such detectors do not have spontaneous-emission noise, and can be made to have no output unless energy is incident on them. A molecular beam can in principle be made to operate in such a way that only molecules which were originally in the lowest state, then absorbed electromagnetic field quanta, will be detected. Future development of molecular beams might make feasible the detection of single radio-frequency quanta with no spontaneous-emission noise.

A power amplifier can in principle be constructed which operates only when ground-state particles are excited. Consider a beam of molecules with total spin of, say, n. In a magnetic field we can have $2n+1$ equally spaced states. First we arrange a molecular beam so that only particles in the lowest state (with $m=-n$) enter an interaction region. Those which absorb microwave quanta will now have $m=-n+1$. All molecules are now removed from the beam except those with $m=-n+1$. These remaining molecules now enter a region in which the magnetic field is slowly dropped to zero, then reversed to its earlier value. The molecules now have $m=n-1$. If these are allowed to enter a second cavity, each molecule can then lose $2n-1$ quanta of the same frequency as the exciting radiation, giving in principle a power gain of $2n-1$. Such an amplifier would not have spontaneous emission noise. However, it would not be a maser. The more general term "quantum mechanical amplifier" should be used to describe it.

CONCLUSION

The saturation field has only a very small effect on the noise of a three-level maser, for the mechanisms considered here. The value $h\nu/k$ which has been given as the limiting equivalent temperature does not appear to be fundamental to all amplifiers, although it does apply to existing maser devices.

It appears that quantum theory does not set a lower limit to the noise temperatures theoretically attainable with microwave detectors and amplifiers, at low temperatures.

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Brownian Movement

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A general analysis of the Brownian movement is given which is not limited to systems having a linear relaxation mechanism. Detailed results are obtained for the case of modest nonlinearity, to which presumably all problems of physical interest are limited.

1. INTRODUCTION

It may perhaps seem presumptuous today to suggest that it is still worthwhile to give an analysis of Brownian movement. However, we wish here also to attempt an analysis of the Brownian motion of a system having a nonlinear relaxation mechanism, and we hope in so doing to give a consistent analysis of Brownian movement which might perhaps be freer of difficulties and possible obscurities than is sometimes the case.

2. ANALYSIS

To be specific, let us consider an elementary electrical circuit with capacity C and resistance R as shown in Fig. 1; the resistance R is assumed to be placed in a thermal bath at temperature T. The condenser is idealized so that its behavior is completely characterized by the charge, q, on it at any instance. We now assume that the resistance R is the seat of random thermal fluctuations of electrical charge, and consequently that the charge q on the condenser will also fluctuate. It follows immediately that the element R must be able to dissipate power at an appropriate rate if a statistical equilibrium is to be maintained; the condenser will then have a mean energy $\bar{E}=\langle q^2\rangle/2C$. The condenser itself being idealized as a purely electromechanical element, this energy is free energy (i.e., available in principle for doing work). It follows then that, no matter what resistance is connected across the condenser, $\bar{E}$ must have the same value, because otherwise we might in principle establish in this way, using a condenser as intermediary, a net flow of power from one resistance to another (both being at the same temperature), which is contrary to the second law of

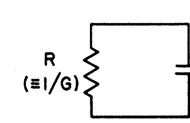

Fig. 1. Simple electrical circuit for discussion of Brownian movement.

thermodynamics. It is of course sometimes said that fluctuations themselves contravene the second law, but this is not relevant for a strict classical application of thermodynamics to steady state situations.

Let us now consider an ensemble of electrical circuits of this type. We shall limit ourselves here to a strictly classical discussion; Nyquist,[1] Weber[2] and Balazs[3] have considered the Brownian movement problem in linear systems when quantum effects become significant. We wish to determine the probability distribution for the charge, q, on the condenser, say $f(q)$. If the resistance R were very large, so that we might talk of very weak coupling of the condenser to the thermal environment, we could invoke the theorems of statistical mechanics and say that

$$f(q)dq=(1/2\pi kTC)^{\frac{1}{2}}\exp(-q^2/2kTC)dq. \qquad (1)$$

However, we wish to treat the general case, and consequently we may only use the result for the mean energy of the condenser:

$$\langle q^2\rangle/2C=kT/2, \qquad (2)$$

since as we have pointed out above this result must be independent of the particular resistance.

The fraction of circuits in the ensemble in statistical equilibrium at any time having a charge q between q and $q+dq$ is by definition $f(q)dq$. Let now the *average* rate of change of charge for this subensemble be given by

$$\langle\dot{q}\rangle=-qG(q)/C. \qquad (3)$$

This equation then *defines* operationally the conductance $G(q)$ and removes an arbitrariness in the meaning of the resistance sometimes present in such discussions. Let the random component of change be $r(q)$; evidently $\langle r(q)\rangle=0$. Now consider the maintenance of statistical equilibrium in the ensemble: given any specific value of q, the average number of circuits whose charge increases upwards through q must equal the average number whose charge decreases downward through q in any interval of time, t, over which we believe that statistical equilibrium is maintained. If this time-interval can be taken as sufficiently short so that only small displacements of charge, δq (small in comparison with the total fluctuation or "spread" of the ensemble), are to be expected—i.e., so that the individual fluctuations in q may be considered as local—then we may write:

$$\tfrac{1}{2}\langle[f(q-\delta q)][r_+(q-\delta q)-G(q)q/C]\rangle + \tfrac{1}{2}\langle[f(q+\delta q)][r_-(q+\delta q)-G(q)q/C]\rangle=0. \qquad (4)$$

$r_+(q)$ indicates that we select only those systems with $r(q)>0$, and correspondingly with $r_-(q)$, while the angular brackets indicate an average over all circuits "neighboring" to q. Then[4]

$$\langle[f(q)qG(q)/C]+\tfrac{1}{2}(d/dq)[f(q)r(q)\delta q]\rangle=0. \qquad (5)$$

Evidently an immediate solution is now only possible if $\langle r(q)\delta q\rangle\equiv\langle\delta q^2(q)\rangle/t$ has some definite limit, for sufficiently small t.[5] That is,

$$\langle\delta q^2(q)\rangle=K(q)t, \quad\text{or}\quad \langle\delta q^2(q)\rangle=2F(q)kTt, \qquad (6)$$

say, which is a generalized form of the Einstein equation. Equation (5) may then be solved to give

$$f(q)=\left[\frac{f(0)F(0)}{F(q)}\right]\exp\left[-\left(\frac{1}{kTC}\right)\int_0^q\left(\frac{qG(q)}{F(q)}\right)dq\right], \qquad (7)$$

where we must normalize so that

$$\int_{-\infty}^{+\infty}f(q)dq=1.$$

As far as we can see, no mandatory general thermodynamic relationship exists which might relate uniquely $G(q)$ and $F(q)$ for arbitrary values of q. We shall see later that this is not necessarily a serious drawback since the only cases likely to be of physical importance will involve only a very modest degree of nonlinearity, which can often be analyzed without further assumptions. Before discussing this, however, we can make some further remarks about the general problem.

3. GENERAL CONCLUSIONS

From Eq. (5), multiplying through by q and integrating, we have

$$-\int_{-\infty}^{+\infty}q\frac{d}{dq}[f(q)F(q)]dq=\frac{1}{kTC}\int_{-\infty}^{+\infty}q^2G(q)f(q)dq \qquad (8)$$

[using also Eq. (6)]. Integrating the left hand side by parts and assuming $f(q)\rightarrow0$ rapidly as $q\rightarrow\pm\infty$, we have[6]

$$\int_{-\infty}^{+\infty}F(q)f(q)dq=\frac{1}{kTC}\int_{-\infty}^{+\infty}q^2G(q)f(q)dq \qquad (9)$$

or

$$\int_{-\infty}^{\infty}F(q)f(q)dq =\int_{-\infty}^{+\infty}q^2G(q)f(q)dq\Big/\int_{-\infty}^{+\infty}q^2f(q)dq. \qquad (10)$$

[1] H. Nyquist, Phys. Rev. **32**, 110 (1928).

[2] J. Weber, Phys. Rev. **101**, 1620 (1956) [see also J. Weber, Phys. Rev. **90**, 977 (1953); **94**, 211 (1954); **96**, 556 (1954)].

[3] N. Balazs, Phys. Rev. **105**, 896 (1957).

[4] The derivation given of Eq. (5) is admittedly rather heuristic; a more formal derivation could be given but, the writer believes, with no greater rigor.

[5] We have assumed here that we need not consider fluctuations moments higher than the second, i.e., $\langle\delta q^2\rangle$. H. A. Kramers [Physica **7**, 284 (1940)] has mentioned the possibility that higher-order moments may contribute, but we believe that this contingency may be neglected, at any rate in cases where modest nonlinearity is concerned with which we are primarily concerned in the present paper. See also Appendix 2.

[6] An alternative and rather direct proof of this result is given in Appendix 2.

Thus we have the entirely general result that

$$\langle F(q)\rangle = \langle q^2 G(q)\rangle / \langle q^2\rangle.$$

In deriving Eq. (5) we have assumed that statistical equilibrium has been achieved; if this is not the case, so that f is now also a function of time, then Eq. (5) will read,

$$\beta(q,t) = -[f(q)qG(q)/C] - kT(\partial/\partial q)[f(q)F(q)],$$

where $\beta(q,t)$ is the net flux of systems "upwards" (i.e., the net fraction per unit time whose charge is increasing through q). We have also, however, $(\partial f/\partial t) = -(\partial\beta/\partial q)$, and thus,

$$(\partial f/\partial t) = (1/C)(\partial/\partial q)[qG(q)f(q)] + kT(\partial^2/\partial q^2)[f(q)F(q)], \quad (11)$$

which is then the generalized Fokker-Planck equation for systems with nonlinearity under our assumptions.

Failing any entirely general theorem connecting the instantaneous values $F(q)$ and $G(q)$, let us consider two particular possibilities. First let us assume that we might set $F(q) = G(q)$; then Eq. (7) reduces to

$$f(q) = [f(0)G(0)/G(q)] \exp(-q^2/2kTC), \quad (12)$$

where $f(0)$ is determined by

$$\int_{-\infty}^{+\infty} f(q)dq = 1.$$

However, we have also to satisfy the general relation

$$\langle q^2\rangle \equiv \int_{-\infty}^{+\infty} q^2 f(q)dq = kTC$$

which will not be true in Eq. (12) for arbitrary $G(q)$. Consequently we must reject equality of $F(q)$ and $G(q)$ in general. Let us now instead make the plausible assumption that the fluctuations are throughout determined by some average taken over the whole physical "state" of R, so that $F(q) = \bar{F}$ say, where $\bar{F}$ is in fact the average conductance defined by Eq. (9) or Eq. (10). Hence from Eq. (7),

$$f(q) = f(0) \exp\left[-(1/kTC\bar{F})\int_0^q qG(q)dq\right] \quad (13)$$

[where again of course $f(0)$ is determined by $\int_{-\infty}^{+\infty} f(q)dq = 1$]. If now we take the special case where $G(q)$ is itself a constant, say $G(q) = G$, then also $\bar{F} = G$, so that $\langle\delta q^2\rangle = 2GkTt$, the Einstein relation, and

$$f(q) = f(0) \exp(-q^2/2kTC) = (1/2\pi kTC)^{\frac{1}{2}} \exp(-q^2/2kTC). \quad (14)$$

Thus (assuming $F = \bar{F}$) the canonical distribution [cf. Eq. (1)] will hold true for *any* degree of damping so long as the damping ("viscosity") is strictly linear. Otherwise it seems unlikely that this will be so, if, indeed, we assume $F(q) = \bar{F}$ as in the foregoing, we should have to use Eq. (13) with Eq. (9). With this assumption, the Fokker-Planck Eq. (11) will read,

$$(\partial f/\partial t) = (1/C)(\partial/\partial q)[qG(q)f(q)] + \bar{F}kT(\partial^2 f/\partial q^2), \quad (15)$$

where, as before,

$$\bar{F} = (1/kTC)\int_{-\infty}^{+\infty} q^2 G(q)f(q)dq.$$

We see here that three conductances, $G(q)$, $q(\partial G(q)/\partial q)$, and $\bar{F}(=\langle q^2G(q)\rangle/\langle q^2\rangle)$, are involved in place of the single parameter G of the linear approximation.

4. MODEST NONLINEARITY

Let us now consider systems with a modest degree of nonlinearity, setting $G(q) = \alpha + \gamma q^2$ where γ is "small."[7] It is not necessary here to make any particular assumption(s) relating $F(q)$ to $G(q)$, such as we have discussed above, except naturally that the nonlinearity in $F(q)$ is also modest and symmetrical in q.

Now by Eq. (10):

$$\langle F(q)\rangle = \left[\alpha\int_{-\infty}^{+\infty} q^2 f(q)dq + \gamma\int_{-\infty}^{+\infty} q^4 f(q)dq\right] \Big/ kTC. \quad (16)$$

The first term in the numerator has the value αkTC quite generally; and in the second term we can evidently use the unperturbed (canonical) distribution for $f(q)$ as an adequate approximation. That is to say, we do not require to know specifically $f(q)$ to deal with modest nonlinearity of this type, and we then have

$$\langle\langle\delta q^2\rangle\rangle \equiv 2\langle F(q)\rangle kTt = 2\alpha kTt + 6\gamma k^2T^2Ct. \quad (17)$$

In an earlier paper,[8] the writer approached the problem of the Brownian movement of systems with a nonlinear relaxation mechanism by proposing the following hypothesis: an ensemble of circuits subject to Brownian movement will behave in such a way that the average charge in a subensemble having a given initial charge will relax as would the charge in a *single* circuit having the same initial charge, presumed to be free from Brownian movement. This hypothesis is now seen to be untenable [cf. Eq. (19) below] (Polder[9] had already voiced his objections). However, it is interesting that this hypothesis, as we foresaw at that time, showed the general behavior to be expected, since for small t we also found then,

$$\langle\langle\delta q^2\rangle\rangle = 2\alpha kTt + 6\gamma k^2T^2Ct.$$

[See Eq. (15) of reference 8 with $t\to 0$.]

We now inquire about the frequency-spectrum of the fluctuations. In order to do this, as discussed in our

[7] We are thus restricting ourselves in this section to a *symmetrical* nonlinear conductance $G(q)$; some results for an *asymmetrical* nonlinear conductance will be given in the following section where we consider a specific example.

[8] D. K. C. MacDonald, Phil. Mag. **45**, 63 (1954).

[9] D. Polder, Phil. Mag. **45**, 69 (1954).

earlier paper[8] on this subject, we wish to determine the quantity $\langle\langle q(t)q(0)\rangle\rangle$ (for all values of t) averaged over the entire ensemble in equilibrium. Using our previous notation we can write (essentially following Langevin[10])

$$(dq/dt)+(\alpha q/C)+(\gamma q^3/C)=r(q), \quad \equiv A_q(t), \text{ say}, \quad (18)$$

and thus obtain an approximate solution under the usual assumption (see, e.g., Uhlenbeck and Ornstein[11]) that the correlation time of $A(t)$ is very short compared with C/α. This we have in fact already used in Eq. (5) above, having seen that the relation $\langle\delta q^2\rangle\propto t$ for $t\ll C/G$ is a necessity for the maintenance of statistical equilibrium. After a little analysis, outlined in Appendix 1, we find

$$\begin{aligned}\langle q(t)\rangle \doteqdot q(0)\exp(-\alpha t/C)-(\gamma q^3(0)/2\alpha)\exp(-\alpha t/C)\\ \times[1-\exp(-2\alpha t/C)]-3\gamma q(0)kT\exp(-\alpha t/C)\\ \times\{t-(C/2\alpha)[1-\exp(-2\alpha t/C)]\}. \quad (19)\end{aligned}$$

This equation is of interest in itself. If $\gamma=0$, we have simply $\langle q(t)\rangle=q(0)\exp(-\alpha t/C)$ the familiar result in the conventional theory of linear Brownian movement; the Brownian movement is not manifest in the average relaxation. However in the general nonlinear case the Brownian movement makes itself felt *directly* [as evidenced in the third term of Eq. (19)] by spreading out the distribution of the subensemble over the nonlinear relaxation mechanism as time proceeds.

Continuing the analysis, if we average over the whole ensemble in equilibrium, we have

$$\begin{aligned}\langle\langle q(t)q(0)\rangle\rangle=kTC\exp(-\alpha t/C)-(3\gamma k^2T^2C^2/2\alpha)\\ \times[\exp(-\alpha t/C)-\exp(-3\alpha t/C)]-3\gamma k^2T^2C\\ \times\{t\exp(-\alpha t/C)-(C/2\alpha)[\exp(-\alpha t/C)\\ -\exp(-3\alpha t/C)]\}, \quad (20)\end{aligned}$$

using $\langle q^2\rangle=kTC$ and $\langle q^4\rangle\doteqdot 3k^2T^2C^2$ (this approximation is sufficient in the nonlinear term, as pointed out earlier). Hence

$$\begin{aligned}\langle\langle[q(t)-q(0)]^2\rangle\rangle\equiv\langle\langle\Delta q_t^2\rangle\rangle=2kTC[1-\exp(-\alpha t/C)]\\ +(3\gamma k^2T^2C^2/\alpha)[\exp(-\alpha t/C)-\exp(-3\alpha t/C)]\\ +6\gamma k^2T^2C\{t\exp(-\alpha t/C)-(C/2\alpha)\\ \times[\exp(-\alpha t/C)-\exp(-3\alpha t/C)]\}. \quad (21)\end{aligned}$$

Thus

$$\begin{aligned}\frac{d}{dt}\langle\langle\Delta q_t^2\rangle\rangle=2\alpha kT(1+3\gamma kTC/\alpha)\exp[-(\alpha t/C)]\\ -6\alpha\gamma k^2T^2t\exp(-\alpha t/C). \quad (22)\end{aligned}$$

[As $t\to 0$ this yields

$$\left[\frac{d}{dt}\langle\langle\Delta q_t^2\rangle\rangle\right]_{t\to 0}=2\alpha kT[1+(3\gamma/\alpha)kTC],$$

in agreement with Eq. (17).] The spectrum of the current fluctuations is then given by (see previous papers by the author[8,12])

$$W(f)=4\pi f\int_0^\infty \frac{d}{dt}\langle\langle(\Delta q_t)^2\rangle\rangle\sin(2\pi ft)dt;$$

and, using Eq. (22), again after some algebra, we find

$$\begin{aligned}W(f)=4\alpha kT(1+3\gamma kTC/\alpha)\{(\omega C/\alpha)^2/[1+(\omega C/\alpha)^2]\}\\ -24\gamma k^2T^2C(\omega C/\alpha)^2/[1+(\omega C/\alpha)^2]^2, \quad (23a)\end{aligned}$$

where $\omega\equiv 2\pi f$. Corresponding to Eq. (23a) the charge fluctuations are given by $\langle q_f^2\rangle=W(f)/\omega^2$ and the voltage fluctuations by

$$\begin{aligned}\langle V_f^2\rangle=(\langle q_f^2\rangle/C^2)=[W(f)/\omega^2C^2]\\ =(4kT/\alpha)\{(1+3\gamma kTC/\alpha)/[1+(\omega C/\alpha)^2]\}\\ -(24\gamma k^2T^2C/\alpha^2)/[1+(\omega C/\alpha)^2]^2. \quad (23b)\end{aligned}$$

The *total* charge fluctuation is given by

$$\langle q^2\rangle=\int_0^\infty\langle q_f^2\rangle df=\int_0^\infty[W(f)/4\pi^2f^2]df,$$

and it may readily be checked from Eq. (23a) that this yields once more $\langle q^2\rangle=kTC$, so that the whole analysis is self-consistent. If $\gamma=0$ so that we assume strict linearity of the relaxation then Eq. (23b) simply gives

$$\begin{aligned}\langle V_f^2\rangle=4RkT/[1+(\omega CR)^2], \quad \text{or if } \omega CR\ll 1:\\ \langle V_f^2\rangle=4RkT, \quad \text{where } R\equiv 1/G=1/\alpha\end{aligned} \quad (24)$$

the familiar result from Nyquist's theorem.

5. METAL-OXIDE RECTIFIER

An interesting example is provided by one of the models for the metal-oxide rectifying contact discussed by Mott and Gurney.[13] The theory gives for the current i, in terms of the potential difference V across the contact

$$i=(4\pi mek^2T^2/h^3)p\exp(-E_0/kT)[\exp(eV/kT)-1],$$

where e is the electron charge, p is the probability of an electron penetrating the potential barrier, and E_0 is an electron excitation energy. For convenience, let us take p as constant so that we may write $i=A[\exp(eV/kT)-1]$, where A is independent of the potential difference, V. Hence,

$$G(V)\equiv i/V=(A/V)[\exp(eV/kT)-1],$$

i.e.,

$$G(q)=(AC/q)[\exp(eq/kTC)-1]. \quad (25)$$

Now we shall see *a posteriori* that the nonlinearity involved is rather slight (as will presumably generally be the case physically), and consequently we may write $G(q)\approx\alpha+\beta q+\gamma q^2$, where $\alpha=eA/kT$, $\beta=e^2A/$

[10] P. Langevin, Compt. rend. **146**, 530 (1908).
[11] G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. **36**, 823 (1930).
[12] D. K. C. MacDonald, Phil. Mag. **40**, 561 (1949).
[13] N. F. Mott and R. W. Gurney, *Electronic Processes in Ionic Crystals* (Clarendon Press, Oxford, 1940), p. 181.

$2k^2T^2C$, and $\gamma = e^3A/6k^3T^3C^2$. Now $\langle\dot{q}\rangle = -qG(q)/C = -(\alpha q/C) - (\beta q^2/C) - (\gamma q^3/C)$, and since $\langle\dot{q}\rangle$ must equal zero in equilibrium, it is clear that $\langle q\rangle$ must now differ from zero. To the first approximation,

$$\begin{aligned}\langle q\rangle &= -\beta kTC/\alpha \\ &= -e/2,\end{aligned} \tag{26}$$

in the present case (emphasizing that the asymmetry is certainly very slight!). If we now make the assumption that $F(q) = \bar{F}$ (cf. Sec. 3 above) then we can readily derive a first approximation to $\langle q^3\rangle$, obtaining

$$\langle q^3\rangle = -5\beta(kTC)^2/\alpha, \tag{27}$$

so that to the second approximation

$$\langle q\rangle = -e(1-5e^2/6kTC)/2. \tag{28}$$

We may remark immediately that the term $5e^2/6kTC$ is extremely small in all cases that are of practical interest, at any rate at present [e.g., $C=1$ $\mu\mu$f (10^{-12} f); $T=1$°K; $5e^2/6kTC \approx 10^{-3}$].

We can also now derive the average fluctuations, since

$$\begin{aligned}\langle\langle\delta q^2\rangle\rangle &= 2t\langle q^2G(q)\rangle/C \quad [\text{see Eq. (9)}] \\ &= 2t(\alpha\langle q^2\rangle + \beta\langle q^3\rangle + \gamma\langle q^4\rangle)/C \\ &\approx 2\alpha kTt\{1-[(5\beta^2/\alpha^2)-(3\gamma/\alpha)]kTC\},\end{aligned} \tag{29a}$$

and in the present case this gives

$$\langle\langle\delta q^2\rangle\rangle \approx 2eAt(1-3e^2/4kTC). \tag{29b}$$

It is clear again that the deviation from linear theory is small in all practical cases.

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I should like here to express my appreciation of correspondence and discussion in the past with Professor Einstein on the foundations of Brownian movement. I am particularly grateful to Dr. T. H. K. Barron for many invaluable discussions and comments on the manuscript. I should also like to thank Dr. J. Weber for earlier correspondence, and Dr. J. S. Dugdale for discussions on this topic in the past.

APPENDIX 1

We have

$$(dq/dt) + (\alpha q/C) + (\gamma q^3/C) = A_q(t). \tag{18}$$

Neglecting at first the nonlinearity, we have as the solution of Eq. (18):

$$q(t) = q(0)\exp(-\alpha t/C) + \exp(-\alpha t/C)\int_0^t \exp(\alpha x/C)A(x)dx.$$

Hence, to the next approximation, we have

$$\begin{aligned}(dq/dt) + (\alpha q/C) &= A_q(t) - [\gamma q^3(0)/C]\exp(-3\alpha t/C) \\ &- [3\gamma q^2(0)/C]\exp(-3\alpha t/C)\int_0^t \exp(\alpha x/C)A(x)dx \\ &- [3\gamma q(0)/C]\exp(-3\alpha t/C)\int_0^t\int_0^t \exp[\alpha(x+y)/C)] \\ &\times A(x)A(y)dxdy - (\gamma/C)\exp(-3\alpha t/C) \\ &\times \int_0^t\int_0^t\int_0^t \exp[\alpha(x+y+z)/C] \\ &\times A(x)A(y)A(z)dxdydz.\end{aligned}$$

Solving again and making use of the usual assumptions about $F(t)$ (e.g., Uhlenbeck and Ornstein[11]), we have

$$\begin{aligned}\langle q(t)\rangle &= q(0)\exp(-\alpha t/C) - [\gamma q^3(0)/2\alpha]\exp(-\alpha t/C) \\ &\times [1-\exp(-2\alpha t/C)] - 3\gamma q(0)kT\exp(-\alpha t/C) \\ &\times \{t-(C/2\alpha)[1-\exp(-2\alpha t/C)]\},\end{aligned}$$

which is Eq. (19) in the text.

APPENDIX 2

Consider, as before, a simple circuit as in Fig. 1 and let $q(t)$ be the charge on the condenser C at time t. Then

$$\langle[q(t)-q(0)]^2\rangle = \langle q^2(t)\rangle + q^2(0) - 2\langle q(t)\rangle q(0),$$

where the angular brackets denote averages over a subensemble of such circuits all having the same charge $q(0)$ at $t=0$. Thus,

$$\begin{aligned}\langle[q(t)-q(0)]^2\rangle_{t\to 0} &\doteq \langle q^2(t)\rangle - q^2(0) - 2q(0)\langle\dot{q}(0)\rangle t, \\ &\qquad \text{for sufficiently}^{14}\text{ short } t \\ &= \langle q^2(t)\rangle - q^2(0) + 2q^2(0)G(q(0))t/C \\ &\qquad \text{for } t \ll C/G(q(0)),\end{aligned}$$

where, as in the text, $G(q)$ is defined by $\langle\dot{q}\rangle = -qG(q)/C$. If now we average over all values of $q(0)$ and assume that there is statistical equilibrium so that $\langle\langle q^2(t)\rangle\rangle = \langle q^2(0)\rangle$, then

$$\langle\langle[q(t)-q(0)]^2\rangle\rangle_{t\to 0} = 2t\langle q^2G(q)\rangle/C,$$

which is equivalent to Eq. (9) in the text.

[14] The assumption that a short enough time t can be chosen so that higher order terms such as $t^2\langle\ddot{q}(0)\rangle/2$ *can* be neglected in comparison with $t\langle\dot{q}(0)\rangle$ appears to correspond to the neglect of higher order fluctuational moments in Kramers' discussion (see reference 5).