Distribution Functions for Noncommuting Operators*

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The properties of a certain class of functions $f(\alpha',\beta',\gamma',\cdots)$ of noncommuting observables α, β, γ, $\cdots$ are investigated. They have all the usual properties of distribution functions, except that they are complex. They satisfy simple Markoff-type stochastic equations and permit the calculation of the expectation values unambiguously. Conversely, quantum theory can be formulated in terms of such distribution functions having the prescribed properties.

1. INTRODUCTION

IN von Neumann's density matrix formalism of quantum mechanics, the state of a system is described by the statistical operator P, satisfying the following conditions: $P^\dagger=P$, Tr $P=1$, $(\ |P|\ )\geqslant 0$ for any state $|\ )$. Then $P-P^2\geqslant 0$. Equality in the last equation holds for pure states. The expectation value of any observable G in the state $|\ )$ corresponding to P is given by $G=(\ |G|\ )=\mathrm{Tr}(GP)$, $G^\dagger=G$. The equation of motion of P is $i\hbar P=[H,P]$.[1]

As an extension of this formalism we shall investigate the properties of a simple operator F, defined below, and a class of bilinear forms derived from F which will constitute the quantum-mechanical analog of the joint density operator and density distributions for any set of noncommuting observables. A spectral resolution of the observables and the relativistic and nonrelativistic equations of motion of the distributions are given, showing a Markoff-type stochastic equation between space-like surfaces. The expectation values of the observables will be found unambiguously in the usual manner. However, there are some fundamental differences as compared with the classical stochastic processes. The theory applies to "mixtures" as well as to pure states so that all information which the quantum theory gives can be obtained from a knowledge of F. In this sense the theory of these distribution functions is one of the, by now many, equivalent formulations of quantum theory.

In the special case of nonrelativistic quantum theory without spin, where all observables can be written as a function of coordinates and momenta, a real phase space distribution function has already been given by Dirac[2] and Wigner and Szilard,[3] studied in detail by Moyal,[4] Takabayasi,[5] and used in several applications.[6] Recently Bopp[7] and Uhlhorn[8] have also discussed the problem of phase space distributions in quantum mechanics.

However, the Dirac-Wigner-Szilard distribution function is valid only for a very limited class of operators, namely those operators which can be written as

$$G=\int \exp[i(\boldsymbol{\sigma}\cdot\mathbf{p}+\boldsymbol{\tau}\cdot\mathbf{q})]\xi(\boldsymbol{\sigma},\boldsymbol{\tau})d\boldsymbol{\sigma}d\boldsymbol{\tau},$$

where $\xi(\boldsymbol{\sigma},\boldsymbol{\tau})$ is the Fourier transform of the corresponding c-number function, i.e.,

$$G(\mathbf{p},\mathbf{q})=\int \exp[i(\boldsymbol{\sigma}\cdot\mathbf{p}+\boldsymbol{\tau}\cdot\mathbf{q})]\xi(\boldsymbol{\sigma},\boldsymbol{\tau})d\boldsymbol{\sigma}d\boldsymbol{\tau}.$$

It cannot be used to find expectation values of some simple and important operators such as the commutator $[p,x]$ or H^2 (a quantity important in statistical mechanics) or M^2, etc. This fact does not seem to be realized by some of the authors who have used this distribution function. This situation is remedied in the present approach by the use of a class of "equivalent" distributions functions. This is another quantum effect besides the nonpositivity of the distribution functions.

The distribution functions studied here show very clearly the fundamental distinctions between classical and quantum-mechanical distribution functions (Sec. 3) and obey stochastic equations with simpler kernel functions (Sec. 4). The general theory can be applied to any quantum-theoretical case and to field theory. Distribution functions corresponding to Feynman-amplitudes are also given (Sec. 4).

2. DISTRIBUTION FUNCTIONS

Let α, β, $\cdots$, ρ, σ be N different complete commuting sets of observables. We denote their normalized eigenfunctions by $|\alpha')$, $|\beta')$, $\cdots$ $|\sigma')$ and a general normalized state of the system, for the time being, by $|\ )$. α, β, $\cdots$ may also represent the same complete set at different times. We define the operator F (in dyadic form) by

$$F=|\ )(\alpha'|\beta')(\beta'|\gamma')\cdots(\rho'|\sigma')(\ |, \qquad (1)$$

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and form the bilinear functions

$$f(\alpha',\beta',\cdots,\sigma')=(\sigma'|F|\alpha')=(\,|\alpha')(\alpha'|\beta')(\beta'|\gamma')\cdots(\rho'|\sigma')(\sigma'|\,). \quad (2)$$

It is seen immediately that the integration of f over all the sets of observables except one gives the probability distribution of this set in the state $|\,)$.[9] Actually for a given system of complete sets we have a whole class of $N!$ functions having this property, namely functions obtained from $f(\alpha',\beta',\cdots,\sigma')$ by permutations of the arguments. Among them are the complex bilinear functions obtained from Hermitian conjugate operators

$$F^\dagger=|\,)(\beta'|\alpha')(\gamma'|\beta')\cdots(\sigma'|\rho')(\,|, \quad (3)$$

i.e.,

$$f(\sigma',\cdots,\beta',\alpha')=(\alpha'|F^\dagger|\sigma')=(\,|\sigma')(\sigma'|\rho')\cdots\times(\beta'|\alpha')(\alpha'|\,)=f^*(\alpha',\beta',\cdots\sigma'). \quad (4)$$

Thus, Eq. (4) expresses a symmetry of the distribution functions to the effect that reading the arguments backwards is equivalent to taking the complex conjugate.

The distribution functions are correctly normalized,

$$\int f(\alpha',\beta',\cdots,\sigma')d\alpha'd\beta'\cdots d\sigma'=1, \quad (5)$$

and under successive integrations they reduce as follows

$$\begin{gathered}\int f(\alpha',\beta',\cdots\rho',\sigma')d\sigma'=f(\alpha',\beta',\cdots\rho'),\quad\cdots,\\ \int f(\alpha',\beta',\gamma')d\gamma'=f(\alpha',\beta'),\\ \int f(\alpha',\beta')d\beta'=f(\alpha'),\\ f(\alpha',\alpha',\gamma')=f(\alpha',\gamma'),\\ f(\alpha',\alpha')=f(\alpha'),\\ \int G(\alpha'')f(\alpha',\alpha'',\beta')d\alpha''=G(\alpha')f(\alpha',\beta'),\end{gathered} \quad (6)$$

where the final functions $f(\alpha')$ are the positive real probability densities of a single complete set. From this relation it follows that the real part of $f(\alpha',\beta')$ itself could also be taken as a kind of joint distribution function and that

$$\int \mathrm{Im}f(\alpha',\beta')d\alpha'=\int \mathrm{Im}f(\alpha',\beta')d\beta'=0. \quad (7)$$

[9] For a statistical mixture of states $|s')$ with *a priori* probabilities $p_{s'}$ the definition of F is $F=\sum_{s'}p_{s'}|s')(\alpha'|\beta')\cdots(\rho'|\sigma')(s'|$. All the results in this paper apply also to this case unless stated otherwise. However, the introduction of "mixtures" into quantum theory is purely formal and phenomenological since the probabilities $p_{s'}$ are unknown, except in equilibrium. In general, if a mixture is given by its statistical operator P, then F is defined by $F=P(\alpha'|\beta')(\beta'|\gamma')\cdots(\rho'|\sigma')$.

According to Eq. (4), $\mathrm{Re}f(\alpha',\beta')$ is a symmetric, $\mathrm{Im}f(\alpha',\beta')$ an antisymmetric function of its argument. However, $\mathrm{Re}f$ is not necessarily positive nor must it remain positive in time. Since our aim is not to reduce quantum mechanics to classical statistics—in fact it will be seen that this is not possible—but to give a simple and closest possible analog of joint distribution functions, we shall prefer to use f rather than its real part. The reason is that, as we shall see, f satisfies a much simpler stochastic equation than $\mathrm{Re}f$.

From Eqs. (1) and (3) one gets that

$$FF^\dagger=F^\dagger F=|a|^2P\geqslant 0;\quad a=(\alpha'|\beta')\cdots(\rho'|\sigma'), \quad (8)$$

since von Neumann's density operator P can be written in dyadic form as $P=|\,)(\,|$. F is therefore a normal operator, and as such, its eigenfunctions corresponding to distinct eigenvalues are orthogonal. Indeed the eigenfunction of F is the state vector $|\,)$, the eigenvalue being $a=(\alpha'|\beta')\cdots(\rho'|\sigma')$. For mixtures, the eigenfunctions are $|s')$ with the corresponding eigenvalues $\lambda_{s'}=p_{s'}a$. We also note that since $FP=F$,

$$\mathrm{Tr}F=(\,|F|\,)\equiv\bar{F}=\textstyle\sum_{s'}\lambda_{s'}. \quad (9)$$

We see further from Eq. (2) that the distribution functions are such that two noncommuting quantities cannot have definite values, for if the system is in the state $|\alpha'')$, i.e., $f(\alpha',\beta')$ is proportional to $\delta(\alpha'-\alpha'')$, then its β'-dependence is necessarily proportional to $(\beta'|\alpha'')$. This expresses the uncertainty principle which quantitatively formulated in terms of mean square deviations reads

$$\Delta^2(\alpha)\Delta^2(\beta)\geqslant\left|\int\alpha'\beta'\,\mathrm{Im}f(\alpha',\beta')d\alpha'd\beta'\right|^2+|\tfrac{1}{2}\,\mathrm{Tr}[(\alpha\beta+\beta\alpha)P]-\bar{\alpha}\bar{\beta}|^2, \quad (10)$$

and

$$\tfrac{1}{2}\,\mathrm{Tr}[(\alpha\beta+\beta\alpha)P]=\int\alpha'\beta'\,\mathrm{Re}f(\alpha',\beta')d\alpha'd\beta'.$$

Thus, the imaginary part of $f(\alpha',\beta')$ is purely of quantum-mechanical origin and is responsible for the uncertainty principle, for the second term in Eq. (10) is also true for dependent random quantities in classical statistics. We expect, therefore, that in the limit to classical statistics, the imaginary part of f vanishes and that the real part approaches the classical distribution function.

3. EXPECTATION VALUES AND SPECTRAL RESOLUTION

For the expectation value of any quantity which is of the form $G=(\alpha^m\beta^n\gamma^k\cdots\rho^r\sigma^s)_{\mathrm{order}}$, where the parentheses indicate any arbitrary order of the operators in the product, we can easily prove the relation

$$(\,|G|\,)\equiv\bar{G}=\sum_{\alpha'\beta'\cdots\sigma'}G(\alpha',\beta',\cdots\sigma')f(\alpha',\beta',\cdots,\sigma'), \quad (11)$$

where $G(\alpha',\cdots,\sigma')$ is the function obtained from the operator G by replacing the operators α, β, $\cdots$, σ by the corresponding c-numbers and *the arguments of f have to appear in the same order as the corresponding operators appear in G*. Thus, each member of the class of equivalent distribution functions will be used in finding the mean values. Equation (11) is proved as follows:

$$\begin{aligned}\bar{G}=(\ |G|\ )&=\sum_{\alpha',\sigma'}(\alpha'|\alpha^m\beta^n\cdots\sigma^s|\sigma')(\ |\alpha')(\sigma'|\ )\\&=\sum_{\alpha',\sigma',\beta',\rho'}\alpha'^m\sigma'^s(\beta'|\beta^n\cdots\rho^r|\rho')(\ |\alpha')(\alpha'|\beta')\\&\qquad\times(\rho'|\sigma')(\sigma'|\ )=\cdots\text{ etc.}\\&=\sum_{\alpha',\cdots,\sigma'}\alpha'^m\beta'^n\cdots\rho'^r\sigma'^s f(\alpha',\beta',\cdots,\rho',\sigma').\end{aligned}$$

If some of the operators are repeated in G, for example, $G=\alpha\beta\alpha$, then in Eq. (11) we have to sum over repeated observables as though they were different quantities in the order in which they appear:

$$G=\alpha\beta\alpha;\quad \bar{G}=\sum_{\alpha'\beta'\alpha''}\alpha'\beta'\alpha''f(\alpha',\beta',\alpha'')d\alpha'd\beta'd\alpha''.$$

However, using the commutator $C=[\alpha,\beta]$ we can reduce this equation as follows:

$$\begin{aligned}G&=\alpha\beta\alpha=\alpha^2\beta-\alpha C;\\\bar{G}&=\sum_{\alpha'\beta'}\alpha'^2\beta'f(\alpha'\beta')-\sum_{\alpha'c'}\alpha'c'f(\alpha',c').\end{aligned}$$

In this way, various identities between different f functions and between the real and imaginary part of f can be obtained. For example, in the case of coordinates and momenta, $[x,p]=i\hbar$ we get

$$\int G(x')p'\,\mathrm{Im}f(x',p')dx'dp'=\frac{\hbar}{2}\int G'(x')\,\mathrm{Re}f(x',p')dx'dp',$$

$$\int G(p')x'\,\mathrm{Im}f(x',p')dx'dp'=\frac{\hbar}{2}\int G'(p')\,\mathrm{Re}f(x',p')dx'dp'.$$

Equation (11) is also valid for the functions of ordered noncommuting observables and for a presumably larger class of functions, introduced by Dirac,[10] containing the former. In each term in the expansion of functions of ordered operators, for example, observables relating to different times, the order of the operators is one and the same such that only one f-function is necessary. Dirac's functions are defined as follows:

$$G(\alpha,\beta')|\beta')=G(\alpha,\beta')|\beta');\quad G(\alpha,\beta')|\alpha')=G(\alpha',\beta')|\alpha').$$

[10] P. A. M. Dirac, Revs. Modern Phys. **17**, 195 (1945).

Hence,

$$\begin{aligned}(\ |G|\ )&=\sum_{\alpha'\beta'}G(\alpha',\beta')(\ |\alpha')(\alpha',\beta')(\beta'|\ )\\&=\sum_{\alpha'\beta'}G(\alpha',\beta')f(\alpha',\beta').\end{aligned}$$

The expectation value of arbitrary functions which are defined in terms of their power series expansions can be given by using Eq. (11) for each term. From Eq. (11) we get for the general term

$$(\ |G^n|\ )=\int G(\alpha',\beta')G(\alpha'',\beta'')\cdots G(\alpha^{(n)},\beta^{(n)})\times f(\alpha',\beta',\alpha'',\beta'',\cdots\alpha^{(n)}\beta^{(n)})d\alpha'd\beta'\cdots d\alpha^{(n)}d\beta^{(n)}.$$

Introducing the commutator $C=[\alpha,\beta]$ we can derive in this general case, the relation

$$\bar{G}=\int G(\alpha',\beta')\,\mathrm{Re}f(\alpha',\beta')d\alpha'd\beta'+\bar{R},\qquad(12)$$

where R is an operator depending on C, α, β, and is zero when C is zero (see Appendix). Thus, in the transition to classical mechanics, the quantum-mechanical correction term R approaches zero and $\mathrm{Re}f$ goes over to the classical probability density function. This is in agreement with and supports the statement made at the end of Sec. 2.

Furthermore, we can give by the uniqueness of the expression for $\bar{G}$ a spectral resolution of G in terms of the corresponding classical functions. Introducing the generalized projection operators

$$E_{\alpha'\beta'}=|\alpha')(\beta'|;\quad E_{\alpha'\beta'}{}^\dagger=E_{\beta'\alpha'}=|\beta')(\alpha'|,\quad(13)$$

with the property

$$E_{\alpha'\beta'}E_{\alpha''\beta''}=(\beta'|\alpha'')E_{\alpha'\beta''},$$

we get from Eq. (11)

$$\begin{aligned}G&=\sum_{\alpha'\beta'\cdots\sigma'}G(\alpha',\beta',\cdots,\sigma')aE_{\alpha'\sigma'};\\a&=(\alpha'|\beta')\cdots(\rho'|\sigma').\end{aligned}\qquad(14)$$

For arbitrary operators, we again have to use the power-series expansion and apply (14) for each term. Thus, if G is any simple product, a general function $F(G)$ of G can be written[11]

$$F(G)=F(\sum_{\alpha',\cdots\sigma'}G(\alpha',\cdots\sigma')aE_{\alpha'\sigma'}),$$

for example,

$$e^{\alpha+\beta}=\exp\left[\int(\alpha'+\beta')(\alpha'|\beta')E_{\alpha'\beta'}\right].$$

[11] Another spectral resolution, in nonrelativistic mechanics, has been given in R. P. Feynman, Phys. Rev. **84**, 108 (1951). This paper contains a detailed discussion of functions of ordered operators mentioned above.

4. EQUATION OF MOTION

For a time-dependent state vector $|t)$ we write

$$F(t)=|t)(\alpha'|\beta')\cdots(\rho'|\sigma')(t|. \tag{15}$$

Then in the virtue of the Schrödinger equation $(\partial/\partial t)|t)=-(i/\hbar)H(t)|t)$ and its conjugate complex, we can differentiate $F(t)$ and obtain

$$\frac{\partial F(t)}{\partial t}=-\frac{i}{\hbar}[H(t),F(t)];$$
$$\frac{\partial F^\dagger}{\partial t}=-\frac{i}{\hbar}[H(t),F^\dagger(t)]. \tag{16}$$

Using the unitary transformation connecting the state vectors at different times

$$|t_2)=U(t_1,t_2)|t_1),\quad U(t_1,t_3)=U(t_2,t_3)U(t_1,t_2),$$

we can get also the equation of motion in integral form:

$$F(t_2)=U(t_1,t_2)F(t_1)U^\dagger(t_1,t_2). \tag{17}$$

The corresponding equations for f can be obtained by means of Eq. (2)

$$\frac{\hbar}{i}\frac{\partial}{\partial t}f(\alpha',\cdots\sigma')=(\alpha'|\beta')\sum_{\alpha''}(\alpha''|H|\alpha')\frac{f(\alpha'',\beta',\cdots\sigma')}{(\alpha''|\beta')}-(\rho'|\sigma')\sum_{\sigma''}(\sigma'|H|\sigma'')\frac{f(\alpha',\cdots,\rho',\sigma'')}{(\rho'|\sigma'')}, \tag{18}$$

or, taking, for instance, $\alpha=x$ and $\beta=p$

$$\frac{\hbar}{i}\frac{\partial f(x',p')}{\partial t}=(x'|p')\left[H\left(x',\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\{f(x',p')(p'|x')\}-H\left(\frac{\hbar}{i}\frac{\partial}{\partial x},x\right)\{f(x',p')(p'|x')\}\right], \tag{18'}$$

and

$$\begin{aligned}f(\alpha',\beta',\cdots,\rho',\sigma';t_2)&=\sum_{\alpha'',\sigma''}\frac{(\alpha'|\beta')(\rho'|\sigma')}{(\alpha''|\beta')(\rho'|\sigma'')}(\alpha''|U^\dagger|\alpha')(\sigma'|U|\sigma'')\times f(\alpha'',\beta',\cdots,\rho',\sigma'';t_1)\\&=\sum_{\alpha'',\sigma''}K(\alpha'',\beta',\cdots\rho',\sigma'',t_1;\alpha',\beta',\cdots\rho',\sigma',t_2)\times f(\alpha'',\beta',\cdots\rho',\sigma'',t_1).\end{aligned} \tag{19}$$

Under arbitrary canonical transformations represented by unitary operators V, Eqs. (17) and (19), give the transformation of F and f if U is replaced by V. Equation (19) is a simple Markoff-type stochastic equation whose kernel is related to the Green's functions $(\alpha'|U|\alpha''),\cdots$ of the system. The kernel satisfies the relation

$$K(\alpha'',\sigma'',t_1;\alpha',\sigma',t_3)=\sum_{\alpha''',\sigma'''}K(\alpha'',\sigma'',t_1;\alpha''',\sigma''',t_2)\times K(\alpha''',\sigma''',t_2;\alpha',\sigma',t_3), \tag{20}$$

and can be interpreted as the analog of complex transition probabilities. The probabilities in configuration space, i.e., $|\psi|^2$, or in momentum space alone do not, as is well known, obey a simple stochastic equation of the type (19). We remark that for scattering problems U will be replaced by the S-matrix and that, as operator equations, (15) and (17) are also valid in interaction representation.

If the operators α, β, $\cdots$, σ relate to different times, then the f-functions represent distribution functions along the "trajectory" of the system, i.e., distribution functions corresponding to Feynman amplitudes. Such functions have been introduced by Dirac.[10] For example, in $f(\alpha',\beta')$, α' may represent the coordinates at time t', β' the coordinates at time t''. More generally, we can introduce the analog of "phase-space" transition probabilities. Consider three phase-space points α',β' at time t' (point 1); α'',β'' at time t'' (point 2); and α''',β''' at time t''' (point 3). Then

$$f(\alpha',\beta',t';\alpha'',\beta'',t'')\equiv f(1,2)$$

is equal to the kernel of Eq. (19) provided f at each point is normalized to one, i.e., $f(\alpha',\beta')=1$; otherwise

$$f(1,2)=K(\alpha'\beta't',\alpha''\beta''t'')/f(\alpha',\beta').$$

These transition functions satisfy the equation

$$f(1,2)=\sum_3\frac{f(1,3)f(3,2)}{f(3)}. \tag{21}$$

In the relativistic case the state vectors depend not only on time t but on space-time points x_μ and satisfy $(\partial/\partial x_\mu)|\ )=-(i/\hbar)P_\mu|\ )$, where P_μ is the energy-momentum operator. Equation (16) becomes in this case

$$\frac{\partial F}{\partial x_\mu}=-\frac{i}{\hbar}[P_\mu,F]. \tag{22}$$

Again using the unitary transformation connecting the states between the space-like surfaces σ_1 and σ_2,

$$|\sigma_2)=U_{21}|\sigma_1),$$

we get instead of (17) and (19)

$$F(\sigma_2)=U_{21}F(\sigma_1)U_{21}^\dagger,$$

$$f(\sigma_2)=\sum_{\alpha''\rho''}\frac{(\alpha'|\beta')(\delta'|\rho')}{(\alpha''|\beta')(\delta'|\rho'')}\times(\rho_{\sigma_1}'|\rho_{\sigma_2}'')(\alpha_{\sigma_2}''|\alpha_{\sigma_1}')f(\sigma_1). \tag{23}$$

In field theory f will be functionals of the field variables, or space-time functionals. They may be useful in the theory of Green's functions or propagators which are the vacuum expectation values of the products of ordered field operators and can be calculated, in principle, according to Eq. (11).

If α and β represent coordinate and momenta, in nonrelativistic mechanics and H is independent of time, Eq. (19) can be written in terms of the wave functions in coordinate and momentum space as follows:

$$f(x,p,t)=\int (x|n)(n|x')(p'|n')(n'|p) \times\exp\left[-\frac{i}{\hbar}t(E_n-E_{n'})\right]\exp\left[\frac{i}{\hbar}(p'x'-px)\right] \times f(x',p',0)dndn'dx'dp', \quad (24)$$

where $H|n)=E_n|n)$. For instance, for a free particle, we get

$$f(\mathbf{x},\mathbf{p},t)=\left(\frac{m\hbar}{2\pi it}\right)^{\frac{3}{2}}\int\exp\left[\frac{i}{\hbar}\frac{m}{2t}\left((\mathbf{x}-\mathbf{x}')+\frac{t}{m}p'\right)^2\right] \times f(\mathbf{x}',\mathbf{p}',0)\delta(\mathbf{p}-\mathbf{p}')d\mathbf{x}'d\mathbf{p}'. \quad (25)$$

In the case of nonrelativistic mechanics it is easy to see that $f(x,p,t)$ in pure states also satisfies the differential equation

$$\frac{\partial f}{\partial x}\frac{\partial f}{\partial p}-f\frac{\partial^2 f}{\partial x\partial p}=0, \quad (26)$$

which may be taken as the equation characterizing the pure states.

It remains only to characterize the stationary states. In this case f is independent of time and the problem of finding f_0 reduces to solving the homogeneous integral equation of the second kind:

$$f_0(\alpha',\cdots,\sigma')=\int Kf_0(\alpha'',\cdots\sigma'')d\alpha''d\sigma''. \quad (27)$$

The kernel is, in principle, known if $U(t_1,t_2)$ is given. For example, $f_0=$constant is a stationary solution of (25). The differential equation (18), on the other hand, with the left hand size equal zero, separates and we get the usual time-independent Schrödinger equations in coordinate and momentum spaces.

5. DISCUSSION AND CONCLUSIONS

We have investigated the properties of certain functions of noncommuting quantities which have all the properties of the classical joint density functions except that of being always non-negative. These complex functions allow the unambiguous calculation of expectation values in the usual form and satisfy relatively simple equations of motion. Conversely, functions having these properties provide a formulation of quantum mechanics in terms of distribution functions rather than probability amplitudes. These functions have rather remarkable symmetry properties, and it would be perhaps a complicated mathematical problem to see to what extent these properties uniquely determine, in more general cases than the stationary states, the distribution functions. It may also be of interest to study the important theorems of probability theory, such as ergodic theorems and central-limit theorem, for the case of these complex distribution functions.

Physically, f may represent a stationary wave, a wave packet, or even a mixture. In applying (19) we have to choose f in agreement with (26) and the boundary conditions imposed on it by the symmetry properties and the uncertainty principle. From given wave packets, new ones may be obtained by canonical transformations. In this case Eq. (19) is again valid, U being independent of time, for instance $U=e^{iaP}$ or $U=e^{i\omega M_z}$.

Finally, we mention that such distribution functions may be useful in quantum hydrodynamics or in statistical mechanics where one desires to develop quantum equations in analogy to the classical equations in terms of distribution functions.

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APPENDIX

A function of the two noncommuting operators can be expanded as follows:

$$G=\sum_{n=0}^{\infty}\sum A_{n_1n_2\cdots n_{n+1}}\alpha^{n_1}\beta^{n_2}\alpha^{n_3}\beta^{n_4}\cdots\beta^{n_n}\alpha^{n_{n+1}},$$

the second summation is taken over all $n_1,n_2,\cdots,n_{n+1}$ such that $\sum_{k=1}^{n+1}n_k=n$ and contains $\sum_{r=0}^{n}\binom{n}{r}=2^n$ terms. Let $[\alpha,\beta]=C$. Then

$$[\alpha^n,\beta]=\alpha^{n-1}C+\alpha^{n-2}C\alpha+\alpha^{n-3}C\alpha^2+\cdots+C\alpha^{n-1}\equiv C^{(n)},$$
$$[\alpha^n,\beta^m]\equiv C^{(n,m)}=\beta^{m-1}C^{(n)}+\beta^{m-2}C^{(n)}\beta+\cdots+C^{(n)}\beta^{m-1}.$$

With the help of $C^{(n,m)}$ we can commute once all α's to the right and once to the left. Taking then the average value we get Eq. (12) where the operator R is given by

$$R=\tfrac{1}{2}\sum_{n=0}^{\infty}\sum A_{n_1n_2\cdots n_{n+1}}\sum_{r=0}^{n/2-1}\{\beta^{n_2+n_4+\cdots+n_{2r}} \times C^{(n_1+n_3+\cdots+n_{2r+1},\,n_{2r+2})}\alpha^{2r+3}-\alpha^{n_1+n_3+\cdots+n_{2r+1}} \times C^{(n_{2r+3},\,n_2+n_4+\cdots+n_{2r+2})}\}\beta^{2r+4}\alpha^{2r+5}\cdots\beta^{n_n}\alpha^{n_{n+1}},$$

where the second summation is as above.