Theory of Photoproduction of Pions from Nucleons*

J. ENOCH,† R. G. SACHS, AND K. C. WALI
University of Wisconsin, Madison, Wisconsin
(Received July 3, 1957)

A resonance theory of the pion-nucleon system based on a very general static model is extended to give an expression for the pion photoproduction amplitude. It is found that for photon energies less than 500 Mev there are just two important terms in the amplitude, one of them describing direct photoejection of the pion and the other, photoexcitation of the $J=\frac{3}{2}$, $I=\frac{3}{2}$ resonance state of the nucleon. The first term is estimated by means of a weak-coupling, finite-source theory which is made gauge-invariant by introducing line currents in the source. Comparison with the threshold π^+ production leads to a coupling constant (without recoil correction) $f^2=0.049$. The behavior of the cross section at high energies does not appear to be consistent with this result. Much better agreement with the data above threshold is obtained for the uncorrected coupling constant $f^2=0.07$. However, serious discrepancies within the data make it impossible to resolve this difficulty at present.

The behavior of the photoejection part of the π^+ cross section above the resonance is found to be sensitive to the form of the source function. With $f^2=0.049$ and a Gaussian source function, infinite cutoff comes closest to fitting the data. However a considerably better fit is obtained with $f^2=0.07$ and a cutoff of about 4 pion masses. The resonance term is sensitive to the form of the pion-nucleon phase shift α_{33} above the resonance. We find that the simple one-level form for the energy dependence of α_{33} (Chew-Low curve) is not adequate to account for the data for high-energy photoproduction of pions; it is necessary to take account of the residue of higher resonances. Good agreement with the resonance term is obtained in terms of just two constant parameters, $\mathfrak{M}$, the strength of the resonance matrix element, and Q, the contribution of higher resonances to the phase shift. We find no supporting evidence for a previously suggested S wave, $I=\frac{1}{2}$ resonance.

I. INTRODUCTION

THERE is good reason to hope that the extensive data[1] on pion photoproduction which have recently become available will provide some insight into the nucleon structure. In general, the small magnitude of the electromagnetic coupling constant means that a system is not greatly disturbed by its interaction with electromagnetic radiation. Matrix elements for transitions produced by the radiation depend primarily on the structural characteristics of the system in its stationary states. Hence electromagnetic transitions may be used to study the structure of a system. The photoproduction of pions is one of the simpler processes involving such transitions. We shall attempt here to provide for this process a theory relating it rather directly with the structural properties of the nucleon.

The basis for this theory has already been given[2] in connection with a similar treatment of pion-nucleon scattering. It has been shown that the general characteristics of the scattering do not depend strongly on the dynamical details of the theory, such as the form of the interaction, and we shall find that a similar conclusion applies to the photopion problem. The results are expressed in terms of a few constant (or nearly constant) parameters, each one of them having a rather direct physical interpretation. The values of these parameters could be calculated, at least in principle, for any specific form of interaction. However, we prefer to use them as the basis for a phenomenological treatment, to determine the few parameters from experiment, and to make use of their rather direct physical meaning to interpret their values in terms of structure of the states of the nucleon.

To carry through this program, it has been necessary to neglect nucleon recoil and nucleon-antinucleon pairs; otherwise the assumptions are very broad. Hence the general aspects of the theory should include any finite static model of the nucleon. However, many detailed assumptions such as the use of the one level approximation, are not given any justification here, but they are made as simplifying assumptions which are found to give a reasonable fit to the data. Their true justification can come only from a detailed dynamical theory which serves to determine all the parameters.

Detailed dynamical treatments of the photoproduction of pions have been presented by Chew and Low[3] for the static model, and by Ross[4] for a model including some recoil and nucleon pair effects. Their work has the advantage that it is more fundamental in principle and that it involves fewer parameters, but it must be recognized that hidden parameters are introduced by the use of approximations which are not always clearly justified.

Chew and Low have not presented a detailed comparison of their results with experiment[5] but they find agreement with the important trends already suggested by the more phenomenological analyses of the data[6];

* Work supported in part by the U. S. Atomic Energy Commission and in part by the University Research Committee with funds provided by the Wisconsin Alumni Research Foundation.

† Present address: Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

[1] D. C. Oakley and R. L. Walker, Phys. Rev. **97**, 1283 (1955); Goldschmidt-Clermont, Osborne, and Scott, Phys. Rev. **97**, 188 (1955); Tollestrup, Keck, and Worlock, Phys. Rev. **98**, 220 (1955); Walker, Teasdale, Peterson, and Vett, Phys. Rev. **98**, 210 (1955); Beneventano, Bernardini, Carlson-Lee, Stoppini, and Tau, Nuovo cimento **4**, 323 (1956); McDonald, Peterson, and Corson, Phys. Rev. **107**, 577 (1957); L. J. Koester, Jr., and F. E. Mills, Phys. Rev. **105**, 1900 (1957).

[2] R. G. Sachs, Phys. Rev. **95**, 1065 (1954).

[3] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1579 (1956).

[4] M. Ross, Phys. Rev. **103**, 730 (1956).

[5] A detailed comparison of the theory with experiment has been made by M. J. Moravcsik, Phys. Rev. **105**, 267 (1957).

[6] M. Gell-Mann and K. M. Watson, *Annual Review Nuclear Science* (Annual Reviews, Inc., Stanford, 1954), Vol. 4, p. 219.

the energy dependence of the matrix element is dominated by the well-known p wave $J=\frac{3}{2}$, $I=\frac{3}{2}$ resonance state of the nucleon-pion system operating against a background of direct photoproduction having the same form as would be expected for weak pion-nucleon coupling. Ross obtains similar results and finds them to be in moderately good agreement with the data, which he analyzed in considerable detail.

The most inclusive phenomenological analysis of the data has been carried out by Watson, Keck, Tollestrup, and Walker.[7] The theoretical basis for their work is limited to the most general principles such as invariance of the interactions under time reversal and unitarity of the S matrix. From these principles, Watson has demonstrated that the complex phases of the photoproduction matrix elements are determined by the pion-nucleon scattering phase shifts. The phase shifts having been determined from the scattering, the photopion matrix elements involve only real functions of the energy, one function corresponding to each multipole moment. The energy dependence of these functions is determined from the data, but only after use has been made of certain reasonable assumptions concerning the trends of these functions. In particular, use is made of the notion that the P-wave production is enhanced in the $J=\frac{3}{2}$, $I=\frac{3}{2}$ state.

Our results take on a similar form except that the energy dependence of the multipole moments is now given explicitly and the empirical parameters are constants having a direct physical interpretation. We shall find that there are some significant differences between our results and those of Watson, Keck, Tollestrup, and Walker, especially for the higher energies. Nevertheless we obtain a good fit to the data (occasionally by making judicious use of assigned experimental errors) by means of a very simple expression for the matrix element.

II. REVIEW OF THE THEORETICAL BASIS

We first review some of the pertinent features of the resonance theory[2] of pion-nucleon scattering which is the basis of our treatment of the photoproduction. The theory takes as its starting point the notion that in any finite, static theory of the pion-nucleon system, the state vector of the nucleon ground state may be expanded in terms of states of the free pion field. It is expected that the pion field associated with a single nucleon in the ground state extends only to small distances from the center-of-mass of the system. This is taken into account by expanding the fields in terms of a specially chosen set of functions in place of the Fourier expansion normally used for this purpose. The special set is actually divided into two sets, the two together forming the complete set replacing the plane waves. One of these is a discrete set of "bound" functions, $\varphi_{\mathbf{q}}$, having the property that they vanish at least exponentially for distances large compared to the size of the pion proper field. The other is the continuously infinite set of "unbound" functions, $\varphi_{\mathbf{p}}$, orthogonal to the bound functions, which when combined with the bound functions leads to a complete set. In general the labels $\mathbf{q}$ and $\mathbf{p}$ comprise four quantum numbers and we shall later find it convenient to choose two of these to specify the orbital angular momentum and another the charge state of the pion. Thus $\mathbf{q}\equiv(q,l,m,t)$ and $\mathbf{p}\equiv(p,l,m,t)$ where l is the orbital angular momentum, m its projection, $t=\pm1$, 0 is the pion charge, q is a discrete index (radial quantum number) labeling the bound states, and p is a continuous index which may be interpreted as the magnitude of the momentum of a pion in the asymptotic region.

A method for determining the bound set has not been specified, nor is it crucial for our purpose. It becomes important only when an attempt is made to carry out complete calculations starting from a specific interaction. However one property of the set is required, that the state vector of the free nucleon can be described to a very good approximation in terms of the bound set, the unbound set does not enter into the description of this state vector. This objective could be met to a very good approximation in, for example, a finite source theory by defining the bound functions as the complete set in a sphere of given radius, centered on the nucleon. For a radius larger than, say, a pion Compton wavelength, such a set should be capable of giving a good description of the ground state.

If the pion field operators are expanded in terms of the functions $\varphi_{\mathbf{q}}$ and $\varphi_{\mathbf{p}}$, the coefficients $a_{\mathbf{q}}^*$, $a_{\mathbf{p}}^*$ and $a_{\mathbf{q}}$, $a_{\mathbf{p}}$ in the expansion are creation and annihilation operators for pions in the states $\varphi_{\mathbf{q}}$ and $\varphi_{\mathbf{p}}$. The Hamiltonian for the pion field in the presence of a nucleon is presumed to be a sum of bilinear and linear forms in the field operators which are now expanded in terms of the $a_{\mathbf{q}}$, $a_{\mathbf{p}}$, etc.; hence the Hamiltonian becomes the sum of three operators

$$H=H_B+T_u+V, \tag{1}$$

where H_B contains only the operators $a_{\mathbf{q}}$ and $a_{\mathbf{q}}^*$, T_u is a bilinear expression in the $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^*$ arising from the free pion-field Hamiltonian, and V is the remainder of the Hamiltonian. V contains terms of the form $a_{\mathbf{q}}^*a_{\mathbf{p}}$ and $a_{\mathbf{p}}^*a_{\mathbf{q}}$ arising from the free-field Hamiltonian; other contributions arise from the interaction. In the usual case of a linear interaction, V is linear in the operators $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^*$, and we shall make use of its linearity.

The characteristic value problem,

$$H_B\chi_\lambda=E\chi_\lambda, \tag{2}$$

is now considered. H_B is a Hermitian operator given as a function of the discrete set of operators $a_{\mathbf{q}}$ and $a_{\mathbf{q}}^*$. Hence the spectrum of characteristic values, E_λ, is discrete and they are real. The corresponding χ_λ are

[7] Watson, Keck, Tollestrup, and Walker, Phys. Rev. **101**, 1159 (1956).

state vectors which may be written as functions of the occupation numbers in states $\varphi_{\mathbf{q}}$. The set of unbound states $\varphi_{\mathbf{p}}$ does not appear in this problem in any way. In general the χ_λ will not be states of a definite number of pions. Furthermore they are not expected to be stationary states of the physical system since the complete Hamiltonian, Eq. (1), has not been included in Eq. (2). However the lowest state χ_0 is expected to be stationary (or at least nearly so), because of the manner in which the $\varphi_{\mathbf{q}}$ have been defined, namely, by the condition that the ground state vector of the nucleon and hence of the full Hamiltonian H, be describable in terms of the $\varphi_{\mathbf{q}}$. Thus χ_0 is identical with, or nearly identical with the state vector of a free nucleon.

From this statement we may derive a test of the adequacy of any particular choice of the set $\varphi_{\mathbf{q}}$. The nuclear state vector must be a solution of the equation

$$H\chi_0=E_0\chi_0, \tag{3}$$

which, on comparison with Eqs. (2) and (1) gives

$$(T_u+V)\chi_0=0.$$

Now since T_u is just a bilinear form in the $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^*$ (ordered so as to eliminate zero-point energy) while χ_0 involves only the bound states we have

$$T_u\chi_0=0.$$

It follows that V must satisfy the condition

$$V\chi_0=0, \tag{4}$$

and this is a condition on the functions $\varphi_{\mathbf{q}}$. The form of the condition may be easily established when V is linear in the operators $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^*$. Then it is a linear combination of expressions having the form $\Omega(\mathbf{p})a_{\mathbf{p}}^* +\Omega^*(\mathbf{p})a_{\mathbf{p}}$, where $\Omega(\mathbf{p})=\Omega[\mathbf{p};a_{\mathbf{q}},a_{\mathbf{q}}^*]$ is an operator depending on the bound operators. Since $a_{\mathbf{p}}\chi_0=0$ quite generally, Eq. (4) now reads

$$\Omega[\mathbf{p},a_{\mathbf{q}},a_{\mathbf{q}}^*]\chi_0=0. \tag{5}$$

This is a condition on the $\varphi_{\mathbf{q}}$, as stated. For any given Hamiltonian H and choice of the $\varphi_{\mathbf{q}}$, the decomposition Eq. (1) may be carried out. The operators $\Omega(\mathbf{p})$ are thereby determined as is the state χ_0, hence the condition Eq. (5) may be tested. Presumably, if the $\varphi_{\mathbf{q}}$ have been well chosen, Eq. (5) will be nearly satisfied; if not, another choice of the $\varphi_{\mathbf{q}}$ is required. We assume henceforth that a good choice has been made.

So far, no mention has been made of the manner in which the $\varphi_{\mathbf{p}}$ are to be fixed. Our procedure is to consider the one-pion states which are solutions of the characteristic value equation

$$T_u\Psi_{\mathbf{p}}=p_0\Psi_{\mathbf{p}}, \tag{6}$$

where $p_0=(p^2+m_\pi^2)^{\frac{1}{2}}$, m_π being the mass of the pion, which will hence forth be taken equal to one, as are c and $\hbar$. It has been shown[2] that this equation leads to a set of algebraic equations determining the $\varphi_{\mathbf{p}}$ in terms of the $\varphi_{\mathbf{q}}$. An important property of the $\varphi_{\mathbf{p}}$ are the phase shifts $\eta_l(p)$ associated with pion orbital angular momentum l. It is shown in reference 2 that the η_l may be expressed explicitely in terms of the functions $\varphi_{\mathbf{q}}$ of angular momentum l. It turns out that the η_l are negative for small momenta as would be expected since they express the condition of orthogonality to the bound field, which is an influence similar to that of a repulsive potential.

There are both outgoing and incoming solutions to Eq. (6) denoted by $\Psi_{\mathbf{p}}^\pm$, respectively. To fix the phase of each of these functions we set

$$\Psi_{\mathbf{p}}^\pm=e^{\pm i\eta_l}\Psi_{\mathbf{p}} \tag{7}$$

where the phase of $\Psi_{\mathbf{p}}$ is such that under the operation of (Wigner) time reversal[8]

$$\Psi_{plm}\rightarrow-(-1)^m\Psi_{pl,-m}.$$

The radial part of the function $\varphi_{\mathbf{p}}$ of a pion in the state $\Psi_{\mathbf{p}}$ is therefore a real function.

The part of the Hamiltonian,

$$H_0=H_B+T_u,$$

which describes the uncoupled bound and unbound fields, may be used to provide a basis for the expansion of the complete solutions of the problem. We shall be interested in the characteristic functions χ_λ (no unbound pions) of H_0, and the characteristic functions $\psi_{\lambda\mathbf{p}}^\pm =e^{\pm i\eta_l}\psi_{\lambda\mathbf{p}}$ which are essentially products of the χ_λ and $\Psi_{\mathbf{p}}^\pm$. The corresponding characteristic values of H_0 are E_λ and $E_\lambda+p_0$, respectively.

In the treatment of both scattering of a single pion and its photoproduction, we are concerned with the solution $\Phi_{\mathbf{p}}$ of the equation

$$H\Phi_{\mathbf{p}}=p_0\Phi_{\mathbf{p}}, \tag{8}$$

which has the asymptotic form of the wave function of an unbound pion of energy p_0 in the presence of a nucleon in the ground state χ_0. For the sake of simplicity we have now chosen the energy scale so that

$$E_0=0. \tag{9}$$

Equation (8) may conveniently be replaced by the integral equations[9]

$$\Phi_{\mathbf{p}}^\pm=\left\{1+\frac{1}{p_0-H_\pm\pm i\epsilon}V +\frac{1}{p_0-H_0\pm i\epsilon}V\frac{1}{p_0-H_\pm\pm i\epsilon}V\right\}\psi_{0\mathbf{p}}^\pm, \tag{10}$$

where the $\pm$ indicates a solution satisfying the outgoing or incoming wave conditions, respectively. The operator

[8] R. G. Sachs, Phys. Rev. **87**, 1100 (1953).

[9] K. M. Brueckner and K. M. Watson, Phys. Rev. **90**, 699 (1953).

$H_\pm$ appearing in Eq. (10) is

$$H_\pm = H_0 + V\frac{1}{p_0 - H_0 \pm i\epsilon}V. \quad (11)$$

According to Eq. (10), the asymptotic form of the function $\Phi_p^\pm$ in the one (unbound) pion channel, has a phase shift δ in addition to the phase shift η_l of the unbound wave. For a channel of given orbital angular momentum l, total angular momentum $J=l\pm\frac{1}{2}$, and total isotopic spin I, the total phase shift is then

$$\alpha_\gamma = \eta_\gamma + \delta_\gamma,$$

where we have introduced the condensation $\gamma \equiv (l,J,I)$. Examination of the asymptotic form of the function reveals that[10]

$$e^{i\delta_\gamma}\sin\delta_\gamma = -\pi p p_0 \left\langle \psi_{0p\gamma},\, V\frac{1}{p_0 - H_+ + i\epsilon}V\psi_{0p\gamma}\right\rangle. \quad (12)$$

If V is the linear form in $a_{\mathbf{p}}^*$ and $a_{\mathbf{p}}$:

$$V = \int d\mathbf{p}(p/p_0)^l[\Omega(\mathbf{p})a_{\mathbf{p}}^* + \Omega^*(\mathbf{p})a_{\mathbf{p}}], \quad (13)$$

with $\mathbf{p} \equiv (p,l,m,t)$ and $\int d\mathbf{p}$ a shorthand notation for $\sum_{l,m,t}\int p^2 dp$, we may make use of Eq. (5) to obtain

$$e^{i\delta_\gamma}\sin\delta_\gamma = -\pi p p_0 (p/p_0)^{2l} \times \sum_{\lambda,\lambda'} g_{\lambda l}^* g_{\lambda' l}\left\langle \chi_\lambda, \frac{1}{p_0 - H_+ + i\epsilon}\chi_{\lambda'}\right\rangle, \quad (14)$$

where λ and λ' are limited to bound states having angular momentum J, isotopic spin I, and parity $(-1)^{l+1}$. The coefficients $g_{\lambda l}$ are given by

$$g_{\lambda l} = \langle \chi_\lambda, \Omega^*(\mathbf{p})\chi_0\rangle. \quad (15)$$

Note that, according to Eq. (5), $g_{0l}(p)\equiv 0$. It is likely, as suggested in reference 2, that $g_{\lambda l}$ is a slowly varying function of p over the range of energies of interest here, namely, below the cutoff in the interaction.

III. FORM OF THE MATRIX ELEMENT

Because of the small size of the electromagnetic coupling, the transition matrix for photoproduction is simply given by the matrix element of the electromagnetic interaction between the ground state, χ_0, of the nucleon and the final state $\Phi_{\mathbf{p}}$. The electromagnetic interaction, $\mathcal{E}$, is proportional to the current-density operator which can safely be assumed to be a sum of linear and bilinear forms in the field operators. Therefore, when $\mathcal{E}$ is expanded in terms of the operators $a_{\mathbf{q}}$, $a_{\mathbf{p}}$, etc., it has the form:

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_1^* + \mathcal{E}_2 + \mathcal{E}_2^* + \mathcal{E}_2', \quad (16)$$

with

$$\mathcal{E}_1 = \int d\mathbf{p}\,\mathcal{C}_1(\mathbf{p})a_{\mathbf{p}}^*, \quad (17a)$$

$$\mathcal{E}_2 = \int d\mathbf{p}\int d\mathbf{p}'\,\mathcal{C}_2(\mathbf{p},\mathbf{p}')a_{\mathbf{p}}^* a_{\mathbf{p}'}^*, \quad (17b)$$

$$\mathcal{E}_2' = \int d\mathbf{p}\int d\mathbf{p}'\,\mathcal{C}_2'(\mathbf{p},\mathbf{p}')a_{\mathbf{p}}^* a_{\mathbf{p}'}. \quad (17c)$$

The expressions $\mathcal{E}_0$, $\mathcal{C}_1(p)$ are operators in the bound field given as functions of the $a_{\mathbf{q}}^*$ and $a_{\mathbf{q}}$, $\mathcal{E}_0$ being of second order and $\mathcal{C}_1$ of first order in $a_{\mathbf{q}}^*$ and $a_{\mathbf{q}}$. On the other hand the expressions $\mathcal{C}_2$ and $\mathcal{C}_2'$ are expected to be simple functions of the variables p and p'.

Watson[11] has shown that the cross section is determined by the matrix element

$$E(\mathbf{p}) = \langle \Phi_{\mathbf{p}}^-, \mathcal{E}\chi_0\rangle, \quad (18)$$

between the incoming wave solution, $\Phi_{\mathbf{p}}^-$ of Eq. (8) and the ground state χ_0. From Eq. (10), the matrix element is found to have the form

$$E(\mathbf{p}) = \langle \psi_{0\mathbf{p}}^-, \mathcal{E}\chi_0\rangle + \left\langle V\psi_{0\mathbf{p}}^-, \frac{1}{p_0 - H_+ + i\epsilon}\mathcal{E}\chi_0\right\rangle + \left\langle \psi_{0\mathbf{p}}^-, V\frac{1}{p_0 - H_+ + i\epsilon}V\frac{1}{p_0 - H_0 + i\epsilon}\mathcal{E}\chi_0\right\rangle. \quad (19)$$

Now since $a_{\mathbf{p}}\chi_0 = 0$, in general, it can be seen from Eq. (17) that the terms $\mathcal{E}_1^*$, $\mathcal{E}_2^*$ and $\mathcal{E}_2'$ may be dropped from Eq. (19) when $\mathcal{E}$ is replaced by Eq. (16). Furthermore, only the term $\mathcal{E}_1$, contributes to the first and third term of Eq. (19) since these matrix elements require that the number of unbound pions change by an odd number. Similarly, only $\mathcal{E}_0$ and $\mathcal{E}_2$ contribute to the second term of Eq. (19) which now becomes

$$E(\mathbf{p}) = \langle \psi_{0\mathbf{p}}^-, \mathcal{E}_1\chi_0\rangle + \left\langle \psi_{0\mathbf{p}}^-, V\frac{1}{p_0 - H_+ + i\epsilon}V\frac{1}{p_0 - H_0 + i\epsilon}\mathcal{E}_1\chi_0\right\rangle + \left\langle V\psi_{0\mathbf{p}}^-, \frac{1}{p_0 - H_+ + i\epsilon}(\mathcal{E}_0 + \mathcal{E}_2)\chi_0\right\rangle. \quad (20)$$

Some simplification of Eq. (20) may be attained by calculating the matrix element into a pion state $\Phi_{p\gamma}$ of energy $p_0 = (p^2+1)^{\frac{1}{2}}$ and orbital angular momentum l combined with the spin of the nucleon to form a total angular momentum J. Similarly, I is the total isotopic spin of the system and $\gamma \equiv (l,J,I)$, as before. These

[10] The following differs from the treatment given in reference 2 only in its use of running rather than standing waves. The running wave solution gives the simplest form for a matrix element, such as the photoproduction matrix element, while the standing wave gives the simplest form for the phase shift.

[11] K. M. Watson, Phys. Rev. 88, 1163 (1952).

functions are formed simply by inserting for ψ_{0p} in Eq. (10) the usual linear combinations $\psi_{0plJI}=\psi_{0p\gamma}$ of products of χ_0 and Ψ_p. With the understanding that one such state is being considered, we now treat separately each of the three terms in Eq. (20).

Since $\mathcal{E}_1$, creates a single unbound pion, we may expand $\mathcal{E}_1\chi_0$ in the complete set of one (unbound) pion states $\psi_{\lambda p\gamma}$, namely:

$$\mathcal{E}_1\chi_0=\sum_{\lambda,\gamma}\int p^2dpD_{\lambda,\gamma}(p)\psi_{\lambda p\gamma}. \quad (21)$$

The first term of Eq. (20) is then simply

$$\langle\psi_{0p\gamma}^-,\mathcal{E}_1\chi_0\rangle=e^{i\eta_l}D_{0\gamma}(p). \quad (22)$$

This term will be referred to as the *direct* matrix element since it corresponds to direct photoejection of the pion from the proper field of the nucleon into the unbound state. An estimate of the direct matrix element will be made in the next section.

The expansion Eq. (21) may also be inserted into the second term of Eq. (20), with the result

$$\left\langle\psi_{0p\gamma}^-,\, V\frac{1}{p_0-H_++i\epsilon}V\frac{1}{p_0-H_0+i\epsilon}\mathcal{E}_1\chi_0\right\rangle$$
$$=\sum_\lambda e^{i\eta_l(p)}P\int\frac{p'^2dp'}{p_0-p_0'-E_\lambda}D_{\lambda\gamma}(p')$$
$$\times\left\langle\psi_{0p\gamma},\, V\frac{1}{p_0-H_++i\epsilon}V\psi_{\lambda p'\gamma}\right\rangle$$
$$-i\pi e^{i\eta_l(p)}\Bigg\{p'p_0'\sum_\lambda D_{\lambda\gamma}(p')$$
$$\times\left\langle\psi_{0p\gamma},\, V\frac{1}{p_0-H_++i\epsilon}V\psi_{\lambda p'\gamma}\right\rangle\Bigg\}_{p_0'=p_0-E_\lambda}, \quad (23)$$

where P indicates that the principal part of the integral is to be taken. Comparison with Eqs. (12) and (14) shows that, if $g_{\lambda l}$ may be treated as a constant, the terms with $\lambda=0$ in Eq. (23) have the simple form

$$-e^{i\delta_\gamma}\sin\delta_\gamma e^{i\eta_l(p)}\left[\frac{1}{\pi pp_0}P\int\frac{p'^2dp'}{p_0-p_0'}\times\left(\frac{p'p_0}{pp_0'}\right)^l D_{0\gamma}(p')-iD_{0\gamma}(p)\right].$$

An evaluation of these terms has been made from the estimates of the direct matrix element $D_{0\gamma}$, which are given in the next section. The integrated term is found to be quite small compared to other contributions to the matrix element, hence we shall neglect it. Furthermore, there is no reason to expect that the other ntegrated terms in Eq. (23) are any more important than this one, and all such terms will be dropped from consideration henceforth. Equation (23) then becomes

$$\left\langle\psi_{0p\gamma}^-,\, V\frac{1}{p_0-H_++i\epsilon}V\frac{1}{p_0-H_0+i\epsilon}\mathcal{E}_1\chi_0\right\rangle$$
$$\approx ie^{i\alpha_\gamma}\sin\delta_\gamma D_{0\gamma}(p)-i\pi\Bigg[p'p_0'e^{i\eta_l}\sum_{\lambda\neq0}D_{\lambda\gamma}(p')$$
$$\times\left\langle\psi_{0p\gamma},\, V\frac{1}{p_0-H_++i\epsilon}V\psi_{\lambda p'\gamma}\right\rangle\Bigg]_{p_0'=p_0-E_\lambda}. \quad (24)$$

Now $\langle\psi_{0p\gamma},\, V[1/(p_0-H_++i\epsilon)]V\psi_{\lambda p'\gamma}\rangle_{p_0'=p_0-E_\lambda}$ is proportional to the matrix element for pion inelastic scattering with production of the nucleon excited state λ. At the energies under consideration here, the process evidently has a very small cross section compared to the elastic scattering cross section.[12] Hence it seems reasonable to drop all but the first term of Eq. (24), with the result

$$\left\langle\psi_{0p\gamma}^-,\, V\frac{1}{p_0-H_++i\epsilon}V\frac{1}{p_0-H_0+i\epsilon}\mathcal{E}_1\chi_0\right\rangle\approx ie^{i\alpha_\gamma}\sin\delta_\gamma D_{0\gamma}(p). \quad (25)$$

Turning now to the last term in Eq. (20), we note that $\mathcal{E}_0\chi_0$ contains only bound states so that

$$\mathcal{E}_0\chi_0=\sum_\lambda B_{\lambda0}\chi_\lambda. \quad (26)$$

If we make use of Eqs. (13) and (5) for V, we obtain

$$\left\langle V\psi_{0p\gamma}^-,\,\frac{1}{p_0-H_++i\epsilon}\mathcal{E}_0\chi_0\right\rangle=e^{i\eta_l}(p/p_0)^l\times\sum_{\lambda,\lambda'}g_{\lambda l}^*B_{\lambda'0}\left\langle\chi_\lambda,\,\frac{1}{p_0-H_++i\epsilon}\chi_{\lambda'}\right\rangle, \quad (27)$$

where the sum includes only those states[13] λ, λ' having parity $(-1)^{l+1}$, angular momentum J, and isotopic spin I. The similarity between Eq. (27) and Eq. (14) is striking. In particular the fact that there is only one known resonance state of the nucleon in the energy range of interest suggests limiting the sums appearing in both equations to just the one term corresponding to this resonance. Then we have

$$\left\langle V\psi_{0p\gamma}^-,\,\frac{1}{p_0-H_++i\epsilon}\mathcal{E}_0\chi_0\right\rangle\approx-(1/\pi pp_0)(p/p_0)^l\times(B_{\lambda0}/g_{\lambda l})e^{i\alpha_\gamma}\sin\delta_\gamma. \quad (28)$$

[12] R. S. Margulies, Phys. Rev. **100**, 1255 (1955).

[13] Note that $\lambda=0$ does not occur because $g_{0l}=0$. However, $\lambda'=0$ might appear in the sum. But $H_+\chi_0=(H_0+V(p_0-H_0+i\epsilon)^{-1}V)\chi_0=0$ according to Eqs. (4) and (9); hence $[p_0-H_++i\epsilon]^{-1}$ connects the state χ_0 only with itself.

Finally we have the term involving $\mathcal{E}_2$ in Eq. (20). Since $\mathcal{E}_2$ creates a pair of (unbound) pions and [by virtue of Eq. (5)], $V\psi_{0p}^-$ is a linear combination of the χ_λ, the matrix element is a sum of terms of the form $\langle\chi_\lambda, (p_0-H_++i\epsilon)^{-1}\psi_{0p'p''}\rangle$, the last factor being a two-pion state. This quantity also occurs as one of the terms in the matrix element for inelastic scattering of pions which, as we have noted before,[12] seems to be a negligible effect. Therefore the $\mathcal{E}_2$ contribution is dropped from consideration.

We now collect together our results Eqs. (22), (25), and (28) to obtain in place of Eq. (20) the approximate matrix element

$$E_\gamma(p)\approx e^{i\alpha_\gamma}[D_{0\gamma}(p)\cos\delta_\gamma - (1/\pi p p_0)(p_0/p)^l(B_{\lambda 0}/g_{\lambda l})\sin\delta_\gamma], \quad (29)$$

for the transition into a state of parity $(-1)^{l+1}$, angular momentum J, isotopic spin I, and pion energy p_0. The interpretation of this rather simple result is straightforward enough; the first term corresponds to direct photoejection of the pion, an effect analogous to the photoelectric effect in atoms, while the second term arises from photoexcitation of the resonance state of the nucleon followed by its decay via emission of a pion. The general form of the resonance term is similar to that conjectured by Watson[14] who assumed, however, that the resonance contribution would be proportional to $\sin\alpha_\gamma$ rather than $\sin\delta_\gamma$. The factor $\cos\delta_\gamma$ appearing with the direct term is required for purposes of normalization. This requirement has also been noted by Ross,[4] who introduced α_γ rather than δ_γ in the cosine.

Since the result Eq. (29) forms the basis of all our considerations, we summarize here the approximations used to reduce Eq. (20) to this form. (1) The one-level approximation has been made. (2) The resonance scattering of virtual pions described by the complicated terms of Eq. (23) have been neglected. (3) Terms relating to the inelastic scattering of pions have been dropped.

IV. ESTIMATE OF THE DIRECT TERM

In the absence of nucleon recoil, the direct term vanishes for π^0 production so we consider here the photoproduction of positive pions from protons. According to Eq. (22), the desired quantity is the matrix element of $\mathcal{E}_1$ between the ground state, χ_0, of a nucleon and a state consisting of χ_0 multiplied by an unbound pion function. Equation (17a) shows that this is proportional to $\langle\chi_0,\mathcal{C}_1\chi_0\rangle$ where $\mathcal{C}_1$, is a first order polynominal in a_q and a_q^*. If χ_0 is expressed as a linear combination of states of a definite number of bound pions, $\langle\chi_0,\mathcal{C}_1\chi_0\rangle$ contains terms corresponding to overlap of states differing by, at most, one bound pion. If we now assume that the zero-pion contribution to χ_0 is dominant, as is suggested by the mirror theorem on nucleon magnetic moments,[8] the important contributions to the direct term will come from the zero- and one-pion states contained in χ_0.

[14] K. M. Watson, Phys. Rev. 95, 228 (1954).

The result suggests that the form of the matrix element may be similar to the one obtained in a weak-coupling approximation to the static model, since that approximation yields contributions just from the zero and one pion states. We shall, therefore, use the result of the weak coupling theory for the direct matrix element. Since it is necessary to use a finite source in the static model, we shall have in addition to the coupling constant, another parameter at our disposal, the source size. In fact, the shape of the source function could also be used to parameterize the problem but it seems unlikely that the results will be sensitive to shape for pion energies in the range of interest.

Both parameters may be determined from pion photoproduction data. It is well known that the Kroll-Ruderman theorem[15] or, equivalently, the Siegert theorem[16] may be used to obtain the coupling constant from the cross section for photoproduction of positive pions at threshold. Furthermore, we shall see that the energy dependence of this cross section on the high-energy side of the resonance is sensitive to the range of the source function.

The use of a finite source size requires some care in order that results obtained be gauge-invariant. As pointed out by Capps and Holladay,[17] line currents must be introduced in the source in such a way that charge and current are conserved locally. We have made use of their form of the interaction [reference 17, Eq. (5)] to obtain for the matrix element for production of a positive pion of momentum $\mathbf{k}$ by a photon of momentum $\boldsymbol{\omega}$ and polarization $\mathbf{e}$, the expression

$$D^+(\mathbf{k})=\frac{ief}{(\pi k_0\omega)^{\frac{1}{2}}}\left\{\left[2\frac{(\boldsymbol{\sigma}\cdot[\boldsymbol{\omega}-\mathbf{k}])(\mathbf{e}\cdot\mathbf{k})}{1+(\mathbf{k}-\boldsymbol{\omega})^2} +(\mathbf{e}\cdot\boldsymbol{\sigma})\right]S(\mathbf{k}-\boldsymbol{\omega})+(\mathbf{k}\cdot\boldsymbol{\sigma})(\mathbf{e}\cdot\mathbf{k})G(\mathbf{k}\cdot\boldsymbol{\omega})\right\}, \quad (30)$$

where the units are $c=\hbar=m_\pi=1$ and $k_0=(k^2+1)^{\frac{1}{2}}$. $S(\mathbf{k})$ is the source function normalized so that

$$S(0)=(2\pi)^{-\frac{3}{2}},$$

and having the representation in configuration space:

$$s(\mathbf{r})=(2\pi)^{-\frac{3}{2}}\int d^3k e^{i\mathbf{k}\cdot\mathbf{r}}S(\mathbf{k}). \quad (31)$$

The function $G(\mathbf{k},\boldsymbol{\omega})$ is given by

$$G(\mathbf{k},\boldsymbol{\omega})=(2\pi)^{-\frac{3}{2}}(\mathbf{e}\cdot\mathbf{k})^{-1}\int d^3r e^{-i\mathbf{k}\cdot\mathbf{r}} \times[1-\exp(i\boldsymbol{\omega}\cdot\mathbf{r})]s(\mathbf{r})(\mathbf{e}\cdot\mathbf{r})(\boldsymbol{\omega}\cdot\mathbf{r})^{-1}. \quad (32)$$

[15] N. M. Kroll and M. A. Ruderman, Phys. Rev. 93, 326 (1954).
[16] R. H. Capps, Phys. Rev. 99, 926 (1955).
[17] R. H. Capps and W. G. Holladay, Phys. Rev. 99, 931 (1955).

The terms proportional to $S(\mathbf{k}-\boldsymbol{\omega})$ in Eq. (30) give the usual weak-coupling result[18] while the contributions determined by $G(\mathbf{k},\boldsymbol{\omega})$ arise from the line currents in the source.

We shall make use of the Gaussian source function

$$S(\mathbf{k})=(2\pi)^{-\frac{3}{2}}\exp(-\beta^2k^2). \quad (33)$$

Then, from Eqs. (31) and (32) we find[19]

$$G(\mathbf{k},\boldsymbol{\omega})=-\beta(2^{\frac{1}{2}}\pi\omega)^{-1}\exp\{-\beta^2[k^2-(\mathbf{n}\cdot\mathbf{k})^2]\} \times\{\Phi[\beta(\mathbf{n}\cdot\mathbf{k})]-\Phi[\beta(\mathbf{n}\cdot\mathbf{k}-\mathbf{n}\cdot\boldsymbol{\omega})]\}, \quad (34)$$

where $\mathbf{n}$ is the unit vector in the direction of $\boldsymbol{\omega}$ and

$$\Phi(x)=(2\pi)^{-\frac{1}{2}}\int_0^x \exp(-t^2)dt.$$

Having fixed the shape, we see that the matrix element Eq. (30) contains only the two undetermined parameters: f, the coupling constant, and β, the range of the source function.

The direct term defined by Eq. (22) refers to production of an unbound wave which differs from the plane wave occurring in the matrix element Eq. (30) because of the condition of orthogonality to the bound field. Therefore, even within the limitations of the model, Eq. (30) gives only an estimate of the desired matrix element. According to the weakly coupled static model, the bound pion is in a pure P state. Hence, only the partial wave with $l=1$ has an orthogonality condition imposed on it and is not represented correctly by the plane wave. Since the P-wave production is in any case dominated by the resonance term, this error in estimating the direct term does not seem to be of great importance.

V. COMPLETE MATRIX ELEMENT

The matrix element $T_I(\mathbf{p},\mathbf{e},\boldsymbol{\omega})$ for photoproduction of a pion of definite (asymptotic) momentum $\mathbf{p}$ is to be obtained as an appropriate linear combination of the partial wave matrix elements given by Eq. (29). For this purpose, D^+, as given by Eq. (30), must be analyzed into its partial wave components $D_{0\gamma}$ and then each component must be multiplied by $e^{i\alpha_\gamma}\cos\delta_\gamma$ before recombining. Since the only significant resonance in the low-energy region is the $J=\frac{3}{2}$, $I=\frac{3}{2}$ P-wave resonance, we shall modify only this term. Each S-wave term should also be corrected by a factor $e^{i\alpha_\gamma}$, where α_γ is the observed total phase shift, but these phases are rather small for that energy range in which the S waves play an important role and they will be neglected.

The P-wave contribution may be put in the form

$$E_{33}(p)=D_{033}[e^{i\eta_1}+ie^{i\alpha_{33}}\sin\delta_{33}] -(\pi p^2g_{\lambda1})^{-1}B_{\lambda0}e^{i\alpha_{33}}\sin\delta_{33}, \quad (35)$$

where γ has been replaced by the traditional symbol, 33, for the $l=1$, $J=\frac{3}{2}$, $I=\frac{3}{2}$ state.

The second term has the same dependence on phase shifts as the resonance term in Eq. (29). We have calculated the coefficient[20] D_{033} and compared it with the strength of the resonance term obtained directly from the experimental data (Sec. VII). For the entire energy range, this coefficient is quite small (usually less than 10% of the resonance term) so it has been dropped. By dropping the small phase η_1 in Eq. (35) we reduce all direct contributions to $T_I(\mathbf{p},\mathbf{e},\boldsymbol{\omega})$ to the form appropriate for an undisturbed plane wave.

It is now necessary to evaluate the energy dependence of the coefficient $B_{\lambda0}/g_{\lambda1}$ appearing in Eq. (35). We assume that $g_{\lambda1}$ is essentially constant for energies small compared to the cutoff appearing in the source function, *i.e.*, for $p<\beta^{-1}$ and that will include our entire range of interest. The matrix element $B_{\lambda0}$ is defined by Eq. (26). Since it is a matrix element between bound states (*i.e.*, between the ground state and excited state of the nucleon) it is independent of p. However the electromagnetic interaction $\mathcal{E}_0$ depends on ω, the photon energy. It contains a factor $\omega^{-\frac{1}{2}}$ arising from the expansion of the vector potential into creation and annihilation operators. Furthermore we know that the resonance process is due to either a magnetic dipole or electric quadrupole transition. The angular distribution of photoproduced neutral pions indicates that the resonance term is dominated by the magnetic dipole transition which will therefore be assumed to give the only contribution.[21] In the long wavelength approximation, this is a first-order effect in ω. Hence, aside from higher order retardation effects, the matrix element is proportional to $\omega/\omega^{\frac{1}{2}}=\omega^{\frac{1}{2}}$ and we write

$$B_{\lambda0}/g_{\lambda1}\sim\omega^{\frac{1}{2}}. \quad (36)$$

If the dependence on angles and photon polarization for a magnetic dipole transition is now included, the complete matrix element for photoproduction of positive pions from protons is

$$T^+(\mathbf{p},\mathbf{e},\boldsymbol{\omega})=D^++\pi^{-\frac{3}{2}}e\mathfrak{M}\omega^{\frac{1}{2}}p^{-2}e^{i\alpha_{33}}\sin\delta_{33}\{2(\mathbf{n}\times\mathbf{e}\cdot\boldsymbol{\kappa}) +i\boldsymbol{\sigma}\cdot[(\mathbf{n}\times\mathbf{e})\times\boldsymbol{\kappa}]\}, \quad (37)$$

where D^+ is given by Eq. (30) *et seq.* and $\mathfrak{M}$ is a real[14] matrix element between nucleon states, which is presumed to have only a slight energy dependence due to retardation effects. The unit vector in the direction of $\mathbf{p}$ is $\boldsymbol{\kappa}$ and, as before, $\mathbf{n}=\boldsymbol{\omega}/\omega$.

Since there is no direct production of neutral pions

[18] L. L. Foldy, Phys. Rev. **76**, 372 (1949).

[19] The integration is carried out easily in Cartesian coordinates adapted to the orthogonal vectors $\mathbf{e}$ and $\boldsymbol{\omega}$.

[20] We note here that the line currents do not contribute to the magnetic-multipole matrix element as a consequence of our simple choice of radial flow of the currents. Current flowing radially does not produce a moment about the origin. All line current contributions to the electric moments are just those required to satisfy the Siegert theorem (see reference 16) and by means of the Siegert theorem every electric moment could be obtained without making explicit use of the form of the electromagnetic interaction.

[21] See McDonald, Peterson, and Corson, reference 1.

in the no recoil approximation, the corresponding transition matrix element is

$$T^0(\mathbf{p},\mathbf{e},\boldsymbol{\omega})=2^{\frac{1}{2}}\pi^{-\frac{3}{2}}e\mathfrak{M}\omega^{\frac{1}{2}}p^{-2}e^{i\alpha_{33}}\sin\delta_{33}\{2(\mathbf{n}\times\mathbf{e}\cdot\boldsymbol{\kappa})+i\boldsymbol{\sigma}\cdot[(\mathbf{n}\times\mathbf{e})\times\boldsymbol{\kappa}]\}. \quad (38)$$

Equations (37) and (38) are very similar to those given by Watson.[14] In particular he conjectured that the resonance term would have a form similar to that obtained here. There are two notable differences. First, our resonance term is proportional to $\sin\delta_{33}$ while he assumed that the term would behave as $\sin\alpha_{33}$. Second, Watson gave a frequency dependence independent of ω, while we find it to be proportional to $\omega^{\frac{1}{2}}$. This latter difference is of considerable importance to the fitting of data at high energy.

The resonance term involves just the one strength parameter $\mathfrak{M}$ in addition to the phase shifts. $\mathfrak{M}$ will be assumed constant although retardation effects (or, in other words, the effect of the finite size of the nucleon charge distribution in both the ground and excited states) would be expected to lead to a decreasing trend in $\mathfrak{M}$ with increasing energy. Actually the phase shifts are not easily established since both α_{33} and δ_{33} are needed. The former may be taken directly from the data but the latter will require some interpretation of the data. We shall manage to carry through this interpretation by means of one additional constant parameter relating to contributions from the tails of distant resonances, and thereby describe the photoproduction in terms of a total of four constant parameters: the coupling constant f, the range of the source function β, the matrix element $\mathfrak{M}$ and the fourth parameter to be described in detail below.

VI. COMPARISON OF THE DIRECT TERM WITH EXPERIMENT

In order to take some account of nucleon recoil, we shall interpret $\mathbf{p}$ and $\boldsymbol{\omega}$ as momenta in the center-of-mass system. The differential cross section is then given by

$$\sigma(\theta)=\tfrac{1}{4}(2\pi)^4(1+\omega/M)^{-1}(1+p_0/M)^{-1}pp_0\times\textstyle\sum_{\mathbf{e}}\mathrm{Tr}|T(\mathbf{p},\mathbf{e},\boldsymbol{\omega})|^2 \quad (39)$$

the trace being with respect to nucleon spins. The angle θ is measured between $\boldsymbol{\omega}$ and $\mathbf{p}$. When Eq. (37) is inserted for T, we find the positive-pion photoproduction cross section

$$\sigma^+(\theta)=2(2\pi)^3e^2\frac{pp_0}{(1+\omega/M)(1+p_0/M)}\times\Bigg\{\frac{f^2}{\omega p_0}\bigg[\bigg(1-\frac{p^2\sin^2\theta}{2p_0^2(p_0-p\cos\theta)^2}\bigg)S^2(\mathbf{p}-\boldsymbol{\omega})+\tfrac{1}{2}p^4\sin^2\theta G^2(\mathbf{p},\boldsymbol{\omega})-\frac{p^2\sin^2\theta}{p_0(p_0-p\cos\theta)}G(\mathbf{p},\boldsymbol{\omega})S(\mathbf{p}-\boldsymbol{\omega})\bigg]$$
$$+2\frac{f\mathfrak{M}}{p_0^{\frac{1}{2}}}\frac{\sin\delta_{33}}{\pi p^2}\cos\alpha_{33}\bigg[\frac{1}{2}\frac{p\sin^2\theta}{p_0-p\cos\theta}-\cos\theta\bigg]S(\mathbf{p}-\boldsymbol{\omega})+\mathfrak{M}^2\omega\bigg(\frac{\sin\delta_{33}}{\pi p^2}\bigg)^2(\tfrac{3}{2}\sin^2\theta+1)\Bigg\}, \quad (40)$$

where the functions $S(\mathbf{p}-\boldsymbol{\omega})$ and $G(\mathbf{p},\boldsymbol{\omega})$ are to be obtained from Eqs. (33) and (34). The cross section for neutral-pion photoproduction may be obtained from Eq. (38). The result is

$$\sigma^0(\theta)=4(2\pi)^3e^2\frac{pp_0\omega}{(1+\omega/M)(1+p_0/M)}\mathfrak{M}^2\bigg(\frac{\sin\delta_{33}}{\pi p^2}\bigg)^2\times(\tfrac{3}{2}\sin^2\theta+1). \quad (41)$$

The observed[21] angular distribution of photoproduced neutral pions at various energies seems to agree with the form Eq. (41). That is the basis for our neglect of the electric quadrupole contribution to the resonance term. The comparison of the energy dependence of Eq. (41) is reserved for the next section. We shall assume here that Eq. (41) may be used as a means for obtaining, from the data on π^0 photoproduction, the resonance terms appearing in Eq. (40). For example, we may use the 90° (c.m.) cross section

$$\sigma^0(\pi/2)=5W\omega\mathfrak{M}^2(\sin\delta_{33}/\pi p^2)^2, \quad (42)$$

where W is the weight factor

$$W=2(2\pi)^3e^2pp_0(1+\omega/M)^{-1}(1+p_0/M)^{-1}. \quad (43)$$

Then, since $\mathfrak{M}$ is known to be a real number, Eq. (40) may be rewritten as

$$\sigma^+(\theta)=\frac{f^2W}{\omega p_0}\bigg[\bigg(1-\frac{p^2\sin^2\theta}{2p_0^2(p_0-p\cos\theta)^2}\bigg)S^2+\tfrac{1}{2}p^4\sin^2\theta G^2-\frac{p^2\sin^2\theta}{p_0(p_0-p\cos\theta)}GS\bigg]$$
$$\pm2f\bigg[\frac{W\sigma^0(\pi/2)}{5\omega p_0}\bigg]^{\frac{1}{2}}\cos\alpha_{33}\bigg[\frac{1}{2}\frac{p\sin^2\theta}{p_0-p\cos\theta}-\cos\theta\bigg]S+\tfrac{1}{5}\sigma^0(\pi/2)(\tfrac{3}{2}\sin^2\theta+1). \quad (44)$$

The coupling constant f may be determined from the threshold behavior of the cross section. From Eqs. (40) and (43) we find

$$[\sigma^+(\theta)/p]_{p=0}=2.32\times10^{-28}f^2\ \mathrm{cm^2/sterad}. \quad (45)$$

Experimentally, Beneventano, Bernardini, Carlson-Lee, Stoppini, and Tau[1] find

$$[\sigma^+(\theta)/p]_{p=0}=11.3\times10^{-30}\ \mathrm{cm^2/sterad},$$

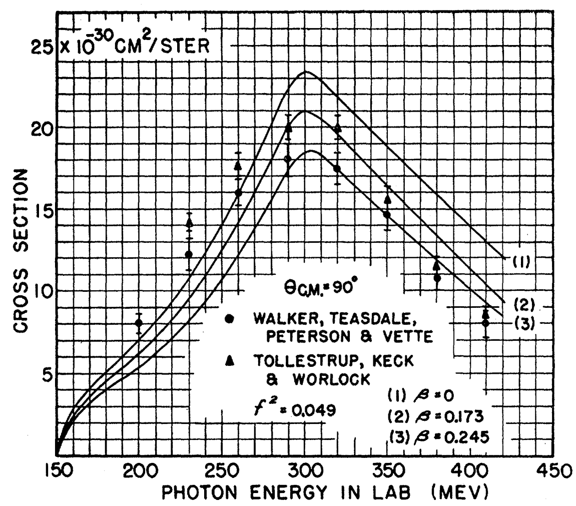
which leads to the value

$$f^2=0.049, \quad (46)$$

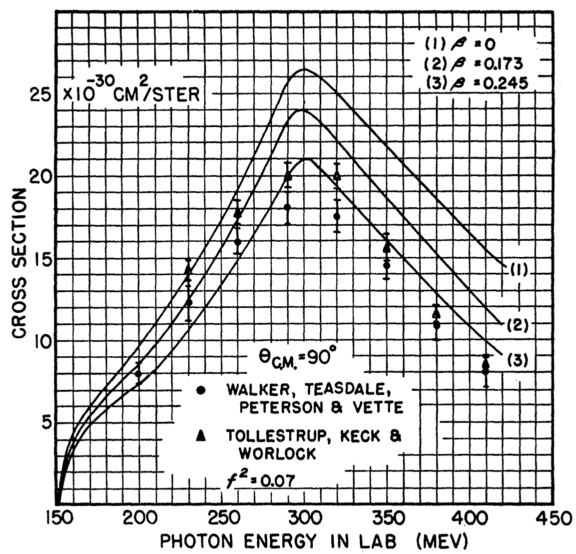
for the coupling constant. This result differs from values frequently quoted[5] partly because no attempt has been made to incorporate additional recoil corrections into our evaluation. We estimate that recoil corrections to the π^+ photoproduction should be obtained[22] by setting $f^2=(1+m_\pi/M)^{-2}f_R^2$, where f_R is the coupling constant corrected for recoil. This gives $f_R^2=0.065$ which is in closer agreement with other values. However the π^- to π^+ ratio obtained in this way[22] is just $(1+m_\pi/M)^2=1.32$ which does not agree with that obtained from the π^- photoproduction on deuterons by Beneventano *et al.*,[1] nor with the value obtained by detailed balance from the Panofsky ratio combined with the $\pi^-+p\rightarrow\pi^0+n$ cross section.[23] These matters are important for the interpretation of f^2 but they do not affect the further analysis of the data since the correction occurs as a constant factor in the direct matrix element.

Our comparison of the energy dependence of the direct term with experimental results is based on Eq. (44). We have substituted the values of $\sigma^0(\pi/2)$ observed at several energies into the equation and have thereby calculated $\sigma^+(\theta)$ at each of these energies. The calculation was first performed at $\theta=\pi/2$ in order to determine the one remaining parameter, β, the range of the source function. Figure 1(a) shows the data at $\theta=\pi/2$ in comparison with results of the calculation for $f^2=0.049$ and various values of β. For no value of β is the agreement particularly good, but the behavior below resonance seems to favor $\beta=0$, i.e., infinite cutoff. Because the agreement is not very good and because of the uncertainties concerning the value of f^2, we have also calculated $\sigma^+(\pi/2)$ for $f^2=0.07$. Figure 1(b) shows the results in this case. A reasonably good fit is obtained over the entire energy region for the rather low cutoff $\beta^{-1}\approx 4$.

The differential cross section at several different angles has been calculated as a function of energy by means of Eq. (44) for the two cases $f^2=0.049$, $\beta=0$ and $f^2=0.07$, $\beta=0.245$ with the results[24] shown in Fig. 2.



(a)



(b)

FIG. 1. (a) Experimental values[1] of π^+ photoproduction cross sections at $\theta_{c.m.}=90°$ compared with the theoretical curves for $f^2=0.049$ and for different values of the parameter β determining the source size. (b) Experimental values[1] of π^+ photoproduction cross sections at $\theta_{c.m.}=90°$ compared with the theoretical curves for $f^2=0.07$ and for different values of the parameter β determining the source size.

The available data are also indicated in these figures. Although large systematic errors in the data are suggested by the discrepancies between groups of investigators, it seems rather clear that the choice $f^2=0.049$ cannot be brought into accord with observation. Except for the lack of agreement with the threshold behavior, the choice $f^2=0.07$, $\beta=0.245$ seems adequate. No firm conclusion can be drawn from these results at the present time in view of the experimental uncertainties.

[22] The argument is as follows: The direct matrix element may be expressed as a sum of multipole terms. The displacement of the pion charge density from the center-of-mass of the system occurring in the multipole moment operators is reduced by a factor $(1+m_\pi/M)^{-1}$ over its value in a nonrecoil theory. Thus the electric dipole moment, for example, is reduced just by this factor. The correction to the other moments is more complicated but each contains the same factor and the remainder of the correction may be obtained by shifting the photon energy to $\omega'=(1+m_\pi/M)^{-1}\omega$. Hence, after summing all multipoles, the direct matrix element is found to contain the factor $(1+m_\pi/M)^{-1}$ and it is to be evaluated at a slightly shifted photon energy. No such correction occurs for π^- production from neutrons since the core carries a charge in that case. We may note that our f_R^2 could not correspond directly to the renormalized coupling constant of Kroll and Ruderman since they find that the recoil correction is divided equally between π^+ and π^- production in first approximation. Applying the correction in the latter fashion leads to a renormalized coupling constant $f^2=0.057$ (compare reference 23).

[23] Cassels, Fidecaro, Wetherall, and Wormald, Proc. Phys. Soc. (London) **70**, 405 (1957).

[24] To obtain the indicated curves, a positive sign has been assigned to the interference term in Eq. (44). The opposite sign leads to results differing markedly from the available data.

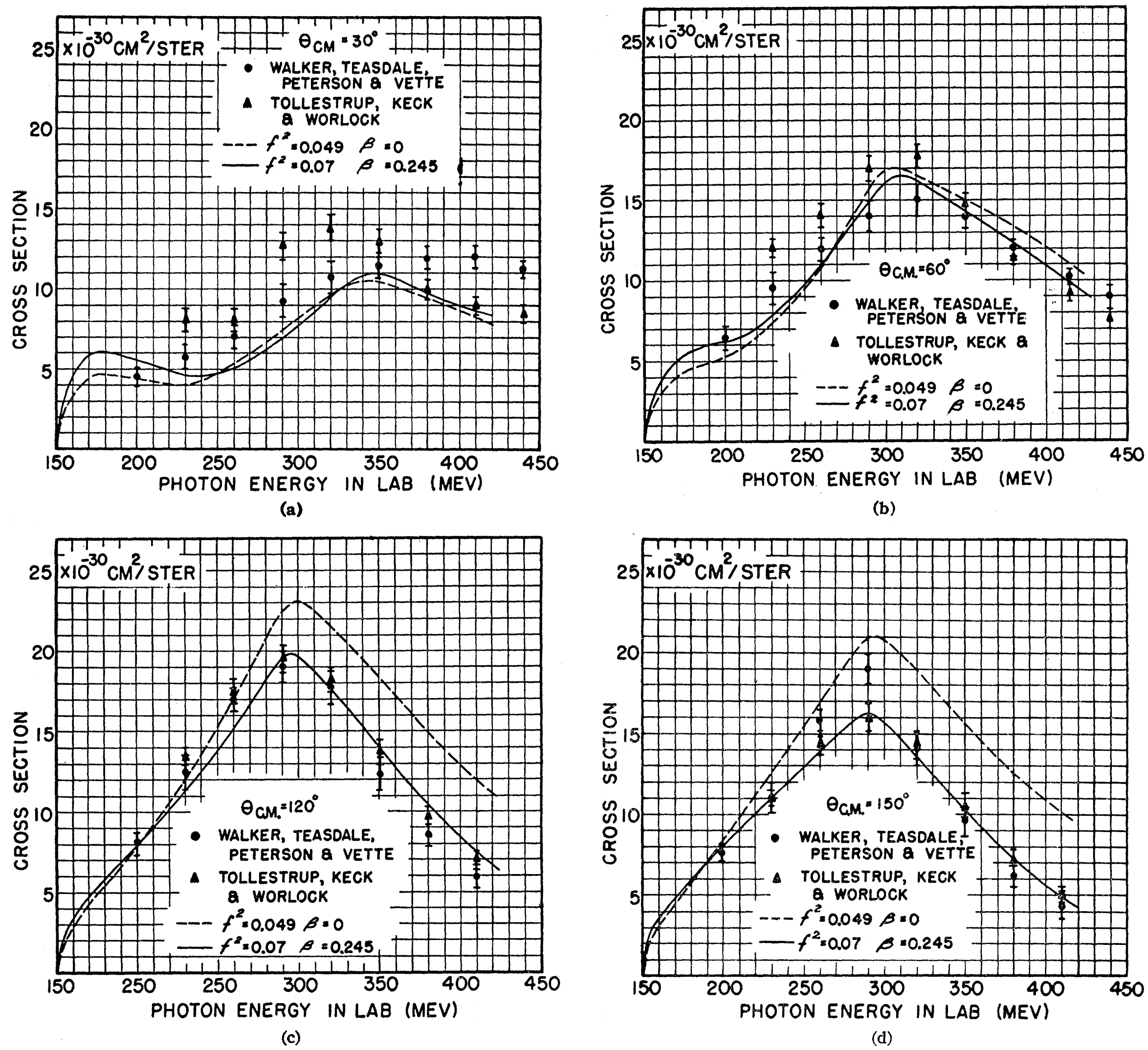


FIG. 2. Experimental values[1] for π^+ photoproduction cross sections at different angles in the center-of-mass system compared with the theoretical curves for indicated values of f^2 and β. The points are taken from the curves fitted to the experimental angular distributions in the center-of-mass system at various energies. The errors shown are the averages of errors in the experimental values at each particular energy.

VII. COMPARISON OF THE RESONANCE TERM WITH EXPERIMENT

The moderately successful treatment of the direct term in the previous section indicates that the relationship between the π^+ and π^0 cross sections may be reproduced by the assumption that an $I=\frac{3}{2}$ magnetic dipole resonance is responsible for all but the direct photoproduction. We have still to establish that the shape of the resonance term is given correctly by the theory. This can be accomplished most easily by comparing Eq. (42) with the measured values of $\sigma^0(\pi/2)$. The comparison will be used to fix the shape of the cross-section curve and to determine the constant $\mathfrak{M}$.

The energy dependence of the cross section can be obtained from Eq. (42) only if the phase shift δ_{33} is known as a function of energy. Since δ_{33} differs from the total phase shift α_{33} by an amount η_1, some determination of η_1, is needed to complete our program. One suggestion[25] is that we set $\eta_1=0$; then $\delta_{33}=\alpha_{33}$, which may be taken directly from the pion-nucleon scattering data. A determination of the shape of $\sigma^0(\pi/2)$ by means of Eq. (42) has been made on this basis, with the results shown in Fig. 3. The experimental points shown in the figure decrease rapidly with energy beyond the peak, and the drop is much faster than the calculated rate.[26]

It seems very likely that the failure is due to our assumption $\eta_1=0$. This phase shift is a measure of the

[25] R. G. Sachs, Phys. Rev. **102**, 867 (1956).

[26] Others (see reference 5) seem to have found it possible to reproduce the data on this basis by using Watson's expression (reference 14) for the resonance term. However, this success probably stems directly from Watson's extra factor $\omega^{-\frac{1}{2}}$ mentioned at the end of Sec. V.

high-momentum components of φ_q and it has been shown[25] that η_1 can be small over the energy range under consideration only if there are appreciable contributions to φ_q from momenta much larger than M, a result seemingly inconsistent with our determination of the cutoff in the source function.

In order to treat the case $\eta_1 \neq 0$, it is necessary to consider the pion-nucleon scattering in some detail. The total phase shift is $\alpha_{33}=\eta_1+\delta_{33}$. In the one-level approximation, δ_{33} is given by

$$\tan\delta_{33}=\frac{\pi p^3}{p_0}\left[\frac{g^2}{E_{33}-p_0}+Q\right], \tag{47}$$

where E_{33}, g, and Q are real constants. Q is an approximate expression for the influence of distant levels.

To fix η_1 in the simplest possible way, we make use of the fact that it depends only on the form of the functions $\varphi_q(\mathbf{k})$ and not at all on the degree to which these bound functions are occupied by pions. Therefore, if we consider a finite source, weak coupling theory, the energy dependence of η_1 could in principle be determined, and it would be independent of the magnitude of the coupling constant f. In the limit $f\rightarrow 0$, we expect that all resonances become remote so that Eq. (47) would be replaced by

$$\lim_{f\rightarrow 0}[\tan\delta_{33}]=\frac{\pi p^3}{p_0}\lim_{f\rightarrow 0}Q.$$

On the other hand, $\alpha_{33}\rightarrow 0$ in this limit. Therefore

$$\tan\eta_1=-\frac{\pi p^3}{p_0}\lim_{f\rightarrow 0}Q.$$

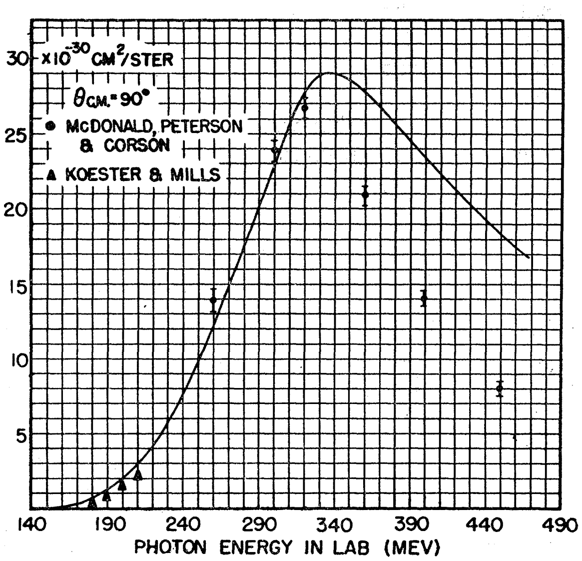


FIG. 3. Comparison of experimental data[1] with Eq. (42), setting $\delta_{33}=\alpha_{33}$ and taking α_{33} directly from scattering data.

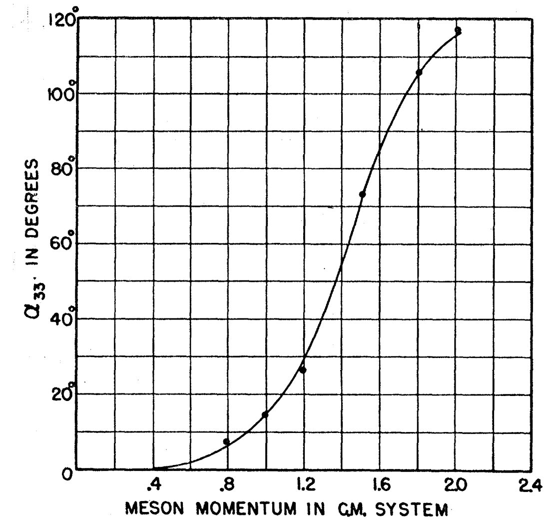


FIG. 4. Comparison of the theoretical curve for α_{33} obtained from Eqs. (47), (48), and (49), as a function of pion momentum in center-of-mass system with experimental data. The points chosen for comparison are taken from H. A. Bethe and F. de Hoffmann, *Mesons and Fields* (Row, Peterson, and Company, 1955), p. 125, Fig. 343.

For finite f, Q expresses the influence of the multitude of remote resonances, all but the one occurring within the low-energy region. Hence we assume that Q does not differ appreciably from its value in the limit $f=0$ and write

$$\tan\eta_1=-\pi p^3 p_0^{-1}Q, \tag{48}$$

where Q is now the constant appearing in Eq. (47).

We have used these equations to fit the values of α_{33} obtained from the data and find that a reasonable fit can be obtained for any value of Q which is not too large. In particular $Q=0$ (which takes us back to $\eta_1=0$) clearly gives the Chew-Low curve.[27] However, as we have seen, this choice of Q is not consistent with the energy dependence of the pion photoproduction cross section. Hence we have used the pion photoproduction cross section combined with the scattering data to determine $\mathfrak{M}$ and Q, the other constants, E_{33} and g, being determined simultaneously. Figures 4 and 5 show the fit to the scattering and π^0 photoproduction, respectively, obtained with the constants

$$E_{33}=1.858, \quad Q=0.02, \qquad g^2=0.0504, \quad \mathfrak{M}=0.037. \tag{49}$$

The results appear to be quite satisfactory.

VIII. CONCLUSION

The basis for our analysis of the pion photoproduction has been a quite general form of the static model of the nucleon. By making use of simple physical arguments we have reduced the pion photoproduction amplitude to the sum of two easily interpreted terms. One of them

[27] G. F. Chew and F. E. Low, Phys. Rev. 101, 1570 (1956).

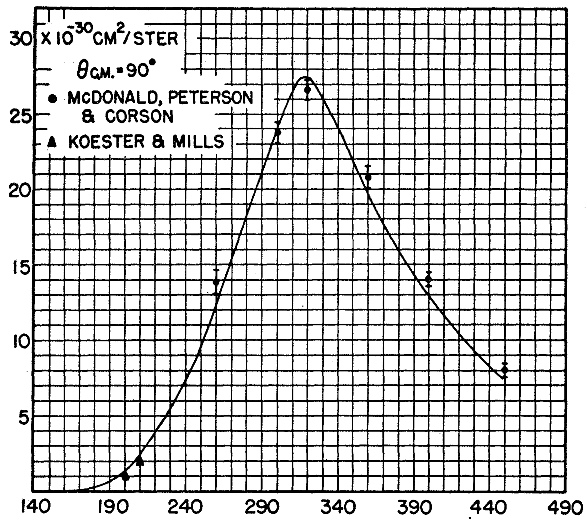


FIG. 5. Comparison of the experimental values[1] of $\sigma^0(\pi/2)$ with Eq. (42) using δ_{33} as obtained from Eqs. (47), (48), and (49).

describes the direct photoejection of pions, and we have used the two-parameter weak-coupling theory to describe that term. The other is the single-resonance term suggested by the existence of a P-wave resonance in pion-nucleon scattering.

Discrepancies between experimental results make it difficult to draw conclusions about the direct term but indications are that the coupling constant $f^2=0.049$ suggested by the threshold production of positive pions is not consistent with the over-all energy dependence. However, with the somewhat larger constant $f^2=0.07$, a cutoff of $\beta^{-1}\approx 4$ leads to a reasonable energy dependence. This cutoff is somewhat smaller than the nucleon mass but it is not an entirely unreasonable value in view of possible contributions of K mesons to nucleon structure.

The relatively small value of the cutoff causes the contributions of the line currents to the direct matrix element to assume some importance. The line current part of the direct matrix element ranges around ten percent of the whole. These terms are, of course, essential to the gauge invariance of the static model.

The fact that the direct term does not seem to give consistent results at both the threshold and higher energy is somewhat disturbing. This may be an indication that the assumed form of the direct term is not correct. In fact, our use of the weak-coupling approximation to describe this term was founded on the notion of a small one-pion probability in the ground state of the nucleon. Although the coupling constant, $f^2=0.049$, suggested by the threshold behavior, is small, the one-pion probability would be very large since a small value of β (implying a large cutoff of the order of the nucleon mass or larger) is required to obtain a reasonable fit to the data below resonance. Thus it is possible that corrections to the weak-coupling result become very important in the neighborhood of the resonance and at larger energies. It is to be noted that for $f^2=0.07$, $\beta^2=0.06$ the one-pion probability is about 20%, so the weak-coupling approach may not have such unfortunate consequences in that case.

An alternative possibility is that the relationship between π^+ and π^0 production is not given correctly by taking into account a single resonance. More reliable data would help to resolve these difficulties.

Comparison of the resonance term with both the π^0 and π^+ production has led us to the conclusion that $Q\neq 0$; a simple one-term resonance formula for $\tan\alpha_{33}$ is not appropriate.[28] This would seem to imply that the success of the Chew-Low plot[27] of $p^3 \cot\alpha_{33}/p_0$ is something of an accident. However, it must be kept in mind that we have neglected retardation corrections to $\mathfrak{M}$, the matrix element of the magnetic moment. These corrections would cause $\mathfrak{M}$ to be energy-dependent in such a way that it should decrease with increasing energy. However, the magnitude of the effect should be small until ω is comparable to or larger than the dominant momentum components of the pion functions in both the ground and excited nucleon states. In order to account for the marked decrease in the resonance cross section beyond the resonance, it would seem necessary to have a cutoff considerably lower than the value $\beta^{-1}=4$ suggested by the direct photoproduction. We think it more reasonable to take $\mathfrak{M}$ to be constant and to assume that the dropoff is due to the influence of distant resonances indicated by the (rather small) value $Q=0.02$.

A number of other approximations have been made in reducing the theoretical amplitude to so simple a form. As far as we can estimate, errors due to these approximations should not amount to more than ten percent in the photoproduction amplitude. However it may turn out that this is a poor estimate and that the neglected terms can account for some of the difficulties.

It should be noted that we have made no effort to analyze the differential cross section for positive-pion photoproduction in terms of S and P waves. Although it is very convenient and has become customary to present the data in these terms, i.e., by writing

$$\sigma(\theta)=A_0+A_1\cos\theta+A_2\cos^2\theta,$$

we find that the direct term contributes strongly to terms of higher order in $\cos\theta$ except at the lowest energies. This clearly means that the D and higher waves make an important contribution to the cross

[28] The direct comparison was made only with the π^0 cross section. However, if the π^0 measurements were in error and the resonance term dropped off slowly with energy, as though $Q=0$, then the curves for π^+ production calculated from Eq. (44) and appearing in Figs. 1(a) and 1(b) would drop off more slowly above the resonance. Hence the discrepancies would be increased and their resolution would require an even larger value of f^2 and lower value of the cutoff.

section, as might be expected when the photon wavelength is smaller than the pion Compton wavelength.[29]

One of our original reasons for undertaking this analysis was to seek evidence for or against the existence of an S-wave pion-nucleon resonance with $I=\frac{1}{2}$, a resonance suggested by our original analysis[2] of the scattering. Despite earlier statements[30] we have found no supporting evidence for the resonance. That may only mean that excitation of the resonance state occurs with a small amplitude. On the other hand, nonlinear coupling of the S waves could account for the scattering[31] without recourse to a resonance so there seems to be little reason to invoke the notion of a resonance at the present time.

[29] This point has been made by Watson *et al.* (reference 7) and, in more detail, by M. J. Moravcsik, Phys. Rev. **104**, 1451 (1956).

[30] J. Enoch and R. G. Sachs, Bull. Am. Phys. Soc. Ser. II, **1**, 168 (1956). The analysis reported here had been based on the assumption that only S- and P-wave pions were important. It turns out that the D and higher waves contribute a large isotropic term to the cross section which eliminates the need for the S-wave resonance.

[31] Drell, Friedmann, and Zachariasen, Phys. Rev. **104**, 236 (1956).

PHYSICAL REVIEW VOLUME 108, NUMBER 2 OCTOBER 15, 1957

Mesonic Atoms: Radiative Yields of the π-Meson L Series*

M. B. Stearns, M. Stearns,† and L. Leipuner‡
Carnegie Institute of Technology, Pittsburgh, Pennsylvania
(Received July 3, 1957)

The total radiative yields of the L series from π-mesonic atoms have been measured for most of the elements ${}_5$B through ${}_{33}$As. The yield curve has a broad maximum of ~70% in the region $12\lesssim Z\lesssim 16$ and decreases at both higher and lower Z values. This decrease is presumably due to competition from direct nuclear absorption at the higher Z's and to nonradiative processes at the lower Z's. The yields are fairly constant for $25\lesssim Z\lesssim 30$, suggesting a possible magic number effect at $Z=28$. The rapid decrease in yield with decreasing Z cannot be attributed to competition between the simple Auger effect and radiative transitions. The simple Auger transition probabilities are about 40 times smaller than the observed values. More complex nonradiative processes are probably involved, such as those proposed by Day and Morrison.

THE radiative yields of the $\pi-L$ series have been studied by the Carnegie Tech[1] and Rochester[2] groups. In this paper we report on more recent measurements of these yields. The experimental setup and techniques used are similar to those described in the preceding articles[3,4] on mesonic x-rays. The corrections to the raw data are similar to those discussed in II,[4] and they were made in an analogous manner.

The $\pi-L$ mesonic x-ray yields were measured for most of the elements between ${}_5$B and ${}_{33}$As inclusive. These elements and their $\pi-L_\alpha$ transition energies are listed in Table I, columns 1 and 2. A $\frac{1}{16}$-in. thick NaI crystal was used as the x-ray detector for the elements B through F. A $\frac{1}{2}$-in. NaI crystal was used for elements F through Si and a 2-in. crystal for Al and all higher Z elements. In addition, a 3-in. diameter×3-in. thick NaI crystal, stopped down to $1\frac{1}{4}$-in. diameter by a lead collimator, was used for measuring the yields of silicon and higher Z elements. Each element was run at least twice. The meson targets, up to and including titanium, were identical to those used in the μ-meson yield work.[4] The target material was packed uniformly inside of a thin hollow Lucite cylinder, 1 in. thick and $2\frac{3}{4}$ in. in

TABLE I. Energies and yields of the $\pi-L$ series.

Element	L_α energy	Absolute L yield per stopped meson	Ratio of higher transitions to total yield
${}_5$B	12.7	<0.06	
${}_6$C	18.4	0.11±0.015	
${}_7$N	25.1	0.18±0.02	0.28
${}_8$O	32.8	0.25±0.02	0.27
${}_9$F	41.6	0.46±0.05	0.21
${}_{11}$Na	62.3	0.66±0.04	0.20
${}_{12}$Mg	74.2	0.67±0.04	0.16
${}_{13}$Al	87.2	0.70±0.05	0.17
${}_{14}$Si	101.2	0.69±0.05	0.17
${}_{15}$P	116.3	0.65±0.04	0.22
${}_{16}$S	132.4	0.68±0.04	0.27
${}_{17}$Cl	149.6	0.61±0.04	0.26
${}_{19}$K	187.1	0.62±0.04	0.21
${}_{20}$Ca	207.5	0.60±0.04	0.22
${}_{22}$Ti	251	0.52±0.05	0.25
${}_{24}$Cr	299	0.38±0.04	0.31
${}_{25}$Mn	325	0.34±0.06	
${}_{26}$Fe	352	0.39±0.04	0.27
${}_{27}$Co	376	0.31±0.06	0.20
${}_{28}$Ni	405	0.36±0.04	0.23
${}_{29}$Cu	435	0.40±0.06	
${}_{30}$Zn	465	0.39±0.05	0.18
${}_{33}$As	~560	<0.19	

* Supported by the U. S. Atomic Energy Commission.

† Present address: General Atomic, San Diego, California.
‡ Now at Brookhaven National Laboratory, Upton, New York.

[1] Stearns, DeBenedetti, Stearns, and Leipuner, Phys. Rev. **93**, 1123 (1954).

[2] Camac, Halbert, and Platt, Phys. Rev. **99**, 905 (1955).

[3] M. Stearns and M. B. Stearns, Phys. Rev. **103**, 1534 (1956), referred to hereafter as I.

[4] M. B. Stearns and M. Stearns, Phys. Rev. **105**, 1573 (1957), referred to hereafter as II.