The energies of levels at 1.34, 1.46, and 1.55 Mev are in excellent agreement with the values recently reported from measurement of $\gamma$ radiation.[5] If proton groups ($A$) and ($B$) do not correspond to states in fluorine, the upper limit of the cross section for excitation of other possible states is 0.1 mb/sterad.

### DISCUSSION

Confirmation has been obtained for states at 6.06, 6.14, 6.93, 7.13, and 8.88 Mev of excitation in $O^{16}$. These include three of the four states usually used to set up the $\alpha$-particle model. An extensive discussion of the 8.88-Mev state and its implications in relation to the $\alpha$-particle model has been given by Wilkinson *et al.*[2] Excited states at 1.34, 1.46, and 1.55 Mev in $F^{19}$ have been confirmed by inelastic proton scattering. No conclusive results were obtained regarding a state in $F^{19}$ at 2.22 Mev, and no evidence was seen for a state at 0.9 Mev.[11] The results presented here are in good agreement with those obtained for $O^{16}$ and $F^{19}$ by Squires *et al.*[12] The state at 1.98 Mev in $O^{18}$ has been observed by inelastic proton scattering, but no evidence was seen for a state at 2.45 Mev. These results might be interpreted as meaning the isotopic spin of this state is not the same as that of the ground state. However, the contradictory results of this and the other recent experiment[10] of $O^{18}+d$ scattering leave the isotopic spin of the 1.98-Mev state in $O^{18}$ in question.

[12] Squires, Bockelman, and Buechner, Phys. Rev. **104**, 413 (1956).

PHYSICAL REVIEW VOLUME 108, NUMBER 1 OCTOBER 1, 1957

# Impulse Approximation for Stripping Reactions*

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(Received September 21, 1956; revised manuscript received December 18, 1956)

The impulse approximation expansion is applied to stripping processes. We find that the first term of the impulse expansion for the stripping problem is identical with the first term of the Born expansion. It is suggested that the impulse approximation argument may provide a better justification for the usual treatment of stripping and pickup reactions than does the Born approximation argument.

## I. INTRODUCTION

THEORETICAL analyses developed to describe deuteron stripping and pickup reactions[1] have had a large measure of success. These treatments all contain two central assumptions. It is first of all assumed that the interactions giving rise to the stripping and pickup reactions all take place in the surface or outside region of the target nucleus. This is called the cutoff assumption. Secondly, it is assumed that these interactions can be treated by Born approximation. It is the second of these two assumptions which we wish to discuss.

It is, at first sight, surprising that the kind of interactions involved in stripping reactions can be satisfactorily treated by Born approximation. It is well known that when these same interactions are involved in elastic scattering processes they cannot be treated by Born approximation. We shall show that the Born approximation expression for the transition amplitude can be derived by means of an impulse approximation.[2] The impulse approximation would appear to be better justified for the stripping problem than the Born approximation, but for neither case has any quantitative estimate been made.

## II. BORN APPROXIMATION TO STRIPPING

Consider a system consisting of three particles which we denote by $n$ (a neutron), $p$ (a proton), and $N$ (a nucleus). Let $n$ interact with $p$ by means of potential $V_{np}$, and let $V_{Np}$ represent the interaction between $N$ and $p$. For simplicity assume there is no interaction between $N$ and $n$. Let $T=T_n+T_p+T_N$ be the kinetic energy operator. The Schrödinger equation for this system is then

$$(E_0-T-V_{Np}-V_{np})\psi_0=0. \qquad (1)$$

To discuss the solutions of the above equation we introduce the solutions of the Schrödinger equations for

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[1] G. F. Chew and M. L. Goldberger, Phys. Rev. **77**, 470 (1950); S. T. Butler, Proc. Roy. Soc. (London) **A208**, 559 (1951); A. B. Bhatia *et al.*, Phil Mag. **43**, 485 (1952); P. B. Daitch and J. B. French, Phys. Rev. **87**, 900 (1952); E. Gerjuoy, Phys. Rev. **91**, 645 (1953); W. Tobocman, Phys. Rev. **94**, 1655 (1954); R. G. Thomas, Phys. Rev. **100**, 25 (1955).

[2] G. Breit, Phys. Rev. **71**, 215 (1947); G. F. Chew, Phys. Rev. **80**, 196 (1950); G. F. Chew and G. C. Wick, Phys. Rev. **85**, 636 (1952); J. Ashkin and G. C. Wick, Phys. Rev. **85**, 686 (1952); S. Epstein, Phys. Rev. **86**, 836 (1952); G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952).

the system in the absence of one or more of the interactions,

$$(E_a - T - V_{Np})\varphi_a = 0, \tag{2}$$

$$(E_a - T - V_{np})\chi_a = 0, \tag{3}$$

$$(E_a - T)\theta_a = 0, \tag{4}$$

and the corresponding Green's functions,

$$G_a = \lim_{\epsilon \to 0} (E_a - T - V_{Np} + i\epsilon)^{-1}, \tag{5}$$

$$\mathcal{G}_a = \lim_{\epsilon \to 0} (E_a - T - V_{np} + i\epsilon)^{-1}, \tag{6}$$

$$g_a = \lim_{\epsilon \to 0} (E_a - T + i\epsilon)^{-1}. \tag{7}$$

The transition amplitude for scattering into the state $\varphi_0$ (elastic scattering, inelastic scattering, or stripping) is given by

$$A = \langle \varphi_0 | V_{np} | \psi_0 \rangle. \tag{8}$$

To evaluate $A$ for stripping, the cutoff assumption requires that we limit the integration over $r_p$, the separation of $p$ and $N$, to the range $R < r_p < \infty$, where $R$ is the range of $V_{Np}$. In addition we make the Born approximation of replacing $\psi_0$ by $\chi_0$, the wave function for the incident deuteron beam. If we want to do a little better we can use the distorted wave Born approximation by replacing $\varphi_0$ and $\chi_0$ with the wave functions $\bar{\varphi}_0$ and $\bar{\chi}_0$, respectively. These wave functions are defined by

$$(E_0 - T - V_{Np} - \bar{V}_{Fn})\bar{\varphi}_0 = 0, \tag{9}$$

$$(E_0 - T - V_{np} - \bar{V}_{ND})\bar{\chi}_0 = 0, \tag{10}$$

where the potentials $\bar{V}_{Fn}$ and $\bar{V}_{ND}$ are chosen so as to give the right elastic scattering. $\bar{V}_{Fn}$ is taken to be a function of the separation of the neutron from the center of mass of $N$ and $p$ while $\bar{V}_{ND}$ depends on the separation of $N$ from the center of mass of $n$ and $p$.

In order to estimate the difference between $\psi_0$ and $\bar{\chi}_0$ we make a perturbation expansion of $\psi_0$ in which $\bar{\chi}_0$ is the leading term. This expansion comes from interating an integral equation for $\psi_0$ in which $\bar{\varphi}_0$ is the inhomogeneous term.

$$\begin{aligned} \psi_0 &= \bar{\chi}_0 + \mathfrak{G}_0(V_{Np} - \bar{V}_{ND})\psi_0, \\ &= \bar{\chi}_0 + \mathfrak{G}_0(V_{Np} - \bar{V}_{ND})\bar{\chi}_0 + \cdots, \end{aligned} \tag{11}$$

where

$$\mathfrak{G}_0 = \lim_{\epsilon \to 0} (E_0 - T - V_{np} - \bar{V}_{ND} + i\epsilon)^{-1}.$$

We see that $\bar{\chi}_0$ will be a good approximation to $\psi_0$ if we can find a $\bar{V}_{ND}$ which essentially cancels $V_{Np}$. This is possible only if $V_{Np}$ is slowly varying over distances of the order of the average separation of the neutron and proton in the deuteron. This condition is not fulfilled for the stripping of deuterons by nuclei.

## III. IMPULSE APPROXIMATION TO STRIPPING

We shall construct an integral equation for $\psi_0$ having the impulse approximation wave function for the inhomogeneity. Iteration of this equation yields the impulse approximation expansion. We start by introducing the integral equation associated with Eq. (3),[3]

$$\chi_a = \theta_a + g_a V_{np}\chi_a, \quad \text{(scattering states)}, \tag{12}$$

$$\chi_\alpha = g_\alpha V_{np}\chi_\alpha, \quad \text{(bound states)}.^4 \tag{13}$$

We next introduce the operator

$$\begin{aligned} \Omega^\dagger &\equiv 1 - \textstyle\sum_a g_a V_{np}|\chi_a\rangle\langle\chi_a| - \sum_\alpha g_\alpha V_{np}|\chi_\alpha\rangle\langle\chi_\alpha| \\ &\equiv 1 - \{gV_{np}\}, \end{aligned} \tag{14}$$

whose Hermitian conjugate will play an important role in the impulse approximation expansion. We use the subscript $a$ to denote the quantum numbers of the scattering eigenstates of $T + V_{np}$ while the subscript $\alpha$ is used to denote the quantum numbers of the bound eigenstates. Comparison of Eq. (14) with Eq. (13) leads to the result

$$\Omega^\dagger = \textstyle\sum_a |\theta_a\rangle\langle\chi_a|. \tag{15}$$

Consequently, the Hermitian conjugate to $\Omega^\dagger$ is just

$$\Omega = \textstyle\sum_a |\chi_a\rangle\langle\theta_a|. \tag{16}$$

$\Omega$ is thus the wave matrix for $T + V_{np}$. Since the $\theta_a$ form a complete set while the $\chi_a$ do not, we have

$$\Omega^\dagger\Omega = \Omega - \{gV_{np}\}\Omega = 1, \tag{17}$$

and

$$\Omega\Omega^\dagger = \Omega - \Omega\{gV_{np}\} = 1 - \textstyle\sum_\alpha |\chi_\alpha\rangle\langle\chi_\alpha| \equiv 1 - P. \tag{18}$$

$P$, the projection operator on to the bound state eigenstates of $T + V_{np}$, satisfies the relation

$$P = [\Omega, \Omega^\dagger] = [\Omega, \{gV_{np}\}]. \tag{19}$$

We shall be concerned with two cases. The first case will be that arising when a neutron is incident on a bound state of the nucleus and proton. The wave function describing this situation will satisfy the following integral equations[5]:

$$\psi_0 = \varphi_0 + G_0 V_{np}\psi_0, \tag{20A}$$

$$\psi_0 = \mathcal{G}_0 V_{Np}\psi_0 = \mathfrak{G}_0(V_{Np} - \bar{V}_{ND})\psi_0. \tag{20B}$$

Here $\varphi_0$ is the wave function for the incident beam. The second case that interests us will be that which results when a deuteron is incident on a nucleus.

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[3] B. A. Lippman and J. Schwinger, Phys. Rev. **79**, 469 (1950); M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953); S. Sunakawa, Progr. Theoret. Phys. Japan **14**, 175 (1955).

[4] The wave function $\chi$, as well as $\theta$, $\varphi$, and $\psi$, depends on the coordinates of all three particles. The adjective "bound" and "scattering" applied to $\chi$ refers to the dependence of $\chi$ on the separation of the neutron from the proton.

[5] B. A. Lippmann, Phys. Rev. **102**, 264 (1956); L. L. Foldy and W. Tobocman, Phys. Rev. **105**, 1099 (1957).

The integral equations for this case are

$$\psi_0=\chi_0+\mathcal{G}_0V_{Np}\psi_0=\bar{\chi}_0+\mathfrak{G}_0(V_{Np}-\bar{V}_{ND})\psi_0, \quad (21A)$$

$$\psi_0=G_0V_{np}\psi_0, \quad (21B)$$

where $\chi_0$ is the wave function describing a plane wave of deuterons and nuclei.

Note that one integral equation does not uniquely determine the wave function of a three-body system. We can add any solution of Eqs. (21) to a solution of Eqs. (20) to get a second solution of Eq. (20A). Iteration of any one particular integral equation, if the resulting series converges, will give a particular solution of the equation.

For the moment consider the case where we have neutrons incident on bound protons. We can combine Eqs. (18) and (20A) to get

$$(1-P+\Omega\{gV_{np}\})\psi_0=\Omega(\varphi_0+G_0V_{np}\psi_0),$$

or

$$\psi_0=P\psi_0+\Omega\varphi_0+\Omega(G_0V_{np}-\{gV_{np}\})\psi_0. \quad (22)$$

From the definition of $\Omega$ we see that $\Omega\varphi_0$ is the expression which results from making a Fourier analysis of $\varphi_0$ and replacing each plane wave in the separation of $n$ and $p$ by an exact scattering wave function for $n$ and $p$. Thus $\Omega\varphi_0$ is the impulse approximation to $\psi_0$ for this case. $P\psi_0$ is the part of $\psi_0$ which asymptotically represents deuterons formed by pickup. $P\psi_0$ is negligible compared to $\Omega\varphi_0$ so we have, to a good approximation,

$$\psi_0\approx\Omega\varphi_0+\Omega(G_0V_{np}-\{gV_{np}\})\Omega\varphi_0+\cdots. \quad (23)$$

Thus we have before us the structure of the higher order corrections to the impulse approximation.

The above case was considered for purposes of illustration. Our primary interest is the case where we have deuterons incident on nuclei. Combining Eqs. (18) and (21$B$) gives

$$(1-P+\Omega\{gV_{np}\})\psi_0=\Omega G_0V_{np}\psi_0,$$

or

$$\psi_0=P\psi_0+\Omega(G_0V_{np}-\{gV_{np}\})\psi_0. \quad (24)$$

Here $P\psi_0$ asymptotically represents the incident deuteron beam plus the elastically scattered deuterons plus the inelastically scattered deuterons. Since the inelastically scattered deuterons are relatively unimportant, we can replace $P\psi_0$ by $\bar{\chi}_0$, the wave function for the incident and elastically scattered deuterons, without making much of an error. Thus

$$\psi_0\approx\bar{\chi}_0+\Omega(G_0V_{np}-\{gV_{np}\})\bar{\chi}_0+\cdots. \quad (25)$$

Just as in the previous case $\Omega\varphi_0$ was the impulse approximation to $\psi_0$, here we regard $\bar{\chi}_0$ as the impulse approximation to $\psi_0$.

The terms which we have neglected on the right sides of Eqs. (23) and (25) can be written $P\mathfrak{G}_0(V_{Np}-\bar{V}_{ND})\psi_0$. These are not neglected simply on the claim that $V_{Np}-\bar{V}_{ND}$ is small as in the Born approximation, but also because of the presence of the projection operator $P$. We have pointed out that

$$P\mathfrak{G}_0(V_{Np}-\bar{V}_{ND})\psi_0$$

represents the inelastically scattered deuterons and is thus negligible in comparison to the incident and elastically scattered deuterons. However, this interpretation has meaning only in the asymptotic region, that is to say, in the region where $V_{Np}-\bar{V}_{ND}=0$. However, as a consequence of the cutoff assumption alluded to in the introduction, we use only the asymptotic part of $\psi_0$, the part where $r_p>R$. Thus the cutoff assumption plays a role in justifying the impulse approximation for stripping.

## IV. DISCUSSION

We have demonstrated that the distorted wave Born approximation and the impulse approximation applied to the stripping reaction give the same result. But the conditions for the validity of these two approximations are quite different. In both instances it is necessary that $V_{Np}$ be slowly varying, but in the Born approximation the rate of variation of $V_{Np}$ is compared to the ratio of the binding energy of the deuteron to the average radius of the deuteron while in the impulse approximation we must compare the rate of variation of $V_{Np}$ to that of $V_{np}$.

The nuclear potential $V_{Np}$ varies by about 50 Mev in a distance of $10^{-13}$ cm while the $V_{np}$ interaction is usually represented by a potential having a range of about $10^{-13}$ cm and a depth of several thousand Mev. Clearly the condition for validity of the impulse approximation has a better chance of being satisfied than that for the Born approximation.

The validity of the impulse approximation for elastic scattering of low-energy neutrons incident on protons bound in molecules has been verified in direct numerical calculations by Breit and Zilsel[6] and by Lippmann.[7] The situation for intermediate incident energies and nuclear binding has not been investigated.

## ACKNOWLEDGMENTS

The author is indebted to L. L. Foldy and R. E. Peierls for many helpful discussions of the contents of this paper. He also wishes to express his gratitude to the staff of the Physics Department of the Case Institute of Technology for their kind hospitality.

[6] G. Breit and P. R. Zilsel, Phys. Rev. **71**, 232 (1947).
[7] B. A. Lippmann, Phys. Rev. **79**, 481 (1950).