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High-Energy Potential Scattering

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Asymptotically in the limit of high energies, successive approximations are obtained for the problem of potential scattering where the total phase shift through the potential is not small. Only the Schrödinger wave equation is treated. The method consists of first applying the stationary-phase approximation to the integral equation and then solving the resulting equation by iteration to derive the asymptotic behavior of the field. In particular, some information about the radiation field is obtained.

1. INTRODUCTION

In order to apply the Born approximation to problems of potential scattering, it is necessary that the velocity of the incident particle be very high. That is, if V and R are rough measures of the depth and range of the potential, and v is the velocity of the incident particle, it is required that

$$|VR/hv|\ll 1, \tag{1.1}$$

in addition to the condition that V be small compared with the kinetic energy of the incident particle. In the study of the scattering of fast electrons by heavy nuclei, for example, Eq. (1.1) is often found to be too restrictive. The situation where V is small compared with the kinetic energy, but where (1.1) is violated, has been studied by Molière, Glauber, Schiff,[1] and others. (Schiff's paper may be referred to for a fuller introduction to this subject and a bibliography of earlier work.) Their results take the form of the first Born approximation with an appropriate phase factor. It is therefore natural to try to determine the next-order approximation. It is the purpose of this paper to give a method of finding all higher order approximations in the simplest case of a smooth potential, and in particular an explicit computation for the second-order approximation in this case. For simplicity the work is limited to the Schrödinger equation.

Certain problems in other branches of physics are mathematically similar to the quantum-mechanical problem of high-energy potential scattering. An example is the scattering at high frequencies of an electromagnetic wave by an obstacle whose relative dielectric constant is close to unity. In such problems the function corresponding to the potential often has finite discontinuities, which greatly complicate the treatment. It is hoped that this will be discussed in a later paper.

A generalized Born series appropriate for the present situation is given in the appendix. This is not believed to be of great theoretical value but may be useful for practical purposes.

2. FORMULATION OF THE PROBLEM

In order to study the problem at hand, it is necessary to formulate it asymptotically (in Poincaré's sense), i.e., in a manner such that a parameter, in this case the wave number, approaches infinity. Let the Schrödinger equation be written as

$$(\Delta+k^2-V')\psi=0, \tag{2.1}$$

where k is the wave number at infinity. This equation has been reduced so that $\hbar^2k^2/2m$ is the energy of the incident particle, and $\hbar^2V'/2m$ is the actual potential energy. Thus (1.1) becomes $|V'R/k|\ll 1$. Therefore, for the present purpose, let the high-energy approximation be defined through the requirements

$$|V'|\ll k^2 \quad\text{and}\quad |V'R|\sim k. \tag{2.2}$$

Accordingly, let

$$V'=kU, \tag{2.3}$$

so that (2.1) becomes

$$(\Delta+k^2-kU)\psi=0. \tag{2.4}$$

This equation is to be solved asymptotically for $k\rightarrow\infty$. The problem is now well defined.

It should be noted in particular that as yet no condition has been imposed on the function U. In this paper U is assumed to possess as many derivatives as needed, but this is by no means necessary. For example, U may be discontinuous as mentioned earlier, or it may have a singularity as in the case of a Yukawa well. This resolves the dilemma which arises when U is required to vary slowly over one wavelength and yet may be discontinuous or singular. Certainly, such a discontinuity or singularity requires special attention; it causes a number of complications that are not considered in this paper.

Let the incident field

$$\psi^{\text{inc}}=e^{-ikx} \tag{2.5}$$

represent particles moving in the negative x direction. If G is the free-space Green's function with a coordinate representation given by

$$G(\mathbf{r}-\mathbf{r}')=\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{2.6}$$

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[1] L. I. Schiff, Phys. Rev. **103**, 443 (1956).

then it follows from (2.4–2.5) that

$$\psi = e^{-ikx} - kGU\psi. \tag{2.7}$$

or

$$U\psi + kUGU\psi = Ue^{-ikx}. \tag{2.8}$$

It is covenient to introduce the quantity

$$J(\mathbf{r}) = U(\mathbf{r})\psi(\mathbf{r})e^{ikx}, \tag{2.9}$$

which is not expected to vary too rapidly. With (2.6), the function J satisfies

$$J(\mathbf{r}) + kU(\mathbf{r}) \int_E \frac{e^{ik(|\mathbf{r}-\mathbf{r}'|+x-x')}}{4\pi|\mathbf{r}-\mathbf{r}'|} J(\mathbf{r}')d\mathbf{r}' = U(\mathbf{r}), \tag{2.10}$$

where E is the entire space. This is the integral equation to be solved asymptotically for $k\to\infty$.

If (2.10) is solved by iteration, the resulting Born series may be integrated by the stationary-phase method term by term and then summed as was carried out by Schiff. Alternatively, it is possible to use the stationary phase method in (2.10) to perform the integration over E, provided $J(\mathbf{r}')$ is a slowly varying function of position. If this is done, the result is an integro-differential equation which may be solved by iteration. Eventually, it is verified directly from the solution (4.12) that $J(\mathbf{r}')$ is indeed slowly varying. This program is carried out in the next sections.

3. STATIONARY-PHASE INTEGRATION

If $J(\mathbf{r}')$ is slowly varying, then the integral,

$$I = \int_E \frac{e^{ik(|\mathbf{r}-\mathbf{r}'|+x-x')}}{4\pi|\mathbf{r}-\mathbf{r}'|} J(\mathbf{r}')d\mathbf{r}', \tag{3.1}$$

may be expanded as a power series in $1/k$ by the method of stationary phase. Even in the Born approximation, there is little occasion to go beyond the third approximation. Therefore, no attempt is made here to write down the general terms, even though this is not difficult.

Let the coordinate system be translated such that the new origin is at $\mathbf{r}$. In this coordinate system, and without changing the notation, it follows from (3.1) that

$$I = \int_E \frac{e^{ik(r'-x')}}{4\pi r'} J(\mathbf{r}')d\mathbf{r}'. \tag{3.2}$$

Let (x', R', ϕ') be the cylindrical coordinates about the x axis. Then

$$I = \tfrac{1}{2}\int_{-\infty}^{\infty} dx' \int_0^{\infty} R'dR' \frac{e^{ik(r'-x')}}{r'} \bar{J}(x',R'), \tag{3.3}$$

where $\bar{J}$ is an average given by

$$\bar{J}(x',R') = \frac{1}{2\pi}\int_{-\pi}^{\pi} J(x',R',\phi')d\phi'. \tag{3.4}$$

For the integral in (3.3), the points of stationary phase lie on the positive half of the x axis. Therefore it is natural to split the integration with respect to x' into two parts, $\int_0^\infty$ and $\int_{-\infty}^0$. Let the two parts be called I_+ and I_-, respectively. The part I_+ is to be considered first.

For I_+, let the variable $\xi' = r' - x'$ be used instead of R'. Thus

$$I_+ = \tfrac{1}{2}\int_0^\infty dx' \int_0^\infty d\xi' e^{ik\xi'} \bar{J}(x',R'). \tag{3.5}$$

Since $k\to\infty$, the presence of the exponential factor suggests the expansion of $\bar{J}(x',R')$ into a power series in ξ'. The first three terms are

$$\bar{J}(x',R') = \bar{J}_0 + 2x'\bar{J}_1\xi' + (\bar{J}_1 + 2x'^2\bar{J}_2)\xi'^2 + \cdots, \tag{3.6}$$

where

$$\bar{J}_i = \frac{\partial^i \bar{J}}{\partial(R'^2)^i}(x',0). \tag{3.7}$$

Note that $\bar{J}(x',R')$ is an even function of R'. With an interpretation according to Abel summability, the substitution of (3.6) into (3.5) gives

$$I_+ = -\frac{1}{2}\left(\frac{1}{ik}\right)\int_0^\infty dx'\bar{J}_0(x') + \frac{1}{(ik)^2}\int_0^\infty dx'x'\bar{J}_1(x') - \frac{1}{(ik)^3}\int_0^\infty dx'[\bar{J}_1(x') + 2x'^2\bar{J}_2(x')] + \cdots. \tag{3.8}$$

Within the order of approximation of (3.8), it is sufficient to use

$$\bar{J}(x',R') \doteq \bar{J}_0(0) + x'\bar{J}_0'(0) + \cdots \tag{3.9}$$

for the evaluation of I_-. With (3.9), I_- is given by

$$I_- = \tfrac{1}{2}\int_0^\infty dx' \int_0^\infty R'dR' \frac{1}{r'} e^{ik(r'+x')}\bar{J}(-x', R') = \frac{1}{4}\frac{1}{(ik)^2}\bar{J}_0(0) - \frac{1}{8}\frac{1}{(ik)^3}\bar{J}_0'(0) + \cdots. \tag{3.10}$$

It only remains to perform a translation of the coordinate system. Note that the functions $\bar{J}_i$ as defined in (3.7) are really functions of the position coordinate. Therefore, it follows from (3.8) and (3.10) that, in a rectangular coordinate system.

$$\begin{aligned} I(x,y,z) = &-\frac{1}{2}\left(\frac{1}{ik}\right)\int_x^\infty dx' J(x',y,z) \\ &+ \frac{1}{(ik)^2}\left[\int_x^\infty dx'(x'-x)\bar{J}_1(x',y,z) + \tfrac{1}{4}J(x,y,z)\right] \\ &- \frac{1}{(ik)^3}\left[\int_x^\infty dx'[\bar{J}_1(x',y,z) + 2(x'-x)^2\bar{J}_2(x',y,z)] + \frac{1}{8}\frac{\partial}{\partial x}J(x,y,z)\right] + \cdots. \end{aligned} \tag{3.11}$$

This is the desired formula for I as defined in (3.1).

4. SOLUTION OF THE INTEGRAL EQUATION

When (3.11) is substituted into (2.10), the following integral equation is obtained:

$$U(x,y,z)=J(x,y,z)-\frac{1}{2i}U(x,y,z)\int_x^\infty dx'J(x',y,z)+\frac{1}{i}U(x,y,z)\frac{1}{ik}\left[\int_x^\infty dx'(x'-x)\bar{J}_1(x',y,z)+\tfrac{1}{4}J(x,y,z)\right]$$
$$-\frac{1}{i}U(x,y,z)\frac{1}{(ik)^2}\left[\int_x^\infty dx'[\bar{J}_1(x',y,z)+2(x'-x)^2\bar{J}_2(x',y,z)]+\frac{1}{8}\frac{\partial}{\partial x}J(x,y,z)\right]+\cdots. \quad (4.1)$$

Since $k\to\infty$, a first approximation to the function J may be obtained from (4.1) by neglecting all terms involving a negative power of k. The result is simply

$$U(x,y,z)=J(x,y,z)-\frac{1}{2i}U(x,y,z)\int_x^\infty dx'J(x',y,z). \quad (4.2)$$

This is essentially a differential equation in x for each y and z. In particular, therefore, the deviation of the direction of motion of the particles from the x axis is at most of the order of $1/k$. The exact solution of (4.2) is

$$J(x,y,z)=U(x,y,z)\exp\left(\frac{1}{2i}\int_x^\infty U(x',y,z)dx'\right). \quad (4.3)$$

This justifies the use of a phase correction in solving problems of high-energy scattering.

Next, Eq. (4.1) may be iterated with (4.3) as the first approximation. It follows from (3.7) that

$$\bar{J}_1(x,y,z)=\tfrac{1}{4}\Delta_t J(x,y,z), \quad (4.4)$$

and

$$\bar{J}_2(x,y,z)=\tfrac{1}{32}\Delta_t^2 J(x,y,z), \quad (4.5)$$

where $\Delta_t=\partial^2/\partial y^2+\partial^2/\partial z^2$ is the transverse Laplacian. Let

$$V(x,y,z)=\frac{1}{2i}\int_x^\infty U(x',y,z)dx', \quad (4.6)$$

so that

$$V'=\frac{\partial V}{\partial x}=\frac{-1}{2i}U(x,y,z). \quad (4.7)$$

Also let

$$f(x,y,z)=1+\frac{1}{2i}\int_x^\infty dx'J(x',y,z), \quad (4.8)$$

so that

$$f'=\frac{\partial f}{\partial x}=\frac{1}{2i}J(x,y,z). \quad (4.9)$$

In particular, if (4.3) is used in (4.8) in order to obtain a first approximation f_1, the result is

$$f_1=e^V. \quad (4.10)$$

This first approximation of f can now be substituted in (4.1) in order to obtain a second approximation to J. It follows from (4.8) that

$$f'+\frac{U}{2i}f+\frac{U}{4k}\left[\int_x^\infty dx'\Delta_t f_1(x',y,z)-f_1'\right]=0, \quad (4.11)$$

and that the boundary condition is $f(\infty,y,z)=1$. The solution of (4.11) is

$$J(x,y,z)=U(x,y,z)e^{V(x,y,z)}\left\{1-\frac{1}{k}\left[\frac{1}{2}U(x,y,z)+\frac{1}{8i}\int_x^\infty U^2(x',y,z)dx'+\frac{1}{2i}\int_x^\infty dx'e^{-V(x',y,z)}\times\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)e^{V(x',y,z)}\right]\right\}. \quad (4.12)$$

The third approximation may be found by another iteration.

Since J does not have an exponential factor involving k, the stationary phase argument used in the last section is justified. With J determined, the total field ψ at any finite point of the space may be found directly from (2.9).

5. RADIATION FIELD

Asymptotically, as $r\to\infty$ for fixed k, the total field is

$$\psi(\mathbf{r})=\psi(r,\hat{r})\sim e^{-ikx}+\frac{e^{ikr}}{r}f^{\text{rad}}(\hat{r}), \quad (5.1)$$

where $\hat{r}$ is the unit vector in the direction of $\mathbf{r}$, and when $f^{\text{rad}}(\hat{r})$ is the radiation field. This can be easily obtained from (2.7) and (4.12). The reason that (2.9) cannot be used is that in general the two limiting operators $r\to\infty$ and $k\to\infty$ do not commute. The result is

$$f^{\text{rad}}(\hat{r})=-\frac{k}{4\pi}\int_E J(\mathbf{r}')\exp(-ikx'+ik\mathbf{r}'\cdot\hat{r})dr'. \quad (5.2)$$

In particular, the forward radiation field is

$$f_0^{\mathrm{rad}}=\frac{k}{4\pi}\int_E J(\mathbf{r})d\mathbf{r}$$

$$=\frac{ik}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dydz\left\{1-\exp\left(\frac{1}{2i}\int_{-\infty}^{\infty}U(x,y,z)dx\right)\left(1-\frac{1}{8ik}\int_{-\infty}^{\infty}U^2(x,y,z)dx\right)\right.$$

$$\left.-\frac{1}{2ik}\int_{-\infty}^{\infty}dx\left[1-\exp\left(\frac{1}{2i}\int_{-\infty}^{x}U(x'y,z)dx'\right)\right]\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\exp\left(\frac{1}{2i}\int_{x}^{\infty}U(x',y,z)dx'\right)+\cdots\right\}. \quad (5.3)$$

From this the total scattering cross section may be found.

Let θ be the angular deviation of the direction of observation from the forward direction. If θ is of the order of $1/k$, then (5.2) gives the radiation field in that direction. On the other hand, if θ is much larger than $1/k$, the right hand side of (5.2) is again an integral involving a rapidly varying phase, and, to be consistent with the previous development, it should again be evaluated by the method of stationary phase. For fixed $\theta \doteq 0$, therefore, the radiation field can at most be of the order of $k^{-(N+1)}$, where N is the number of continuous derivatives that $U(\mathbf{r})$ possesses. This result is indeed physically obvious from a consideration of the momentum transfer.

The present theory is unable to give information about large-angle scattering, which arises principally from irregularities of the potential, such as discontinuities and singularities. Therefore, attempts to get detailed information about the nuclei from data on large-angle scattering at high energies are justified. It is then necessary to treat the more difficult Dirac equation with singularities and discontinuities in the potential.

It is interesting to note that the present treatment, while starting with assumptions that are similar to those used by Schiff, leads to results that are different in form. It is evidently desirable to study this difference before attempting to apply the present method to the Dirac equation.

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APPENDIX. GENERALIZED BORN SERIES

Although the Born series is useful only for the situation $k\gg|V'R|$, it is not a power series in $1/k$. For the case $k\sim|V'R|$, an analogous rapidly convergent series may be obtained by writing (2.10) in the form

$$J(x,y,z)-\frac{1}{2i}U(x,y,z)\int_x^{\infty}dx'J(x',y,z)$$

$$=U(x,y,z)\left[1+\frac{1}{2i}\int_{-\infty}^{x}e^{2ik(x-x')}J(x',y,z)dx'\right.$$

$$\left.-k\int_E\frac{e^{ik(|\mathbf{r}-\mathbf{r}'|+x-x')}}{4\pi|\mathbf{r}-\mathbf{r}'|}[J(x',y',z')-J(x',y,z)]d\mathbf{r}'\right]. \quad (A.1)$$

A generalized Born series may be obtained from (A.1) by using the iterative procedure given in Sec. 4. The resulting series is rapidly convergent if the function $U(x,y,z)$ satisfies a Hölder's condition.