expansion of the $\frac{1}{2}N$ power of the first two terms in Eq. (1.2) namely, $(1+\int_0^\infty p_\epsilon e^{-\epsilon/kT}d\epsilon)^{N/2}$, the integration serving merely as an averaging device. Excitation to higher levels means, of course, additional terms in the system's partition function. For example, if there is one excitation to the higher triplet, this can occur in $11N/2$ different ways, and the contribution to the partition function in the light of Eq. (1.6) will be

$$\frac{1}{2}N\left(\int_0^\infty p_\epsilon' e^{-\epsilon/kT}d\epsilon\right)\left(1+\int_0^\infty p_\epsilon e^{-\epsilon/kT}d\epsilon\right)^{\frac{1}{2}N-1},$$

which would appear in the multinomial expansion of Eq. (1.2). However, if two of the higher levels were excited, these could occur in somewhat fewer than $11^2(\frac{1}{2}N)(\frac{1}{2}N-1)/2$ ways, because this would count cases in which two rotators had a common atom; the expanded exclusion principle begins to operate. These deviations would only become important, however, in the terms of the expansion involving high powers of $\int_0^\infty p_\epsilon' e^{-\epsilon/kT}d\epsilon$, and if the integral is not too large may be neglected.

I wish to thank Professor W. A. Bowers for a number of helpful discussions.

Note added in proof.—We have noted a statement by H. C. Kramers, in *Progress in Low Temperature Physics* (see reference 5), Vol. 2, p. 65, that the phonon specific heat of Kramers, Wasscher, and Gorter[15] is too large, though their total specific heats at 0.8°K and above are about right. Using the new phonon specific heat to obtain C_r changes Table II and the values of m_0 as follows:

T	ϵ/k	$100n/N$	m	m_0
0.8	9.39	0.0041	5.11	7.6
1.1	9.57	0.1030	6.19	7.6
1.4	9.88	0.683	7.94	9.0
1.7	10.4	2.44	11.1	12.1

The consistency with Eqs. (2.8) and (2.10) is improved, and the smaller value of m_0 almost removes the difficulty concerning the number of energy levels. At the lower temperatures $d(\epsilon/k)/dT$ is about 0.6, close to the value in Table II and that given by the Landau-Feynman theory.

PHYSICAL REVIEW VOLUME 108, NUMBER 3 NOVEMBER 1, 1957

Radiation Effects in Shock-Wave Structure

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(Received June 19, 1957)

The equations for shock-wave structure, with the inclusion of radiation effects, are derived. These radiation effects are radiation pressure, radiation energy density, and radiative transfer of energy. Computations have been performed for a diffusion approximation of radiation flux and the neglect of radiation energy density and pressure. The results show that the over-all effect of radiation (under the aforementioned conditions) can be taken as a diminution of the Prandtl number, and that the shock width is larger than when viscosity and heat conduction alone are considered. The radiative contribution to the width of the shock is found to depend primarily on the ratio of the mean free path of radiation to that of the material particles. The proportionate increase in shock width is found to be a function of the Mach number and to increase with it.

Possible application of the above results to shock-wave propagation in a medium of low density is indicated.

1. INTRODUCTION

SACHS[1] has given the Rankine-Hugoniot conditions when the effects of radiation pressure and energy density are included with the hydrodynamics. Sachs examined only the end conditions of such a shock, and did not consider radiative transfer of energy. It is the object of this paper to give an analysis which includes the effects of such radiation terms on the detailed structure of a shock front. The analysis is by the Stokes-Navier equations with the gas possessing its usual viscosity and heat conductivity. The shock will be taken as plane, steady, nonrelativistic, with no superposed electric or magnetic fields.

Radiation pressure and energy density effects are important long before the shock is relativistic. For a shock propagating into air at standard conditions, a Mach number of 10^5 is needed before the shock may be considered relativistic, whereas radiation pressure becomes comparable to material gas pressure behind the shock at a Mach number of about 2×10^2. Again, radiative transport of energy may be important even though the other radiation terms are negligible. For air of atmospheric density, a temperature of a few million degrees must be reached before radiation pressure and energy density become important while radiative transport is significant even at much lower temperatures.

The significance of the results to shock propagation through rarefied atmospheres is also briefly considered.

[1] R. G. Sachs, Phys. Rev. 69, 514 (1946).

2. BASIC THEORY

We shall assume the shock propagating longitudinally (along the x axis), and make the flow time-independent by referring to a coordinate system moving with the shock front. We shall use the suffixes 0 and 1 to denote the physical variables (velocity u, material gas pressure p, density ρ, and temperature T) in front and in back of the shock, respectively. The following equations then describe the flow.

Integration of the equation of continuity gives

$$\rho u=\rho_0 u_0=m, \text{ say.} \tag{1}$$

The Stokes-Navier equation is

$$m\frac{du}{dx}=-\frac{d}{dx}(p+\tfrac{1}{3}aT^4)+\frac{4}{3}\frac{d}{dx}\left(\mu\frac{du}{dx}\right), \tag{2}$$

where μ is the coefficient of viscosity and $\frac{1}{3}aT^4$ is the radiation pressure which is added to the material gas pressure, a being the radiation constant. Equation (2) may be integrated directly to give

$$mu-mC=-(p+\tfrac{1}{3}aT^4)+\tfrac{4}{3}\mu du/dx, \tag{3}$$

where C is an integration constant.

The equation of conservation of energy is

$$m\frac{d}{dx}\left(E+\frac{aT^4}{\rho}\right)=\frac{d}{dx}\left(k\frac{dT}{dx}\right)-(p+\tfrac{1}{3}aT^4)\frac{du}{dx}+\tfrac{4}{3}\mu\left(\frac{du}{dx}\right)^2-\frac{dF}{dx}, \tag{4}$$

where k is the coefficient of thermal conductivity, F is the radiation flux, and aT^4/ρ is the radiation energy per unit mass, which is added to the material internal energy, E. With the aid of Eq. (3), Eq. (4) may be integrated to give

$$m\left(E+\frac{aT^4}{\rho}\right)=k\frac{dT}{dx}+\tfrac{1}{2}mu^2-mCu-F-C_1, \tag{5}$$

where C_1 is another constant of integration.

Finally, we shall take the case of a perfect gas:

$$p=R\rho T,\quad E=C_vT, \tag{6}$$

where R and C_v are constants. C_v is the specific heat at constant volume.

The Rankine-Hugoniot conditions are obtained by assuming uniform conditions in front and in back of the shock. Applying (3) and (5) to the front and the back of a shock gives, respectively,

$$(p_0+\tfrac{1}{3}aT_0^4)+mu_0=mC=(p_1+\tfrac{1}{3}aT_1^4)+mu_1, \tag{7}$$

and

$$m(E_0+aT_0^4/\rho_0)-\tfrac{1}{2}mu_0^2+mCu_0=-C_1=m(E_1+aT_1^4/\rho_1)-\tfrac{1}{2}mu_1^2+mCu_1. \tag{8}$$

In (8) it is assumed that the radiation flux at $x=\pm\infty$ is zero, which will be the case for the diffusion approximation for radiation flux. Equations (7) and (8) along with

$$\rho_0u_0=m=\rho_1u_1 \tag{9}$$

from Eq. (1), are equivalent to the Rankine-Hugoniot equations obtained by Sachs.

The Mach number of the shock may be defined in the usual manner, $M_0=|u_0|/c_0$ where c_0 is the sound speed in the medium in front of the shock. However, in the present situation,

$$c_0^2=\left[\frac{\partial(p_0+\frac{1}{3}aT_0^4)}{\partial\rho_0}\right]_S, \tag{10}$$

the subscript S indicating that the differentiation is to be performed for constant entropy (S). Sachs[1] has shown that, in the case of a perfect gas, Eq. (10) gives

$$c_0^2=\frac{p_0}{\rho_0}\left[\frac{\gamma p_0+16(\gamma-1)aT_0^4/3}{p_0+4(\gamma-1)aT_0^4}\right]\left[1+\frac{4aT_0^4}{3p_0}\right]-\frac{4aT_0^4}{3\rho_0}, \tag{11}$$

where γ is the ratio of specific heats for the material gas. Equation (11) may be further reduced to

$$c_0^2=\frac{p_0}{\rho_0}\left[\frac{\gamma+20(\gamma-1)\eta+16(\gamma-1)\eta^2}{1+12(\gamma-1)\eta}\right], \tag{12}$$

where

$$\eta=\frac{aT_0^4/3}{p_0}=\frac{\text{radiation pressure}}{\text{material gas pressure}}.$$

Note that as $\eta\to 0$, $c_0^2\to\gamma p_0/\rho_0$, which is the usual expression for the material gas. As $\eta\to\infty$, $c_0^2\to\frac{4}{3}(\frac{1}{3}aT_0^4)/\rho_0$ (for $\gamma>1$), indicating that radiation behaves like a perfect gas with $\gamma_{\text{rad}}=\frac{4}{3}$.[2]

We shall assume local thermodynamic equilibrium throughout the shock and shall take the diffusion approximation for the radiation flux, F. Under this approximation, F is given explicitly by

$$F=-\frac{ac}{3\rho\kappa}\frac{dT^4}{dx}=-\frac{4acT^3}{3\rho\kappa}\frac{dT}{dx}, \tag{13}$$

where a is again the radiation constant, c is the velocity of light, and κ is the Rosseland mean absorption coefficient. The derivation and validity of the radiation diffusion approximation is given by Chandrasekhar.[3] In general, it is valid when the temperature does not vary appreciably in a distance of the order of a Rosseland mean free (radiation) path.

Substituting the radiation diffusion approximation

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[2] See S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (The University of Chicago Press, Chicago, 1939), p. 55.

[3] Reference 2, pp. 208–211.

into Eq. (5), we get

$$\left(k+\frac{4acT^3}{3\rho\kappa}\right)\frac{dT}{dx}=m\left(C_vT+\frac{aT^4}{\rho}\right)-\tfrac{1}{2}mu^2+mCu+C_1. \quad (14)$$

Then, from Eqs. (1), (3), and (14) we have

$$\frac{dT}{du}=\frac{m(C_vT+aT^4u/m)-\frac{1}{2}mu^2+mCu+C_1}{(RmT/u+\frac{1}{3}aT^4)+mu-mC}\times\left(\frac{\frac{4}{3}\mu}{k+\frac{4}{3}acuT^3/m\kappa}\right), \quad (15)$$

which is a differential equation in T and u alone; μ, k, and κ are assumed to be functions of T and ρ, and therefore of T and u [from Eq. (1)]. Equation (15) may then be numerically integrated throughout the shock front, the end conditions and constants involved being evaluated from Eqs. (7) and (8). Once T is found as a function of u by integration of (15), u may be found as a function of x by the numerical solution of Eq. (3), giving the velocity profile through the shock. The density and temperature profiles may be subsequently found.

It is to be noted that, with the diffusion approximation, there is no radiation flux at $x=\pm\infty$, since $dT/dx=0$ at these points, i.e., there is no "radiation escape" from the shock region.

3. COMPUTATIONAL EQUATIONS

Computations have been performed for the case when radiation pressure and radiation energy density are negligible, but radiative transport of energy is not negligible in comparison with transport by thermal conduction. In this situation, Eq. (15) may be put into the form:

$$\frac{dT}{du}=\frac{C_vTu-\frac{1}{2}u^3+Cu^2-C_1u/m}{RT+u^2-u}\left(\frac{4\mu}{3k}\right)\times\left(1+\frac{4acuT^3}{3\kappa mk}\right)^{-1}. \quad (16)$$

We shall call the factor

$$g=\left(1+\frac{4acuT^3}{3\kappa mk}\right)^{-1} \quad (17)$$

the radiation broadening factor. The factor $\frac{4}{3}\mu/k$ is proportional to the Prandtl number, which is usually taken as constant with temperature variation. Thus, the radiation broadening factor represents a temperature- and velocity-dependent diminution of the Prandtl number.

We shall introduce the dimensionless variables, $\Pi=T/T_0$ and $U=u/u_0$, and shall take $\mu=\mu_0\Pi^n$, $k=k_0\Pi^n$, $\kappa=\kappa_0(\rho/\rho_0)^\alpha\Pi^\beta=\kappa_0U^{-\alpha}\Pi^\beta$, where n, α and β are constants; μ_0, k_0 and κ_0 are the coefficients of viscosity, heat conductivity, and Rosseland mean absorption, respectively, in the ambient gas in front of the shock. The radiation broadening factor may then be put into the form:

$$g(\Pi,U)=1\bigg/\left(1+\frac{4P}{\delta}\frac{\gamma-1}{\gamma}\frac{c}{\bar{c}_0}\frac{p_{R0}}{p_0}\frac{\lambda_{R0}}{\lambda_0}U^{\alpha+1}\Pi^{3-\beta-n}\right), \quad (18)$$

where

P = Prandtl number $=\mu\gamma C_v/k$,
δ = numerical constant $\simeq\frac{1}{2}$ for rigid elastic spheres,
$\bar{c}_0$ = mean particle velocity in front of shock $=(8RT_0/\pi)^{\frac{1}{2}}$,
p_{R0} = radiation pressure in front of shock $=\frac{1}{3}aT_0^4$,
p_0 = material gas pressure in front of shock,
λ_{R0} = Rosseland mean free (radiation) path in front of shock $=1/\kappa_0\rho_0$,
λ_0 = particle mean free path in front of shock $=\mu_0/\delta\rho_0\bar{c}_0$.

We note that the fraction appearing in the denominator of (18), *viz.*, $(c/\bar{c}_0)(p_{R0}/p_0)(\lambda_{R0}/\lambda_0)$, is essentially a measure of the effects of radiation as compared with material hydrodynamics. We have taken the radiation pressure as negligible in comparison with gas pressure so that the ratio p_{R0}/p_0 will be quite small (but not zero). On the other hand, the ratio $c/\bar{c}_0$ = (velocity of light)/(mean particle velocity) will be quite large. The ratio λ_{R0}/λ_0 will be small when radiation transport effects are negligible, but will grow with the importance of the radiation transport of energy *versus* thermal conduction. The second term in the denominator of (18) may, under these conditions, be very well comparable to, or greater than unity. We thus see that even when radiation pressure and energy density are negligible compared with gas pressure and internal energy, it may not be permissible to neglect the contribution of transport of energy by radiation to the shock structure.[4] The radiation broadening factor $g(\Pi,U)$ goes over asymptotically to unity for the pure gas (no radiation) case, but when radiation effects are important, $g(\Pi,U)$ is temperature- and velocity-dependent and remains sensibly less than unity.

Under the assumptions we have made (diffusion approximation and the neglect of radiation pressure and energy density), Eqs. (7) and (8) give

$$C/u_0=1+1/M_0^2,$$

and

$$-C_1/mu_0^2=\tfrac{1}{2}+1/[(\gamma-1)M_0^2].$$

Upon using these constants and the nondimensional

[4] Sachs arrived at similar conclusions by qualitative physical reasoning. See reference 1.

variables Π and U, Eq. (16) becomes[5]

$$\frac{d\Pi}{dU}=\frac{4P}{3\gamma}\frac{U\{\Pi-\gamma[1+\frac{1}{2}(\gamma-1)M_0^2]+(1+\gamma M_0^2)(\gamma-1)U-\frac{1}{2}\gamma(\gamma-1)M_0^2U^2\}}{\Pi/(\gamma M_0^2)-U[1-U+1/(\gamma M_0^2)]}g(\Pi,U). \quad (19)$$

Also, Eq. (3) becomes

$$\frac{dU}{d\xi}=\frac{3M_0}{4}\left(\frac{\pi\gamma}{2}\right)^{\frac{1}{2}}\times\frac{\Pi/(\gamma M_0^2)-U[1-U+1/(\gamma M_0^2)]}{U\Pi^n}, \quad (20)$$

where $\xi=x/\lambda_0$.

We shall define the shock width according to the formula

$$t_0=(1-U_1)/|dU/d\xi|_{\max}, \quad (21)$$

in terms of mean free particle path, λ_0, in front of the shock, or

$$t_\lambda=t_0/[\tfrac{1}{2}(1+\Pi_1^2U_1)], \quad (22)$$

in terms of mean free path inside the shock (average of mean free particle path in front and in back of the shock).

The Rankine-Hugoniot conditions may be put into the form:

$$U_1=u_1/u_0=(\gamma-1)/(\gamma+1)+2/[(\gamma+1)M_0^2] \quad (23)$$

and

$$\Pi_1=T_1/T_0=\gamma M_0^2U_1[1-U_1+1/(\gamma M_0^2)] \quad (24)$$

in back of the shock. Also we have the normalization, $\Pi_0=1$ and $U_0=1$, in front of the shock.

4. PRANDTL NUMBER

There has been some question regarding the order of magnitude of the Prandtl number to be used for a plasma. The first author, in a previous paper,[5] assumed the shock propagation to be governed mainly by the ions (on account of their greater mass[6]) and used the value 3/4 for the Prandtl number in order to compare with earlier work. Marshall,[7] however, has given reasons for using a low Prandtl number for a plasma, and finds that this would broaden the shock considerably more.

Marshall would be right if there were temperature equilibrium between the electrons and the ions. Even in this case, however, charge separation will produce an electrostatic field which will reduce the thermal conductivity by a factor of the order of one half.[8] Further, there is reason to believe that the ions will be at a higher temperature than the electrons in the shock front so long as ionization equilibrium is not attained.[9] The electrons will also have a lower temperature gradient. These phenomena would reduce (and possibly nullify) the effect noted by Marshall.

In the last line of Table I the authors have included a low Prandtl number in order to estimate its effect on the broadening of the shock front. The further broadening with the lowering of the Prandtl number is much less pronounced with the radiation field than with the pure hydrodynamic shock.

5. NUMERICAL RESULTS

The foregoing analysis is perfectly general and will apply to any gas model. It passes over asymptotically to the pure gas case or pure radiation case depending on the relative magnitudes of the two terms in the denominator of $g(\Pi,U)$ in (18). The transition region is the interesting one, and we shall estimate its character by putting the constant factor in the second term of the denominator of $g(\Pi,U)$ equal to unity. This gives approximately equal weight to collision and radiation broadening. Further, we shall take $\gamma=5/3$ (to compare with earlier results[5]), $n=2.5$, $\alpha=1$, and $\beta=3.5$. The results presumably would not be too sensitive to the particular values given to these constant. The radiation broadening factor then becomes

$$g(\Pi,U)=1/(1+U^2\Pi^4). \quad (25)$$

With this form for $g(\Pi,U)$, Eq. (19) was solved for $M_0=1.5$, 2, 2.5, and 4 with a Prandtl number of 3/4. Also, (19) was solved for $M=1.5$ with a Prandtl number of 3/40 and the same form, (25), for $g(\Pi,U)$.

Solutions of (19) with a Prandtl number of 3/4 are plotted in Fig. 1. Also in Fig. 1 is plotted the solution for $M_0=1.5$ without the inclusion of the radiation broadening factor. Note that the inclusion of radiation

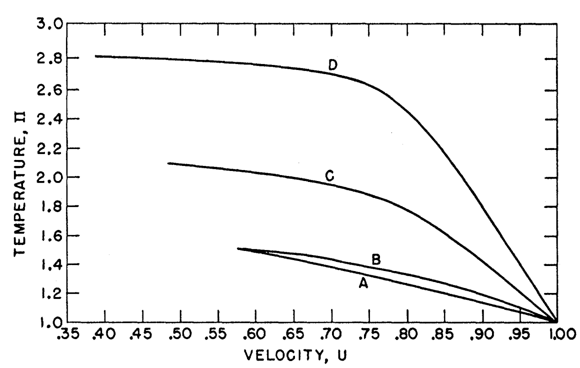

FIG. 1. Variation of temperature with velocity within the shock. $\Pi=T/T_0$ and $U=u/u_0$, where T_0 and u_0 are the temperature and velocity in front of the shock. (A) $M_0=1.5$ (no radiation); (B) $M_0=1.5$ (with radiation); (C) $M_0=2$ (with radiation); (D) $M_0=2.5$ (with radiation).

[5] See H. K. Sen, Phys. Rev. **102**, 5 (1956).
[6] See J. F. Denisse and Y. Rocard, J. phys. radium **12**, 893 (1951).
[7] W. Marshall, Phys. Rev. **103**, 1900 (1956).
[8] L. Spitzer and R. Harm, Phys. Rev. **89**, 977 (1953).
[9] H. Petschek and S. Byron, Ann. Phys. **1**, 270 (1957).

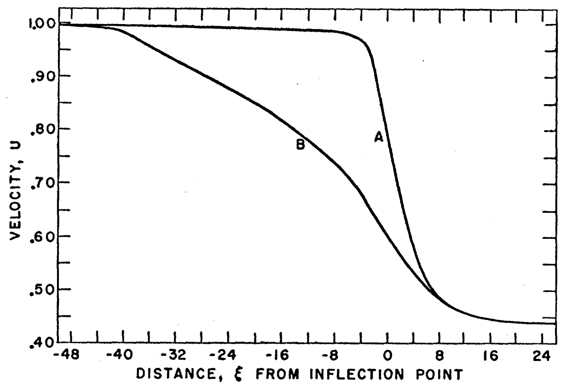

FIG. 2. Variation of velocity within the shock. (A) $M_0=2$ (no radiation); (B) $M_0=2$ (with radiation).

causes the values of Π (*versus* U) to be consistently larger within the shock front than the values without radiation.

From the results of the solution of Eq. (19), Eq. (20) may be solved to give U as a function of ξ. In Fig. 2, the solution of Eq. (20) is plotted both with and without radiation for $M_0=2$ and a Prandtl number of 3/4. Note that the inclusion of the radiation broadening factor (for the parameters we have chosen) considerably broadens the structure of the shock. Radiation has "eaten into" the front part of the shock.

In Table I are presented the results of computation with Eqs. (21) and (22) for the width of the shock. The last three columns of Table I are perhaps the most interesting. They show that, both with and without radiation, the shock width (in terms of mean free path within the shock front) decreases with increasing Mach number (for the region covered by the table). However, the proportional increase in shock width due to radiation increases with increasing Mach number, i.e., radiation becomes increasingly important at higher Mach numbers.

6. CONCLUDING REMARKS

We see from the above analysis and computations, that radiation broadening may be an important factor in shock propagation. In particular, in a highly rarefied atmosphere, the broadening of the shock front due to radiative transfer may be so great as virtually to nullify the shock. Further, the proportionate increase of hydrodynamic shock width due to radiation broadening will act to extend the range of validity of the Stokes-Navier equations towards higher Mach numbers. The effect of radiation (in the diffusion approximation) may be considered as somewhat analogous to a decrease in the Prandtl number.

TABLE I. Width of shock in terms of mean free path. The suffix R refers to inclusion of the radiation broadening factor.

Prandtl No.	M_0	t_0	t_{0R}	t_λ	$t_{\lambda R}$	$t_{\lambda R}/t_\lambda$
0.75	1.5	9.5	27.3	8.3	23.7	2.9
	2	8.5	31.4	5.9	21.7	3.7
	2.5	9.7	40.8	5.0	20.9	4.2
	4	14.7	87.8	2.6	15.7	6.0
0.075	1.5	43.7	75.1	37.9	65.2	1.7

Effort is being made to extend the analysis to the complete radiative-transfer equation (i.e., to situations where the diffusion approximation is invalid). For radiation flux approximations higher than diffusion (use of a Taylor series for which the diffusion approximation is the first term), a differential equation in T and x may be obtained. The form of the equation is, however, quite complicated. It seems best to treat the radiation as a series of flux streams, in the manner of Chandrasekhar.[10]

ACKNOWLEDGMENT

The authors would like to thank Mr. John J. Hanrahan for much of the computational work.

[10] S. Chandrasekhar, *Radiative Transfer* (Oxford University Press, Oxford, 1950), Chap. 3.