.844±0.002	1.250	0.825
All			1.110±0.006				
All but Cf^{252}			1.079±0.007				

as well as a small amount of unpublished data,[17] have been recalculated to obtain the standard deviations of various quantities of interest such as $\sum_0^{\nu} P_n$. In cases where several groups have published data on the same fissioning nuclide, the various sets of data have been weighted together by use of the standard deviations.

The values of σ required to give the best "Gaussian" distribution for each of thirteen different cases are given in Table I, arranged in increasing order of $\bar{\nu}$. The value of σ for a "Gaussian" distribution is uniquely

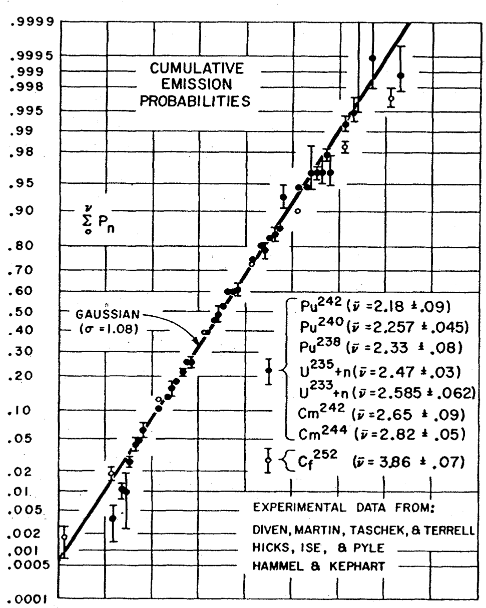

FIG. 3. Experimental cumulative neutron emission probabilities. The straight line represents a "Gaussian" distribution. Experimental data are from references 14–16; standard deviations are shown.

[17] Diven, Martin, Taschek, and Terrell (unpublished).

determined by knowledge of the second moment, $\langle\nu^2\rangle_{Av}=\sum_0^{\infty}\nu^2P_\nu$, for a given $\bar{\nu}$. However, the widths of the experimental distributions are more usefully expressed in terms of the quantities $D=\langle\nu^2\rangle_{Av}-\bar{\nu}^2$ or $\Gamma_2=(\langle\nu^2\rangle_{Av}-\bar{\nu})/\bar{\nu}^2$, either of which also uniquely determines σ for a given $\bar{\nu}$. D is the dispersion or mean square deviation; Γ_2 has the advantage of not depending on knowledge of the efficiency of neutron detection. The experimental values of D and Γ_2 from which σ was obtained are also given in Table I; standard deviations are given for all experimental quantities. The values of D and Γ_2 indicate that the distributions are obviously narrower than Poisson distributions, for which $D=\bar{\nu}$ and $\Gamma_2=1$. It has been pointed out[14,15] that a binomial distribution is a reasonably good representation of the data. For this case $\Gamma_2=1-1/m$, in which m is the maximum number of neutrons which may be emitted. All but the highest values of $\bar{\nu}$ tabulated here would require $m=5$ for a binomial representation, judging from the values of Γ_2. Perhaps the chief disadvantage of this distribution is the fact that the value of m required to fit a set of data reasonably well is apparently not the maximum possible number of neutrons in at least two cases (Cm^{244} and Cf^{252}). All of the experimental values of σ are similar. If Cf^{252}, with $\sigma=1.21\pm0.01$, is excluded, the weighted average of all other data gives $\sigma=1.08\pm0.01$, from which no value of σ except that of Cf^{252} deviates very significantly. The last two columns of Table I give calculated values of D and Γ_2 for a "Gaussian" distribution with $\sigma=1.08$, and these are seen to be reasonably close to the experimental values except in the case of Cf^{252}.

If all the experimental data on neutron emission probabilities were exactly represented by the "Gaussian" of expression (2) with $\sigma=1.08$, all of the points would fall on a single straight line when plotted on the probability scale of Figs. 1 and 2 as functions of $(\nu-\bar{\nu})$.

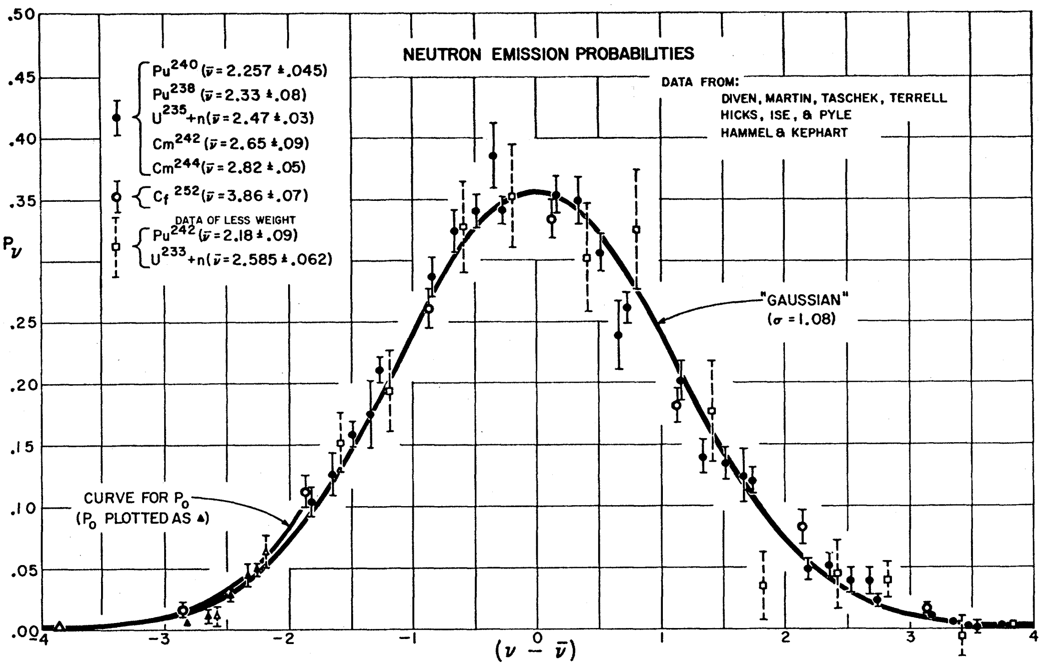

FIG. 4. Experimental noncumulative neutron emission probabilities. The continuous curves are for a "Gaussian" distribution. Experimental data are from references 14–16; standard deviations are shown.

Strictly speaking, the abscissa should be $(\nu-\bar{\nu}+b)$, but b is almost completely negligible for every experimental case. Such a plot is given in Fig. 3 for the eight best sets of experimental data; the other sets have been omitted because of large standard deviations. Only a few points deviate noticeably from the single Gaussian line. The exceptional Cf^{252} points are primarily those corresponding to the emission of more than five neutrons per fission, for which case this simple theory may not be quite adequate. However, there is probably a trend toward slightly larger excitation width (and σ) with increasing average excitation energy (and $\bar{\nu}$), as indicated by the noticeably larger value of σ for Cf^{252}.

The cumulative plot of Fig. 3 brings out most clearly the over-all features of the neutron distributions, but individual values of P_ν are best seen on a noncumulative plot, as in Fig. 4. The standard deviations of the points are, of course, quite different from those of the cumulative distributions. In Fig. 4 the exceptional width of the Cf^{252} distribution is not particularly noticeable. The continuous curves represent the values predicted by Eq. (2) for $\sigma=1.08$. This equation results in a slightly different curve for each value of ν, with an appreciable difference only for P_0 in the experimental range of $\bar{\nu}$. The experimental points do not indicate whether this separate curve for P_0 is necessary but do fall close to the "Gaussian" distribution for $\sigma=1.08$.

WIDTH OF EXCITATION ENERGY DISTRIBUTIONS

On the basis of the considerations described in this paper, it is possible to obtain the width (σE_0) of the distribution of total fission fragment excitation energy from the experimentally observed width (σ) of the distribution of fission neutron numbers. To do so requires an estimate of E_0, the average energy required to release a neutron. One estimate of E_0 may be obtained from the rate at which $\bar{\nu}$ changes with incident neutron energy in neutron-induced fission. The average kinetic energy of the fission fragments does not change noticeably with incident neutron energy, according to data of Wahl,[18] and of Segrè and Wiegand.[19] Except for the complicating factor of slight changes in the fission-fragment mass distribution, this indicates that almost all of the increased excitation of the fissioning nuclide appears as excitation of the fragments. This is also the prediction of Fong's theory.[9] On this basis $d\bar{\nu}/dE_n \cong 1/E_0$, where E_n is the incident neutron energy. This argument is not affected by the presumed emission of a few neutrons before fission for higher excitations, since E_0 should be roughly the same for them as for neutrons emitted after fission. Fowler has made calculations[20] of $d\bar{\nu}/dE_n$ based on this argument.

The observed variation of $\bar{\nu}$ with E_n, as measured by a number of groups, [14,17,20–25] is shown in Fig. 5 for

[18] J. S. Wahl, Phys. Rev. 95, 126 (1954).
[19] E. Segrè and C. Wiegand, Phys. Rev. 94, 157 (1954).
[20] J. L. Fowler (unpublished).
[21] J. Terrell and W. T. Leland (unpublished).
[22] Bethe, Beyster, and Carter, Los Alamos Scientific Laboratory Report LA—1939, August, 1955 (unpublished).
[23] J. M. Blair (unpublished).
[24] Auclair, Landon, and Jacob, Physica 22, 1187 (1956).
[25] D. J. Hughes and R. B. Schwartz, *Neutron Cross Sections*, Brookhaven National Laboratory Report BNL-325, Supplement

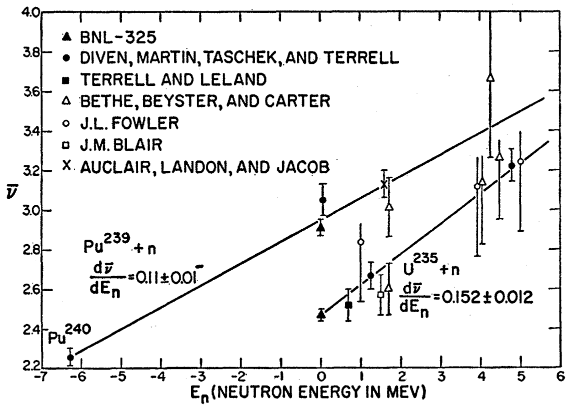

FIG. 5. Experimental variation of the average number of neutrons from fission with incident neutron energy. The straight lines represent least-squares fits. Standard deviations are shown; the experimental data are from references 14, 17, 20–25. Many data points are preliminary.

neutron-induced fission of U^{235} and Pu^{239}. The value of $\bar{\nu}$ for spontaneous fission of Pu^{240} is included at $E_n = -6.28$ Mev, corresponding to the neutron binding energy. Least-squares straight-line fits to the two sets of data in Fig. 5 give E_0 as 6.6±0.5 and 9±1 Mev, for U^{235} and Pu^{239}, respectively. However, it should be remembered that many of the data points shown are preliminary.

Probably a more accurate estimate of E_0 may be obtained from the observed kinetic energy of fission neutrons and from estimates of the neutron binding energies of fission fragments. Fission neutrons are observed to have about 2.0-Mev average energy in the laboratory system,[26,27] which corresponds to approximately 1.4 Mev in the center-of-mass system of a fission fragment, if one assumes prompt isotropic emission of neutrons. Neutron binding energies have been estimated to be about 5.0 and 5.6 Mev by Leachman[1] and by Cohen,[12] so that on this basis $E_0 \cong 6.7$ Mev.

If the value of E_0 is then taken to be 6.7±0.7 Mev, the experimental width ($\sigma=1.08$) of most neutron number distributions gives an rms width of 7.2±0.8 Mev for most excitation energy distributions, or a full width at half-maximum of 17±2 Mev. The corresponding figures for the exceptional case of Cf^{252} ($\sigma=1.21\pm0.01$) are 8.1±0.8 and 19±2 Mev. Actually each of these widths should be considered as a combination of the true excitation width and the width of the distribution of E_0, though this last factor is presumably 1 or 2 Mev and would have an almost negligible effect on the over-all width.

Recent experiments on the distribution of fission-fragment kinetic energies have given reasonably precise information on the distribution of excitation energies for a single mass ratio. Time-of-flight data of Stein,[28] after correction for velocity dispersions, indicate an excitation full width at half-maximum of 13.4±2.6 Mev for thermal fission of U^{235}. Similar data of Milton and Fraser[5] give about 15 to 17 Mev as the width for Cf^{252}. Fission-chamber data of Brunton, Hanna, and Thompson[2] give values of about 14 Mev for U^{235} after correction for dispersion, according to Fong,[9] whose theory predicts about 10 Mev. On the other hand, the recent magnetic-analysis experiments of Cohen, Cohen, and Coley[11] yield a figure of 19.3±1.5 Mev for U^{235}.

All of these experimental widths are for a single mass ratio near the peak. Because of the lack of accurate mass data for primary fission fragments, it is difficult to say how much the average total excitation energy changes as the mass ratio is changed. However, Fong[9] estimates about a 5-Mev increase from symmetrical fission to the most probable mass ratio, which would give an appreciable width to the distribution of total excitation energy even if the excitation for a fixed mass ratio were constant. Thus, the width of the total excitation energy distribution for all possible mass ratios produced in fission should be somewhat wider than for a single mass ratio. The experimental data on numbers of neutrons emitted from fission are consistent with this statement, but do not yield any reliable estimate as to the size of the effect. Furthermore, if it is possible for gamma-ray emission to compete with neutron emission from fission fragments, as suggested by Milton,[7] the neutron distribution would be wider than would be expected from the initial distribution of excitation energy. In this case, the estimate of 17±2 Mev for the excitation energy width should be reduced.

CONCLUSION

It has been shown that, on reasonable assumptions, the distribution of fission neutron numbers should be approximately a "Gaussian," given by Eq. (2), in which the only two variable parameters are $\bar{\nu}$, the average number of neutrons, and σ, the rms width of the total excitation energy distribution in units of E_0, the average excitation energy change per emitted neutron. This conclusion is almost independent of the manner in which excitation energy is divided between two fission fragments and should also hold for fission induced by higher energy neutrons, in which case a few neutrons may be emitted before fission takes place. All experimental data on numbers of fission neutrons are closely approximated by this distribution, with $\sigma \cong 1.08$ in all cases except that of Cf^{252} ($\sigma=1.21\pm0.01$). If E_0 is then taken as 6.7±0.7 Mev, the calculated width of the total excitation energy distribution is consistent with experimental data on this distribution. The rate of change of $\bar{\nu}$ with incident neutron energy

1 (Superintendent of Documents, U. S. Government Printing Office, Washington, D. C., 1957).

[26] B. E. Watt, Phys. Rev. **87**, 1037 (1952).

[27] Cranberg, Frye, Nereson, and Rosen, Phys. Rev. **103**, 662 (1956).

[28] W. E. Stein, Bull. Am. Phys. Soc. Ser. II, **1**, 96 (1956), and private communication.

($d\bar{\nu}/dE_n \cong 1/E_0$) is also reasonably consistent with this value of E_0.

ACKNOWLEDGMENTS

The author is indebted to a number of persons for the privilege of using and quoting their data before publication and wishes to thank Dr. B. C. Diven, Dr. R. B. Leachman, Mr. W. E. Stein, Dr. R. F. Taschek, and Dr. L. Wilets for many valuable discussions, and Mr. Paul E. Harper for many calculations in which an IBM 704 computer was used.

PHYSICAL REVIEW VOLUME 108, NUMBER 3 NOVEMBER 1, 1957

Stripping-Type Nuclear Reactions*†

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(Received June 27, 1957)

An exact formal expression is derived which treats the initial and final states of a nuclear reaction in a symmetric manner. The specific example treated is the (d,n) reaction. The Coulomb interaction is not taken into account. Estimates of some of the terms of the general expression which is obtained provide some indication of the reasons for the similarity of the final results of the various approaches to direct interaction. One of the terms of the development includes effects of compound-nucleus formation. Suggestions are made for a single-particle model of the intermediate state. The process of exchange and heavy-particle stripping is incorporated into the formalism in the Appendix.

I. INTRODUCTION

THE purpose of the present work is to derive an exact, though formal, expression for the cross section of a nuclear reaction of the stripping type. An attempt has been made to strive for conciseness and simplicity in the derivations. Rather than make specific and detailed predictions, the principles behind the argument are stressed and briefly outlined. With this aim in mind, the discussion has been restricted to the (d,n) reaction, though it would apply equally well to the (d,p), (He^3,p), etc., cases. Furthermore, the Coulomb interaction[1] has not been taken into account. Inclusion of this effect does not present any serious difficulties, but the results become somewhat cumbersome to write down. For similar reasons, no detailed treatment of spins or angular momenta has been undertaken either. Appendix A presents a brief discussion of the formal methods by which exchange and heavy-particle stripping[2] can be taken into account.

The point of view in the calculations has been to treat the initial and final states involved in the nuclear reaction on an equal footing. The results obtained in the present paper can be derived by a series of purely formal manipulations from previously known expressions.[3] The present point of view enables one, however, to suggest a reason for the success of the various approximate treatments of deuteron stripping[4] in fitting experimental data. Finally, our results lead naturally to suggestions for ways of including effects of compound-nucleus formation on the cross section.

II. GENERAL EXPRESSION FOR THE (d,n) CROSS SECTION

The notation of the present section follows in its general features that of Lippmann and Schwinger[5] and Chew and Goldberger.[6] We consider only the outgoing-wave solutions $\Psi^{(+)}$, but suppress the superscript $(+)$. We also assume that when the energy E appears in a Green's function, it contains a small positive imaginary part ($E \rightarrow E+i\epsilon$, $\epsilon>0$). Such a choice[5] is equivalent to a boundary condition which selects only outgoing waves at infinity.

From here on, we shall always refer explicitly to the (d,n) reaction. The complete Hamiltonian for this reaction can be written in the form:

$$H=H_0{}^i+H_1{}^i=H_0{}^f+H_1{}^f, \quad (1)$$

where H_0 is the zeroth-order Hamiltonian, and H_1 represents a "perturbation." The superscripts i and f refer to the initial and final state configurations. The operators depend on all the variables of the system, which consists of the target nucleus and the neutron and proton in the deuteron. We have not written down

* Supported in part by the U. S. Air Force, Office of Scientific Research.
† Supported by the U. S. Atomic Energy Commission.
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