small. We have used the value of $r_{0s}=3.85$ and $r_{0t}=0.808$. The results are plotted in Fig. 1.

We have plotted the results only up to $k=0.04$, since we feel that the validity of the effective-range formula can be guaranteed only in regions where $k^2 \ll 1/r_0^2$.

5. COMPARISON WITH OTHER RESULTS AND CONCLUSIONS

There are no experimental results for the scattering of electrons by hydrogen atoms in the vicinity of zero energy; Bederson, Hammer, and Malamud[2] have obtained a cross section of 65π at 1.5 electron volts. The relationship of their results to ours is shown in Fig. 2.

Of the calculations that have been made, the one by Massey and Moiseiwitsch[3] is closest to the one reported here. However, the authors do not emphasize the scattering at zero energy, nor do they report any results in the neighborhood of zero energy. It is difficult to estimate how to extrapolate their results. However, the values of $k \cot\delta$ calculated by them fall on a straight line both for the singlet and for the triplet case. If one extends this straight line to zero energy, one obtains a value of 64.6π, in contrast to our value of 76.6π. The higher value that we have obtained seems more in accord with the experimental results.

One other observation should be made. The results of McDougall,[3] Chandrasekhar and Breen[3] and Kato[3] for the scattering of an electron by the Hartree field of the hydrogen atom give a value of the zero-energy cross section of about 350π, which has always seemed too large. Since these authors do not include an exchange interaction in the Hartree-Fock sense, it is reasonable to suppose that their calculation is an approximation to the singlet cross section. This means that their value must be weighted with the statistical weight of $\frac{1}{4}$. The triplet cross section, as we have seen, is very small at zero energy and therefore the value obtained by considering the scattering by the Hartree field at the atom seems surprisingly good.

6. ACKNOWLEDGMENTS

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Nuclear Level Densities*

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By using the statistical methods originally due to Bethe, the predictions for the densities of nuclear energy levels at excitation energies around 8 Mev are examined for two different versions of the shell model. A crude method is used to take into account the effects of shell structure. The assumed form of the theoretical expression for the density of nuclear energy levels is employed to analyze the data from slow-neutron resonance experiments and from fast (n,γ) cross sections. In contrast to earlier results, for the necessary potential radius, it is found that either the static diffuse potential with a radius of $\sim 1.2\times 10^{-13}\times A^{\frac{1}{3}}$ cm, or the diffuse velocity-dependent potential based on the Johnson-Teller model with a radius of $\sim 1.4\times 10^{-13}\times A^{\frac{1}{3}}$ cm, leads to fair agreement with the above experiments. In each case the values of the thickness of the surface layer on the nuclear potential and the magnitude of the spin-orbit coupling are taken to be those previously found to give close agreement with the experimental shell-model level sequences.

The level-density expressions used here lead to an energy dependence which is in even stronger disagreement with those derived from various excitation function and inelastic scattering experiments than the empirical formula of Blatt and Weisskopf. It is argued that this anomaly may cast more light on the use of the statistical theory of nuclear reactions than on the validity of the expression for nuclear level densities.

I. INTRODUCTION

IGO and Wegner,[1] and others,[2] have pointed out that there exists an anomaly in the various measurements of nuclear level densities: different nuclear-reaction experiments give evidence about the energy dependence and the dependence on mass number A which seems contradictory. However, the statistical theory of nuclear reactions[3] is employed to analyze these experiments so that it is far from certain which of the many assumptions involved is breaking down. Besides the steadily increasing evidence for "direct interactions"[2,4] or noncompound-nucleus processes,

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[1] G. Igo and H. E. Wegner, Phys. Rev. **100**, 1309 (1955).

[2] See, e.g., Brookhaven Conference on the Statistical Aspects of the Nucleus, January, 1955, BNL-331 (C-21) (unpublished).

[3] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).

[4] Austen, Butler, and McManus, Phys. Rev. **92**, 350 (1953); R. M. Eisberg and G. Igo, Phys. Rev. **93**, 1039 (1954); R. M. Eisberg, Phys. Rev. **94**, 739 (1954).

there is some question[5] from the measurement of angular distributions at different energies as to whether the statistical assumption may not also be invalid. As a separate complication, the more direct measurements of level densities[6,7] provide stronger evidence for shell effects than can be ascribed purely to differences in binding energies of the compound nucleus.

In view of these discrepancies it seems worthwhile to re-examine the more direct measurements in order to determine whether they can be reasonably well fitted by the simplest theoretical expression based on independent-particle shell models and more recent radial measurements[8,9] than those considered desirable in earlier work.[10,11] This is of particular interest because of the now quite accurate measurements of nuclear level densities which come from a count of the slow-neutron resonances at excitation energies roughly equal to E_n, the separation energy of the last neutron in the compound nucleus. For with the advent of improved velocity selectors, such as the Brookhaven fast chopper,[12] resolution is now sufficiently high often to validate the assumption that no appreciable number of levels remain undetected in a given energy region.[13] Somewhat less direct values are derived from cross-section measurements of Hughes *et al.*[7] for fast radiative capture of unmoderated fission neutrons of an effective energy of one Mev. These give a measure of level densities at an excitation energy of (E_n+1) which is still direct compared, for example, to excitation-function measurements largely because the radiation widths, unlike the particle widths, vary little from one level to another.[14] The advantage of considering both types of experiment is that whereas one alone will test the prediction of the theory for the absolute number of levels, both combined provide some sort of check of the energy dependence.

In Sec. 2 we shall summarize briefly the assumptions that have usually been made, and are made in this paper. Little attempt is made to justify them. Since Bethe's original work[10] a considerable amount has been done on these lines; but we shall be concerned in investigating the present empirical justification for this general approach. In Sec. 3 we outline the methods used to analyze the experimental data, and the special methods employed in the theory to make use of two particular models[15,16] which, since they give a good description of shell-model assignments[17] for the ground state, might be expected to provide a reasonable starting point for a calculation in which we wish to retain shell effects. In Sec. 4 we discuss the results and their possible implications.

II. ASSUMPTIONS

The basic assumption made is that the number of excited nuclear levels in a given energy region is obtained, to a good approximation, by the number of ways in which nucleons can be excited to independent-particle levels such that the sum of the single-particle excitation energies is equal to the total excitation energy of the nucleus.[18] Although it is not necessary that the nuclear wave functions resemble those of the independent-particle configurations, it must be assumed that the configuration interactions do not spread over too wide an energy band. It should also be emphasized that even though this type of calculation is based on independent-particle models the true independent-particle or "single-particle" levels excited by an incoming nucleon form only a minute fraction of the total number of nuclear levels calculated.

The major computational assumption[19] is that the methods of statistical mechanics[20] can be used to evaluate the combinational problem. It is secondarily assumed that this is a case of strong Fermi degeneracy[21] which enables one to simplify the calculation by use of an asymptotic expansion.[22] Thirdly, it is assumed that, at least for heavy nuclei, it is justifiable to replace the discrete independent-particle energy levels by an energy density distribution of such levels.[23] Of these assumptions, the last is the most obviously unsatisfactory: given any specific model it is difficult to define such a continuous distribution. Shell effects will, indeed, only be taken into account in a crude manner in this work. We do, however, avoid the difficulty[24] encountered in

[5] R. M. Eisberg and N. M. Hintz, Phys. Rev. **103**, 645 (1956).

[6] Summaries, and references, for all data on level spacings found from slow-neutron resonances which are used in this article can be found in references 13, 28, and R. S. Carter and J. A. Harvey, Phys. Rev. **95**, 645 (1954); Carter, Harvey, Hughes, and Pilcher, Phys. Rev. **96**, 113 (1954).

[7] Hughes, Garth, and Levin, Phys. Rev. **91**, 1423 (1953).

[8] Hahn, Ravenhall, and Hofstadter, Phys. Rev. **101**, 1131 (1956).

[9] D. L. Hill and K. W. Ford, Phys. Rev. **94**, 1617 (1954); V. L. Fitch and J. R. Rainwater, Phys. Rev. **92**, 789 (1953).

[10] H. A. Bethe, Phys. Rev. **50**, 332 (1936); Revs. Modern Phys. **9**, 53 (1937).

[11] J. Bardeen, Phys. Rev. **51**, 799 (1937).

[12] Seidl, Hughes, Palevsky, Levin, Kato, and Sjöstrand, Phys. Rev. **95**, 476 (1954).

[13] Harvey, Hughes, Carter, and Pilcher, Phys. Rev. **99**, 10 (1955).

[14] J. S. Levin and D. S. Hughes, Phys. Rev. **101**, 1328 (1956).

[15] Ross, Mark, and Lawson, Phys. Rev. **102**, 1613 (1956).

[16] Ross, Lawson, and Mark, Phys. Rev. **104**, 401 (1956).

[17] See, for example, M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley and Sons, Inc., New York, 1955).

[18] Modified, for example, in the articles of J. Bardeen and E. Feenberg, Phys. Rev. **53**, 938 (1938), L. Motz and E. Feenberg, Phys. Rev. **54**, 1055 (1938), and references 20, 24.

[19] R. H. Fowler, *Statistical Mechanics* (Macmillan Company, New York, 1936).

[20] Not employed, for example, by G. Critchfield and S. Oleksa, Phys. Rev. **82**, 243 (1951).

[21] Investigated, for example, by C. Van Lier and G. E. Uhlenbeck, Physica **4**, 531 (1937) and N. Rosenzweig, Phys. Rev. **105**, 950 (1957).

[22] A. Sommerfeld, Z. Physik **67**, 1 (1928).

[23] Suppose the nucleons move in a mean potential $V(r)$. If the energy eigenvalues are ϵ_i with statistical weight g_i (i.e., maximum number of nucleons allowed to fill the level ϵ_i in accordance with the exclusion principle), then it is necessary to choose a $\rho(\epsilon)$ such that sums over the suffix i, weighted by g_i, can be replaced by integrals over $d\epsilon$ weighted by $\rho(\epsilon)$. The continuous function $\rho(\epsilon)$ will be called the average single-nucleon level density.

[24] C. Bloch, Phys. Rev. **93**, 1094 (1954).

the case of discrete levels where the degeneracy of a model level (of given j say) leads in the case of an incompletely filled level to a large number of rearrangements for no change in nuclear energy. This can, of course, be removed by the introduction of two-body interactions. For $A=20$, Bloch[24] does this very elegantly for long-range Majorana forces. But for heavy nuclei and short-range interactions this would involve a calculation not only tedious but far from straightforward. In this region, also, the approximation should be more applicable; and the introduction of the continuous approximation corresponds, in an inexact manner, to an artificial removal of these degeneracies. Nonetheless, it must be pointed out that the results we shall quote are very strongly influenced by the method devised to define the continuous distribution, and should be understood only in terms of this method.

III. ANALYSIS

For practical purposes we require not the total density of levels at an excitation energy U but those of only a given angular momentum J and a single parity, for a nucleus consisting of N neutrons and P protons (mass number $A=N+P$). On the basis of the given assumptions one derives in a standard manner the following expression for the density of nuclear levels (j-j coupling):

$$\rho_J(U)=\frac{1}{2}\frac{(2J+1)}{2(2\pi)^{\frac{1}{2}}\sigma_m^3}\exp\left[-\frac{(J+\frac{1}{2})^2}{2\sigma_m^2}\right]\times\frac{1}{4}\left(\frac{\delta_p^2\delta_n^2}{216\delta^3U^5}\right)^{\frac{1}{4}}\exp\left[\pi\left(\frac{2U}{3\delta}\right)^{\frac{1}{2}}\right],\quad(1)$$

where

$$\frac{1}{\delta}=\frac{1}{\delta_n}+\frac{1}{\delta_p}=\rho_n(\epsilon^0)+\rho_p(\epsilon^0)=\sum_m\rho_{n,m}(\epsilon^0)+\sum_m\rho_{p,m}(\epsilon^0),$$

$$\sigma_m^2=(1/\pi)(6U/\delta)^{\frac{1}{2}}\langle m^2\rangle_{\mathrm{Av}},\quad(2)$$

$$\langle m^2\rangle_{\mathrm{Av}}=(\sum_m\rho_{n,m}m^2+\sum_m\rho_{p,m}m^2)\delta,$$

and $\rho_{n,m}(\epsilon)$, $\rho_{p,m}(\epsilon)$ are, respectively, the average neutron and proton single-particle densities of levels with the z component of total angular momentum equal to m. The quantity ϵ^0 is the nucleon energy of the last filled level when the nucleus is in its ground state, i.e., the maximum Fermi energy.

In (1) the determination of nuclear level densities of a given spin and parity is essentially reduced to the calculation of two quantities, the average neutron and proton spacings at the top of the Fermi distribution and the magnitude of the most probable angular momentum J_m ($=\sigma_m-\frac{1}{2}$). Shortly it will be seen that the latter is relatively insensitive to the model chosen. Thus, although far from a negligible quantity (the reduction in number of levels from angular-momentum restrictions is $\sim$500 for heavy nuclei) it will not be of primary importance in determining which model is in closer accord with experiment. However, the average nucleon level spacing δ is very sensitive to both the model and the model radius chosen. Moreover in (1) this quantity occurs in the exponent. Concomitantly, the excitation energy U also occurs in the exponent. In both sets of experiments U depends on E_n (the separation energy of the last neutron in the compound nucleus). Since uncertainties of the order of 10% often exist in our knowledge of E_n we shall, for this reason, investigate the experimental level densities on the assumption that the form of (1) is correct. Where E_n has not been measured experimentally it is calculated from the semi-empirical formula[25] with corrections from the papers of Harvey[26] and of Wall,[27] so that in general it will be in agreement with those used in the last summarized article.

Since our assumptions hold better for heavy nuclei, we shall concern ourselves largely with data for $A>100$. Greatest weight will be placed on the comprehensive investigation of some twenty nuclides by Harvey *et al.*[13] Although we do not make an exhaustive survey of all existing measurements we shall include various other[6] estimates from data on slow-neutron resonances. Where this errs it is most likely to be on the side of overestimating the level spacing through omission of undetected levels.

The nuclear level density is derived from the fast (n,γ) cross sections by putting

$$\rho_J(E_n+1)=\frac{\sigma(n,\gamma)}{2\pi^2\lambda\!\!\!^{-2}\Gamma_\gamma}\quad\left(\equiv\frac{1}{D_0}\right).\quad(3)$$

Hughes *et al.*[7] took the radiation widths Γ_γ from the estimated smooth A dependence of Heidmann and Bethe.[28] Since then many more radiation widths have been measured and it has been shown that they exhibit a gradual increase before closed shells (most conclusively in the case of $N=126$ and $P=82$ shell) and a fairly abrupt decrease thereafter. Where new experimental values are available,[14,29] we have made fresh estimates of D_0. Further, we have occasionally interpolated between known values. When Pb is the target nucleus we have an outstanding example. The quoted D_0 changes from 61 ev to 1.6×10^3 ev owing to our choice of $\Gamma_\gamma=800$ Mev.

[25] N. Metropolis and G. Reitwiesner, U. S. Atomic Energy Commission Report NP-1980, 1950 (unpublished).
[26] J. A. Harvey, Phys. Rev. **81**, 353 (1951).
[27] N. S. Wall, Phys. Rev. **96**, 664 (1954).
[28] J. Heidmann and H. A. Bethe, Phys. Rev. **84**, 274 (1951).
[29] H. H. Landon, Phys. Rev. **100**, 1414. G. Igo, Phys. Rev. **100**, 1338 (1955).

To deduce δ from experiment, we rewrite (1) as follows:

$$\delta=\frac{2U}{3}\left\{\frac{\pi}{18.478-\ln[D_0(2J+1)(N+P)/U^2(NP)^{\frac{1}{2}}]+1.5\ln\langle m^2\rangle_{\text{Av}}+[(J+\frac{1}{2})^2/2\sigma_m^2]-\ln\delta}\right\}^2. \qquad (4)$$

The right-hand side of (4) originally depended weakly on δ_p and δ_n. For this purpose we have taken the Fermi-Thomas dependence and written

$$N/\delta_p \doteq P/\delta_n = NP/(N+P)\delta.$$

A value of δ can then be assumed for the right-hand side, which is then evaluated. If necessary the process is repeated.

For the value of $\langle m^2\rangle_{\text{Av}}$ we also made the assumption

$$\langle m^2\rangle_{\text{Av}}=\{N\langle m^2\rangle_{\text{Av}\ N}+P\langle m^2\rangle_{\text{Av}\ P}\}/(N+P).$$

To evaluate $\langle m^2\rangle_{\text{Av}\ N}$ [Eq. (2)] the contribution of all neutron levels in a given neutron shell is taken, and this is assumed to be the value of $\langle m^2\rangle_{\text{Av}\ N}$ at the center of the shell. This yields a smooth curve except at $N=24$ $(20-28$ shell). A similar technique is employed for protons. Values of $\langle m^2\rangle_{\text{Av}}$ for N and P can then be read off a plotted curve and must now be automatically the same for any model which yields a Mayer-Jensen type of shell structure.

It remains to define δ in terms of our models. These models are the diffuse static[15] and diffuse velocity-dependent[16] potentials considered by Ross *et al.*, where in each case we shall take those values of the surface thickness and the spin-orbit coupling which yielded the level sequence in closest agreement with ground-state experimental assignments for spins and parities. The value of r_0 (where the nuclear potential radius $r=r_0A^{\frac{1}{3}}$ $\times 10^{-13}$ cm) will be considered variable. From Eqs. (5) of reference 15 and equations of reference 16 it follows that if we change r_0 to r_0' in such a way that the level sequence is unchanged, then we have to a good approximation

$$\delta r_0^2=\delta' r_0'^2.$$

Thus, the final nuclear level density is indeed very sensitive to r_0. The quantity δ is also quite sensitive to the velocity dependence, which is of interest from the viewpoint of either the Johnson-Teller[30] or the Brueckner model.[31] Roughly, for that velocity dependence which is equivalent to a reduced nucleon mass of one-half in the center of the nucleus, we would expect δ to be correspondingly increased by a factor of two.[32] Actually the increase is not so great, because of the very large surface thickness required for the potential.

As in the case of the usual square-well limit for potentials of either kind, and with or without spin-orbit coupling, it is difficult to improve in a consistent manner on the original Fermi-Thomas approximation for δ. The structure of the levels is such that the type of graphical construction employed by Bloch[23] for light nuclei breaks down when applied for larger values of A. Yet it can be seen that the Fermi-Thomas approximation leads to a value roughly between that for a finite square well and that for an infinite square well. It can also be seen that as the nuclear potential is made diffuse the neutron levels tend to cluster at the top of the Fermi distribution. At the same time it becomes necessary to distinguish more carefully between neutrons and protons, for the former effect is counterbalanced in the proton case by an increase in the spacings due to the decreased radius of the net potential[33] (nuclear plus Coulomb). A Fermi-Thomas estimate for trapezoidal wells indicates that the effects on δ_n and δ_p are of similar magnitude and opposite sign.

Thus far there appears to be no great change. But the big difference is that we can now apply a rather different type of averaging, which has, we hope, some sort of physical justification. Both the models have very definite shell structures for both neutrons and protons throughout the periodic table. We shall now average over levels within a shell, and thus take into account the far-reaching effects of shell structure within the shells themselves,[34] despite the inadequacy of the continuous approximation for the treatment of the actual closed-shell nuclei. The average single-neutron level spacing δ_n has been evaluated as follows[35]: A closed-shell nucleus is taken. The sum of the spacing between the first and the last level in the (last) closed shell, plus one-quarter of the shell spacing above and one-quarter of the shell spacing below this shell, is divided by the number of neutrons allowed in the shell. We choose one-quarter instead of one-half because the spacing is at least doubled at the shell edge and here we are attempting to calculate the spacing inside the shell. This gives an average single-neutron level spacing some one-half of a shell thickness below the top of the Fermi distribution. To adjust this to the top of the Fermi distribution, we first calculated the difference in the classical turning points of the motion when the binding energy is changed from the middle to the top of the shell. The change in δ_n is then calculated for a trapezoidal well in WKB approximation, using this

[30] M. H. Johnson and E. Teller, Phys. Rev. **98**, 783 (1955).

[31] K. A. Brueckner, Phys. Rev. **97**, 1353 (1955).

[32] Similarly shown by Bardeen[11] on the basis of the article by J. H. Van Vleck, Phys. Rev. **48**, 367 (1935).

[33] M. H. Johnson and E. Teller, Phys. Rev. **93**, 357 (1954).

[34] Seen as a marked clustering of the levels in these regions, in the nucleon level diagrams of references 15 and 16.

[35] Some apparently unnecessary complications are introduced here. This is because rather lengthy machine calculations are required to obtain eigenvalues for these potentials. Thus, we prefer to make use of solutions already obtained. Even so the position of the highest level in the shell below the one under consideration was, in most cases, estimated by the methods outlined in the articles.[15,16] It is not felt that the accuracy of this calculation is such as to be impaired by these methods.

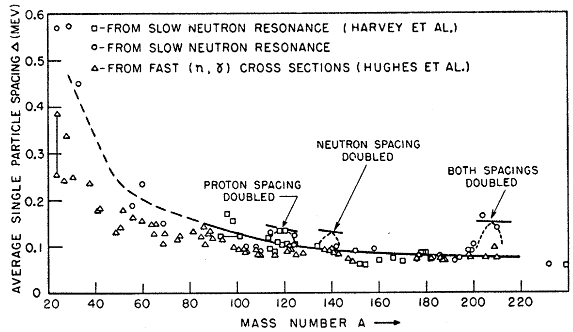

FIG. 1. The curve gives the theoretical value of the average single-nucleon level spacing δ for the static-potential model with $r_0=1.2\times10^{-13}$ cm and the velocity-dependent-potential model with $r_0=1.4\times10^{-13}$ cm. The points are the values derived for δ from experimental data. (δ is defined in the text, and should not be confused with the spacing of independent-nucleon levels of a given j.)

radial change. It is assumed that this estimated change can be applied to δ_n for the diffuse well. We are now in possession of an estimate for δ_n at the top of the closed neutron shells, were the shell spacing not present. Further, all such points can be connected by assuming (as in Fermi-Thomas approximation) a direct dependence on neutron number. These points should then give a fairly good average value for nuclei whose top neutron levels are within a shell. At the shell closure itself this spacing is also shown doubled in the figure, and decreasing again to its normal value within the addition or subtraction of approximately 5 nucleons. The same method is applied to δ_p. In Fig. 1, δ is plotted for the static and velocity-dependent cases with roughly the value of r_0 which gave, in our opinion, the best fit. This was $r_0=1.2\times10^{-13}$ cm for the static potential and $r_0=1.4\times10^{-13}$ cm for the velocity-dependent potential.

IV. RESULTS

In view of the nature of the errors involved, it appears that the experimental points, at least for heavy nuclei, show a fair degree of consistency when analyzed by these methods. The shell effects at $A\sim120$, $A\sim208$ are quite evident even though, probably, they have been somewhat underestimated by keeping the most probable angular momentum unchanged in these regions. It is also noticeable that the points derived from (n,γ) cross sections, where shell effects were first noticed, show far less of these effects than do the points from slow-neutron resonances, despite the new and larger values of Γ_γ around $A\sim208$. But in the shell regions (particularly in the region $A\sim140$) the calculated δ may become larger when more measurements of Γ_γ become available.

It is to be noted that for $140<A<200$ the two sets of experiments give similar values for δ. To put it more strongly, δ could have been deduced directly from the change in level densities of the isotopes in the region $A=180$ where measurements have been made by both methods on the same isotope. In this way a similar value would have been obtained for δ without assumptions about $\langle m^2\rangle_{Av}$. But in the shell region, values of δ obtained from the fast (n,γ) cross sections lie lower than do those from the slow-neutron resonances and also diverge steadily to lower values for light nuclei. There is no case where δ for the same isotope lies significantly higher in the former than the latter case. Even considering that the former are likely to be raised and the latter lowered by further experiments, it seems probable that the disagreement will persist. It is quite interesting that the direction of this disagreement is such that if we had deduced δ everywhere from the change in level density for an energy change of one Mev (as mentioned above) its value would be decreased, i.e., the change with energy is even faster than otherwise predicted. This will be even more strongly in disagreement with the results from the nuclear-reaction experiments, or with the empirical formula of Blatt and Weisskopf. It seems more probable that the discrepancy should be connected with one, or all, of the following reasons: averaging over widely spaced resonances (the disagreement exists only in these regions), insufficient information on Γ_γ or its variation from one resonance to another, or inadequacy of the assumption that only S-wave neutrons are absorbed. It is, however, possible that in the shell regions the apparent average single nuclear spacing really decreases as shell effects are gradually overcome at higher excitation energies. But, for light nuclei, there does not appear to be evidence of this effect where resonance levels are measured over wider energy ranges. The suggestion[36] of a false ground state in the shell regions also runs into difficulty, in that it predicts a slower energy dependence rather than a more rapid one in these regions.

There remains a certain amount of arbitrariness in our approximate choice of radii. In our considerations, greatest weight was placed on the data from slow-neutron resonances. But if the figure is examined closely it might still be asked why slightly larger radii were not chosen, which would have the effect of lowering the curve. Several points lie notably below it: Eu, Tb, Ho, Tm, and several isotopes just below $A=110$. The choice was made mainly because other methods of averaging inside the shell produce smaller values of δ, thus lowering the curve. Or if we take the method even more literally, and crudely associate a width of $(2j+1)\delta_n$ or $(2j+1)\delta_p$ with each filled j level we find the greatest overlaps in those regions where the experimental points lie lowest.

Insofar as the values of the radii are applicable to the ground-state properties, neither radius is unreasonable from the view point of other experiments. The value of $r_0=1.2\times10^{-13}$ cm for the static potential is in better agreement with the Stanford results[8] though a trifle too small. On the other hand, $r_0=1.4\times10^{-13}$ cm for

[36] H. Hurwitz and H. A. Bethe, Phys. Rev. 81, 898 (1951).

the velocity-dependent potential is probably more consistent with values derived from reaction experiments.[37] Were a surface oscillation term also introduced, as suggested empirically by Lang and LeCouteur,[38] the necessary radii would be decreased. The static potential would then have an exceptionally small radius.

There have been some comments in the literature to the effect that the factor $(2J+1)$ in the theoretical expression is not demonstrated in the experimental results. We do not feel that the evidence is conclusive in either direction. Agreement seems slightly better with this factor since many of the nuclei with small values of δ have high J and would show greater discrepancies without this factor. But experimental uncertainties seem too great for a decision on this point at present. Moreover the fairly marked even-odd effect makes a detailed comparison between neighboring isotopes, such as Lu^{175} and Lu^{176} impossible. This even-odd effect is shown clearly by the tin isotopes, molybdenum isotopes, and, more generally, by the fact that far from closed shells the compound nucleus is odd-odd for almost all nuclei with unusually small values of δ. These effects are not, of course, predicted in this statistical theory,[39] though hardly surprising when one considers the pairing energies needed in the shell model. They are also a good argument for the suggestion of Hurwitz and Bethe.[36]

To return briefly to the nuclear-reaction measurements, where the results are quoted in terms of a nuclear level density: the variability of the values found is exemplified by the experiments on Ag^{109} (where however the level density does not refer to the same nucleus in all experiments). Eisberg *et al.*[40] find $a=1.17$ Mev^{-1} for (α,p) reactions with 40-Mev α particles. In excitation-function experiments Bleuler *et al.*[41] find $a=2.4$ Mev^{-1} from (α,n) and $(\alpha,2n)$ cross sections with incident α energies up to 19.5 Mev and Porges[42] finds $a\sim 2$ Mev^{-1} from (α,pn) cross sections. Gugelot[43] obtains $a=8$–10 Mev^{-1} from inelastic scattering. These values can be contrasted with each other, and to our value of a $(=\pi^2/6\delta)=18.6$ Mev^{-1} for Ag^{110} from slow-neutron resonance experiments and $a=20$ Mev^{-1} from fast (n,γ) cross sections, the value of $a=13.8$ Mev^{-1} from the theoretical curve where $A\sim 110$, and the value used by Blatt and Weisskopf[44] of $a\sim 7.6$ Mev^{-1}. We believe that these discrepancies are likely to shed more light on the other assumptions involved in the experiments than on the nuclear level densities. For example, suppose we retain the picture of an intermediate compound state; it is well known that the particle widths are very far from being constant from one level to another[2] (and, in fact, appear to follow an exponential distribution). But if the particle widths vary too much, then the intermediate system will be most likely formed in the states which have large widths and which presumably therefore have a higher probability of decaying into those levels of the residual system corresponding most closely to the levels of the intermediate system. Thus, from each experiment we could obtain only a particularly weighted measure of level densities. This type of argument combined with the general trend and reasonable success of our analysis indicates that a wider analysis of level densities might perhaps only be fruitfully pursued in combination with a correlation to the widths of the levels.

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[37] See e.g., Millburn, Birnbaum, Crandall, and Schecter, Phys. Rev. **95**, 1268 (1954).

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[39] Two points might be noted. First, should we carry our theoretical calculations for $\delta(\epsilon^0)$ to a "logical" conclusion, δ will also depend on E_n in more detail. The direction of this effect is correct, but zero in Fermi-Thomas approximation. Second, the two exceptionally large experimental values of δ near $A=208$ are for odd-odd nuclei.

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