# Binding Energies of Heavy Nuclei

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(Received June 17, 1957)

Nuclear binding energies of heavy nuclei are calculated from a general two-body interaction by using $j$-$j$ coupling shell-model wave functions for the nucleons. The formula obtained can be made to fit experimental data with a high accuracy.

RECENT calculations by Talmi and co-workers[1,2] have revealed a very good agreement between shell-model predictions and experimental data concerning binding energies of light nuclei. As neither the interaction between nucleons nor their radial wave functions are known, a procedure was adopted which did not involve the detailed knowledge of these quantities. Only the following assumptions were made:

I. The wave function describing the nucleus is a shell-model wave function, the single-particle wave functions entering the complete wave function being independent of the number of nucleons in the shell.

II. The residual interaction between nucleons is a two-body charge-independent interaction (this may include central forces, any mutual spin-orbit interaction, tensor forces, etc.).

Thus, the states were characterized by the configuration, the total angular momentum (sometimes briefly referred to as total spin) $J$, the total isotopic spin $T$, and by additional quantum numbers of the generalized seniority.[3,4]

The application of the shell model requires the calcution of the expectation value of the two-body interaction among nucleons in one shell. This expectation value can be expressed as a linear combination of the energies in a two-nucleon configuration, using a method due to Racah.[5] Thus, consider any two-body operator $t_{ij}$. In a configuration of $n$ equivalent nucleons in the state $j$, coupled to total angular momentum $J$, the expectation value will be:

$$(j^nJ|t_{ij}|j^nJ)=\tfrac{1}{2}n(n-1)(j^nJ|t_{12}|j^nJ), \quad (1)$$

where the equality holds because of the equivalence of the nucleons. This expectation value can be further simplified by noting that $t_{12}$ operates on particles 1 and 2 only and that therefore

$$(j^nJ|t_{12}|j^nJ)=\sum_{J_{12}} a(J_{12},J)(j^2J_{12}|t_{12}|j^2J_{12}), \quad (2)$$

where the positive coefficients $a(J_{12},J)$ depend only on the nature of the state $|j^nJ)$ and not on the operators $t_{ij}$, and are simply related to the fractional parentage coefficients.

Since the number $N$ of states in a configuration $j^2$ is finite, it is possible to find $N$ independent operators $t_{ij}^{(1)}, t_{ij}^{(2)}, \cdots t_{ij}^{(N)}$ such that for a given $j$ and every $J_{12}$ one will have

$$(j^2J_{12}|V_{12}|j^2J_{12})=\sum_{k=1}^{N}\alpha_k(j^2J_{12}|t_{12}^{(k)}|j^2J_{12}), \quad (3)$$

where $V_{12}$ is the two-body interaction and the $\alpha$'s are constants which depend on $V$, $j$, and the special choice of $t_{ij}^{(k)}$. Using (1), (2), and (3), one now finds

$$(j^nJ|\sum_{i<j}V_{ij}|j^nJ)=\frac{n(n-1)}{2}\sum_{J_{12}}a(J_{12}J)(j^2J_{12}|V_{12}|j^2J_{12})=\sum_k\alpha_k(j^nJ|\sum_{i<j}t_{ij}^{(k)}|j^nJ). \quad (4)$$

Thus if the arbitrary operators $t_{ij}^{(k)}$ could be chosen simple enough so that their expectation values could be easily evaluated, formula (4) will yield the expectation values for the interaction $V_{ij}$ for any number of particles in any state expressed in terms of at most $N$ parameters $\alpha_k$. These parameters then fully represent the interaction $V_{ij}$ for the specific shell. The number of such known simple operators is not too big, though. One such operator is, for instance, $t_{ik}=(\mathbf{j}_i\cdot\mathbf{j}_k)$ which satisfies

$$(j^nJ|\sum_{i<j}(\mathbf{j}_i\cdot\mathbf{j}_k)|j^nJ)=\tfrac{1}{2}[J(J+1)-nj(j+1)].$$

Another operator is the "Majorana-operator," $M_{12}$, which has the eigenvalue 1 for space symmetrical states and the eigenvalue $-1$ for space antisymmetrical states. The eigenvalue $M$, of the Majorana operator $\sum_{i<k}M_{ik}$ can be calculated, and has the expression[6]

$$M=\tfrac{1}{8}n(16-n)-\tfrac{1}{2}[P(P+4)+P'(P'+2)+P''^2]. \quad (4a)$$

To illustrate the method we take here an example in $LS$ coupling. If two nucleons belonging to a $p$ shell

[1] I. Talmi and R. Thieberger, Phys. Rev. **103**, 718 (1956).
[2] S. Goldstein and I. Talmi, Phys. Rev. **105**, 995 (1957).
[3] G. Racah, "Group Theory and Spectroscopy," mimoegraphed lecture notes, Princeton, 1951 (unpublished).
[4] B. H. Flowers, Proc. Roy. Soc. (London) **A212**, 248 (1952).
[5] G. Racah, *Farkas Memorial Volume* (Research Council of Israel, Jerusalem, 1952).
[6] L. Rosenfeld, *Nuclear Forces* (North-Holland Publishing Company, Amsterdam, 1948), (10.36-6).

interact by spin-independent forces, their interaction energy can assume only three values, according to the three possible values of their resultant momentum. Therefore, by a convenient choice of the three constants $a$, $b$, and $c$, it is possible to express their interaction energy by the formula

$$\langle V\rangle=\langle a+bM_{12}+c(\mathbf{l}_1\cdot\mathbf{l}_2)\rangle.$$

Here $M_{12}$ is the Majorana-operator mentioned above and $(\mathbf{l}_1\cdot\mathbf{l}_2)$ is the scalar product of the angular momenta of the two nucleons. In this case one obtains for the interaction energy of $n$ $p$-shell nucleons

$$\langle V\rangle=\langle\sum_{i<j}|a+bMij+c(\mathbf{l}_i\cdot\mathbf{l}_j)|\rangle$$
$$=\tfrac{1}{2}n(n-1)a+bM+\tfrac{1}{2}c[L(L+1)-nl(l+1)],$$

where $M$ is the expression (4a).

As long as the number of levels in the configuration $j^2$ does not exceed the number of known simple operators, this method can be used directly. However, the method described above cannot be applied directly to those cases where the number of levels exceeds the number of operators available. The method has then to be modified and we have to consider the average energies of groups of states which belong to the same eigenvalues of the known operators. It can be shown[5,7] by group-theoretical methods that, upon taking these averages, the same procedure as before can be used.

Fortunately, inasmuch as ground states are characterized by seniority 1, they are the only members of their group (i.e., the only state which belongs to the same eigenvalues of the known operators) and therefore the energies of ground states can be obtained directly with Racah's method. Thus for the configuration $j^n$ they were given by[1,5]:

$$E(j^nJ)=nA'+\tfrac{1}{2}n(n-1)a'+[T(T+1)-\tfrac{3}{4}n]b'$$
$$+[g(W)-2n(j+1)]c'. \quad (5)$$

Here $A$ is the single nucleon energy (its kinetic energy and its interaction with the closed shells), while the other terms express the mutual interaction. $T$ is the isotopic spin and the quantity $g(W)$ is the eigenvalue of Casimir's operator[3,5]. This last term essentially represents the pairing energy in even-even and odd-even nuclei; its meaning for odd-odd nuclei is less simple.

A slightly different approach is needed in order to get an equation similar to (5) for the heavy nuclei. In this case the protons and neutrons fill in different shells and the isotopic spin formalism loses its value. Therefore it is better to consider proton shells and neutron shells separately. If one of the shells is closed, the handling of the second shell is simple. It is really a special case of Eq. (5), when one has nucleons of one kind only ($T=\frac{1}{2}n$). If neither the proton nor the neutron shells are closed, the situation is not so clear, since there is the question of the coupling scheme.

[7] G. Racah, Phys. Rev. **76**, 1352 (1949).

It is customary to assume in these cases that the neutrons and the protons should be coupled to their lowest state each and the two groups then coupled to each other. This treatment may be justified by noting that the excitation of either protons alone or neutrons alone (for instance in even-even nuclei) requires considerably more energy than that required for a change in the relative orientation of the protons as a whole with respect to the neutrons as a whole (for instance in odd-odd nuclei).

If there are now $p$ protons and $n$ neutrons in the unfilled shells of protons ($j_p$) and neutrons ($j_n$) respectively, we can decompose the energy of this system in the following way

$$E=E(j_p{}^pJ_p)+E(j_n{}^nJ_n)+V_{\rm int}(j_p{}^pJ_p,j_n{}^nJ_n,J). \quad (6)$$

Here $E(j_p{}^pJ_p)$ is the zeroth-order energy of $p$ protons in the shell $j_p$ plus the interaction energy of these protons with each other when coupled to a total angular momentum $J_p$; $E(j_n{}^nJ_n)$ has a similar meaning; and $V_{\rm int}(j_p{}^pJ_p,j_n{}^nJ_n,J)$ is the interaction energy between the protons coupled to $J_p$ and the neutrons coupled to $J_n$ in a state of total angular momentum $J$.

Suppose now we consider a set of isotopes of the same element and let us first choose an element with even $Z$, so that $p$ is even and fixed. By shifting the zero point of the energy, we can get rid of the first term in (6). For the second term, we can use the same method which was used to derive (5), except that the isotopic spin in this case does not add any new information since all nucleons in the state $j_n$ are of the same type. We then obtain

$$E(j_n{}^nJ_n)=nA'+\tfrac{1}{2}n(n-1)a+[g(W)-2n(j+1)]c', \quad (7)$$

where $a=a'+\frac{1}{2}b'$. The last term in (6) can also be easily evaluated. If $p$ is even, then, for the lowest state, $J_p=0$ and hence[8,9]

$$V_{\rm int}(j_p{}^p;\,j_n{}^nJ_n;\,J)=pnV_0, \quad (8)$$

where $V_0$ is independent of $J(=J_n)$, $p$ or $n$.

Summing up these elements, we obtain for the binding energies of a set of isotopes of even $Z$ as a function of the number of neutrons in the unfilled shell an expression of the form

$$E=nA''+\tfrac{1}{2}n(n-1)a+[g(W)-2n(j+1)]c, \quad (9)$$

where $A''=A'+pV_0$. This expression can be still simplified if we note the simple structure of $g(W)$ for states of lowest seniority (zero or one for even or odd $n$, respectively). Thus

$$g(W)=\begin{cases}0 & \text{for } n \text{ even (seniority equals zero)}\\ 2(j+1) & \text{for } n \text{ odd (seniority equals one).}\end{cases}$$

[8] A. de-Shalit, Phys. Rev. **105**, 1528 (1957).
[9] N. Zeldes, Nuclear Phys. **2**, 1 (1956).

Introducing the notation

$$y(n)=\begin{cases}0 \text{ for } n \text{ even}\\ 1 \text{ for } n \text{ odd},\end{cases}$$

we finally find for the energies of the isotopes of an even-$Z$ element as a function of $n$ the expression

$$E=nA+\tfrac{1}{2}n(n-1)a+y(n)D, \qquad (10)$$

with the following relation between the coefficients [compare with Eq. (5)]:

$$A=A'+pV_0-2(j+1)c', \quad a=a'+\tfrac{1}{2}b', \quad D=2(j+1)c'.$$

If we consider the isotopes of an odd-$Z$ element we can no longer replace $V_{\text{int}}$ by the simple expression (8), this expression being correct only if $n$ happens to be even. However, if $n$ is odd one knows[8] that the average of $V_{\text{int}}(j_p{}^pJ_p,j_n{}^nJ_n,J)$ over all possible $J$'s (with fixed $J_p$ and $J_n$) is again equal to $pnV_0$; since the ground state of such an odd-odd nucleus necessarily lies below this average, we can write generally

$$V_{\text{int}}(j_p{}^pJ_p;\, j_n{}^nJ_n,J)=npV_0+y(n)y(p)\epsilon,$$

where $\epsilon$ has the same sign as $V_0$, and although it may depend on $n$ and $p$, Way's[10] rule shows that such a dependence is very weak. Thus the expression (10) holds true also for isotopes of an odd-$Z$ element except that in such cases $D$ should be interpreted as

$$D(\text{for odd } Z)=2(j+1)c'+\epsilon.$$

In some cases two subshells $j_1$ and $j_2$ are being filled simultaneously: $j_1$ fills in with pairs and the odd nucleon fills in $j_2$. Such an effect, mentioned by Mayer and Jensen,[11] is explained by big differences in pairing energy for $j_1$ and $j_2$. For example, in Sn we have for the even isotopes pairs of neutrons in $h_{11/2}$ whereas for the odd isotopes the odd neutron is in the $d_{\frac{3}{2}}$ state.[11] In such a case we would have for the energies of an even number of neutrons the expression

$$E(\text{even})=nA+\tfrac{1}{2}n(n-1)a$$

as before, and for the energies of an odd number $(n+1)$ of neutrons

$$E(\text{odd})=nA+\tfrac{1}{2}n(n-1)a+B+nc,$$

where $B$ is the interaction of the odd nucleon with the core plus its kinetic energy, and $c$ is its interaction with the even nucleons (for instance in the case of Sn, the $h_{11/2}$ nucleons). These formulas can still be written in the form of the formula (10), provided $D$ is interpreted as $B-A$, and $c$ is equal to $a$.

[10] K. Way, in *Proceedings of the Conference held at the Max Planck Institute Mainz*, edited by H. Hintenberger (Pergamon Press, London, 1956).

[11] M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley and Sons, Inc., New York, 1955), p. 69.

It is experimentally known that for a given number of odd neutrons and for states $j_1$ and $j_2$ of the type discussed above, the energies corresponding to the configuration $j_1{}^{2m+1}$ and $j_1{}^{2m}j_2$ (for example in Sn the configuration $h_{11/2}{}^{2m+1}$ and $h_{11/2}{}^{2m}d_{\frac{3}{2}}$) differ by a very small amount compared to the binding energies we are discussing. This would mean that the value of $D$, according to its usual definition, would be close to $B-A$ and the value of $c$ close to the value of $a$ for

$$(n+1)A+\tfrac{1}{2}n(n+1)a+D\approx nA+\tfrac{1}{2}n(n-1)a+B+nc.$$

This implies that

$$A+D\approx B, \quad a\approx c.$$

The results of the calculations show that the value of $D$ obtained in such cases does not differ from the $D$'s obtained in ordinary cases, thus bringing out the consistency between our assumptions and our results.

Instead of considering a set of isotopes, one could of course consider a set of isotones (fixed $n$ and varying $Z$). Similar considerations will yield in these cases also an expression of the type (10). This is true even if we add the Coulomb correction, because this interaction can be represented, too, by three parameters[12]:

$$\beta Z+\tfrac{1}{2}Z(Z-1)a+[\tfrac{1}{2}Z]b.$$

Here $\beta$ represents the Coulomb interaction with the closed shells, $a$ is the Coulomb interaction between $Z$ equivalent protons, and $[\frac{1}{2}Z]=\frac{1}{2}Z$ or $\frac{1}{2}(Z-1)$, whichever is integral. The last term represents the pairing energy resulting from the fact that two protons have a larger probability of being found close together if their spins are oppositely directed. The Coulomb forces between protons oppose this tendency (but, of course, are too weak to prevent pairing). We can combine the Coulomb equation with Eq. (10) and obtain again an expression of the same form with a new meaning for the coefficients.

If we specify the interaction, the coefficients $A$, $a$, and $D$ can be calculated in terms of radial integrals of the potentials[5] (the Slater integrals). However, in order not to introduce special assumptions about the form of the potentials involved we keep these coefficients as free parameters, and check the consistency of experimental data with a three-term expression of the type (10). This may perhaps give better agreement than any conventional first order perturbation calculation, as contributions from higher order may enter. We shall return to this point later.

To check the agreement of our assumptions with the data, we take all available binding energies of ground states where the number of one type of nucleons is held fixed and the other allowed to change (a set of isotopes or isotones). We then fit the data to the linear combination (10), determining the coefficients by a least-squares fit.

[12] B. C. Carlson and I. Talmi, Phys. Rev. **96**, 436 (1954).

TABLE I. Example of experimental and calculated binding energies (B.E.) (in Mev).

| Nucleus | B.E. Exp. | B.E. Calc. | Nucleus | B.E. Exp. | B.E. Calc. |
|---|---|---|---|---|---|
| $_{53}$I relative to I$^{122}$ | | | Pb$^{211}$ | 12.89 | 12.90 |
| I$^{124}$ | 17.1 | 17.0 | Pb$^{212}$ | 18.08 | 18.09 |
| I$^{125}$ | 26.8 | 26.5 | Pb$^{214}$ | 26.98 | 26.98 |
| I$^{126}$ | 33.6 | 33.6 | | | |
| I$^{127}$ | 43.0 | 42.9 | $_{50}$Sn relative to Sn$^{114}$ | | |
| I$^{128}$ | 49.6 | 49.7 | Sn$^{115}$ | 7.9 | 7.5 |
| I$^{129}$ | 58.8 | 58.7 | Sn$^{116}$ | 17.2 | 17.2 |
| I$^{130}$ | 65.3 | 65.3 | Sn$^{117}$ | 24.4 | 24.4 |
| I$^{131}$ | 73.8 | 74.1 | Sn$^{118}$ | 33.6 | 33.7 |
| I$^{132}$ | 80.1 | 80.4 | Sn$^{119}$ | 40.4 | 40.5 |
| I$^{133}$ | 88.7 | 88.9 | Sn$^{120}$ | 49.7 | 49.5 |
| I$^{134}$ | 95.5 | 95.0 | Sn$^{121}$ | 55.8 | 55.9 |
| I$^{136}$ | 107.0 | 109.2 | Sn$^{122}$ | 64.5 | 64.5 |
| | | | Sn$^{123}$ | 70.5 | 70.6 |
| $_{82}$Pb relative to Pb$^{208}$ | | | Sn$^{124}$ | 79.0 | 78.9 |
| Pb$^{209}$ | 3.87 | 3.86 | Sn$^{125}$ | 84.6 | 84.6 |
| Pb$^{210}$ | 9.11 | 9.10 | | | |

The experimental values of the binding energies were taken from several review articles.[13–15] In order to obtain the energy associated with the nucleons in the unfilled shell, we subtracted from the binding energy of every nucleus the binding energy of the nucleus in which the unfilled shell is empty. The comparison with the experimental data was done separately for neutron configurations and proton configurations. Each of these groups was then divided according to the shell being filled, and for each of these subgroups a different set of parameters was determined by the least-squares fit. We disregarded cases in which the experimental material was scarce or where the experimental errors were large.

The agreement is in many cases excellent; the root-mean-square deviation is always less than 1% of the width of the energy range considered. The rms deviation is defined in the usual way as $[\sum_{i=1}^{n}\Delta_i^2/(N-k)]^{\frac{1}{2}}$, where the $\Delta_i$ are differences between the experimental and calculated energies, $N$ is the number of data appearing in the least-squares fit, and $k$ is the number of parameters. This agreement actually means that the rms deviation is almost always within the experimental errors.

We have thus demonstrated the success of a formula which is based on the shell model. The shell model allows for a change in the parameters whenever one passes from one subshell to another; however, our results show that the parameters do not change appreciably, i.e., within the experimental errors they remain the same. A change in the parameters enters only when we pass from one major shell to the other, i.e., when we pass the magic numbers: 50, 82, 126. Even such "submagic" numbers as 38 or 40 do not show up by causing a marked change in the parameters.

TABLE II. Energy parameters of the present model (in Mev) for isotopes.

The binding energies of the isotopes of each element are calculated with respect to the isotope with lowest neutron number as specified in each group of elements. Thus the binding energies of the Ge isotopes with $30=N=40$ are calculated with respect to $_{32}$Ge$_{30}^{62}$ whereas those of the Ge isotopes with $40=N=50$ are calculated with respect to $_{32}$Ge$_{40}^{72}$.

| | $A$ | $a$ | $D$ | Rms deviation in B.E. Mev | % |
|---|---|---|---|---|---|
| $30\leq N\leq 40$ | | | | | |
| $_{28}$Ni | 11.05 | −0.42 | −1.46 | 0.21 | 0.35 |
| $_{29}$Cu | 11.40 | −0.55 | −1.41 | 0.30 | 0.56 |
| $_{30}$Zn | 11.41 | −0.43 | −1.68 | 0.13 | 0.21 |
| $_{31}$Ga | 12.04 | −0.43 | −1.17 | 0.15 | 0.22 |
| $_{32}$Ge | 12.85 | −0.43 | −1.74 | 0.32 | 0.63 |
| $40\leq N\leq 50$ | | | | | |
| $_{32}$Ge | 8.35 | −0.15 | −1.84 | 0.12 | 0.38 |
| $_{33}$As | 9.14 | −0.25 | −1.16 | 0.11 | 0.26 |
| $_{34}$Se | 9.56 | −0.22 | −1.46 | 0.20 | 0.26 |
| $_{35}$Br | 10.11 | −0.24 | −0.98 | 0.18 | 0.22 |
| $_{36}$Kr | 10.60 | −0.26 | −1.67 | 0.15 | 0.21 |
| $_{37}$Rb | 10.91 | −0.21 | −0.90 | 0.14 | 0.29 |
| $_{38}$Sr | 10.96 | −0.15 | −1.57 | 0.22 | 0.56 |
| $N\geq 50$ | | | | | |
| $_{38}$Sr | 7.17 | −0.24 | −0.78 | 0.18 | 0.87 |
| $_{39}$Y | 7.80 | −0.36 | −0.56 | 0.14 | 0.51 |
| $_{40}$Zr | 8.06 | −0.23 | −0.77 | 0.15 | 0.40 |
| $_{41}$Nb | 8.59 | −0.28 | −0.52 | 0.29 | 0.74 |
| $_{42}$Mo | 8.89 | −0.29 | −1.00 | 0.10 | 0.24 |
| $N\geq 64$ | | | | | |
| $_{48}$Cd | 7.84 | −0.22 | −1.34 | ... | ... |
| $_{49}$In | 8.31 | −0.23 | −1.34 | ... | ... |
| $_{50}$Sn | 8.69 | −0.18 | −1.15 | 0.18 | 0.23 |
| $_{51}$Sb | 9.00 | −0.17 | −1.21 | 0.23 | 0.40 |
| $_{52}$Te | 9.11 | −0.14 | −0.92 | 0.42 | 0.51 |
| $_{53}$I | 9.27 | −0.13 | −1.37 | ... | ... |
| $_{54}$Xe | 9.72 | −0.15 | −1.16 | 0.18 | 0.23 |
| $120\leq N\leq 126$ | | | | | |
| $_{81}$Tl | 8.44 | −0.42 | −0.57 | 0.21 | 0.76 |
| $_{82}$Pb | 8.27 | −0.28 | −0.67 | 0.12 | 0.30 |
| $N\geq 126$ | | | | | |
| $_{82}$Pb | 4.561±0.008 | −0.026±0.004 | −0.70 ±0.02 | 0.018 | 0.08 |
| $_{83}$Bi | 4.955±0.044 | −0.092±0.025 | −0.368±0.077 | 0.093 | 0.50 |
| $_{84}$Po | 5.306±0.018 | −0.075±0.008 | −0.817±0.045 | 0.058 | 0.16 |
| $_{85}$At | 5.528±0.066 | −0.049±0.030 | −0.562±0.145 | 0.15 | 0.70 |

In Table I are given some cases of calculated results, which are of special interest. For the Pb isotopes with $N\geq 126$, the experimental errors are especially small, and one can see in this table, that the rms deviation was especially small too. Another example given in Table I is iodine, which illustrates the good agreement so long as one stays within a major shell despite the fact that a number of different subshells are involved in these nuclei; when one passes the major shell (see the example of I$^{136}$ in Table I) the agreement deteriorates.

In the case of the Sn isotopes the formula we used is strictly valid only if all the neutrons fill successively the $h_{11/2}$ subshell and the odd nucleon goes into the $d_{\frac{3}{2}}$ subshell. However, the experimental ground-state spins show that for Sn$^{115}$ and Sn$^{117}$ the odd nucleon goes into the $s_{\frac{1}{2}}$ state. This would suggest that our treatment is not justified; however, one knows that the $d_{\frac{3}{2}}$ level in these nuclei lies fairly close to the ground state,[11] (0.3 Mev for Sn$^{115}$ and 0.2 Mev for Sn$^{117}$), and since the rms deviation is ±0.2 Mev one sees that it does not make any difference whether one compares the theory with

[13] A. H. Wapstra, Physica **21**, 385 (1955).
[14] J. R. Huizenga, Physica **21**, 410 (1955).
[15] *Nuclear Level Schemes*, $A=40$—$A=92$, compiled by Way, King, McGinnis, and van Lieshout, U. S. Energy Commission Report TID-5300 (U. S. Government Printing Office, Washington, D. C., 1955).

the exact state to which it should be applied or to the actual ground state. Of course in cases like the Sn isotopes, for every one of which the $d_{\frac{3}{2}}$ level is known, there is no point in comparing the calculations with the ground state instead of doing so with the actual $d_{\frac{3}{2}}$ state to which they refer. However, in other cases, the position of the level to which the calculation refers may not be known exactly, but nevertheless it can be assumed with great confidence that it lies close enough to the ground state, and the binding energy of the ground state can be used for comparison with the theory so long as one is satisfied with an agreement to within 200–300 kev.

In Table II are summarized the results concerning the neutron configurations. As was mentioned before, when for a fixed number of protons, the number of neutrons changes within a shell, the parameters are left practically unaffected. We have therefore grouped together all isotopes of the same element which had their neutrons in one of the following ranges: $28\leq N\leq 40$, $40\leq N\leq 50$, $50\leq N\leq 82$, $82\leq N\leq 126$, $N\geq 126$, and fitted a set of parameters for each such group. Thus, for instance, the parameters of Pb ($N\geq 126$) appearing in Table II were obtained by a least-squares fit of Eq. (10) using the experimental binding energies quoted in Table I.

The situation with regard to the proton configurations is quite similar; the main differences are that the number of experimental data is smaller and the agreement between the experimental binding energies and the calculated ones is, on the whole, poorer. But the over-all picture remains the same, and so are the conclusions which may be drawn from the results; the best values of the coefficients for the various shells and the rms deviation in the B.E. are given in Table III.

Since the parameters which have been determined are relatively simple functionals of the wave functions belonging to the different shells, it is interesting to see whether they exhibit any regularities. To do so we have first to determine the standard errors on the parameters.[16] we can see from this that, even in cases of small experimental and theoretical errors, some of the parameters are not determined to a very high accuracy. However, it is still possible to obtain a general picture. In particular, the different values of $A$ can be better understood. $A$, by its definition, represents the interaction energy of, say, a neutron in the shell considered with all the closed shells of both protons and neutrons, as well as with the $Z$ protons in the last shell which is not necessarily filled. It is thus very reasonable to assume that as a function of $Z$, $A$ is given to a good approximation by

$$A=A_1+A_2Z.$$

$a$, $D$, and $\epsilon$ are smaller corrections to the energy and thus their dependence on $Z$ can be neglected. With these approximations, one is now led to an expression for the binding energies of nuclei with both $Z$ and $n$ variable, of the form

$$\Delta E=n(A_1+A_2Z)+\tfrac{1}{2}n(n-1)a+[\tfrac{1}{2}n]D+y(n)y(Z)\epsilon, \quad (11)$$

where $\Delta E$ stands for the binding energy of the nucleus $(Z_0+Z, N_0+n)$ relative to that of $(Z_0+Z, N_0)$ (and hence the asymmetry between $Z$ and $n$).

Table IV contains the results of applying Eq. (11) to nuclei with $Z\geq 82, N\geq 126$. The good agreement justifies our assumption on $A$.

It is also gratifying to observe that the values of $|D|$ obtained for odd $Z$ (Table II) are systematically lower than those of even $Z$. Since we have taken the binding energy to be positive, $D$, representing the amount by which odd-$A$ nuclei are less stable than even-$A$ nuclei, is negative, and $\epsilon$ is positive. Since, as was shown, $D(\text{for odd } Z)=D(\text{for even } Z)+\epsilon$, the observed systematics in the values of $D$ are explained, at least qualitatively.

TABLE III. Energy parameters of the present model (in Mev) for isotones.

The binding energies of the isotones, for each $N$, are calculated with respect to the isotone with lowest proton number as specified in each group of elements.

| | $A$ | $a$ | $D$ | Rms deviation in B.E. Mev | % |
|---|---|---|---|---|---|
| $44\leq Z\leq 50$ | | | | | |
| $N=60$ | 8.44 | −0.61 | −1.68 | 0.19 | 0.68 |
| $N=61$ | 8.84 | −0.65 | −0.99 | 0.33 | 0.94 |
| $N=62$ | 8.83 | −0.51 | −1.31 | 0.19 | 0.50 |
| $Z\geq 50$ | | | | | |
| $N=72$ | 7.94 | −0.61 | −1.22 | 0.02 | 0.09 |
| $N=73$ | 8.10 | −0.52 | −1.20 | 0.13 | 0.38 |
| $N=74$ | 8.25 | −0.44 | −0.94 | ··· | <0.1 |
| $Z\geq 82$ | | | | | |
| $N=130$ | 6.01 | −0.45 | −1.09 | 0.09 | 0.46 |
| $N=131$ | 6.06 | −0.35 | −0.78 | 0.10 | 0.52 |
| $N=132$ | 6.45 | −0.37 | −1.05 | 0.05 | 0.22 |
| $N=133$ | 6.85 | −0.40 | −0.71 | 0.24 | 0.96 |
| $N=134$ | 7.12 | −0.39 | −0.89 | 0.08 | 0.42 |

TABLE IV. Example of experimental and calculated B.E. (in Mev), using Eq. (11). ($_{82}$Pb relative to $Pb^{208}$, $_{83}$Bi relative to $Bi^{209}$, $_{84}$Po relative to $Po^{210}$, $_{85}$At relative to $At^{211}$.)

| Nucleus | B.E. Exp. | B.E. Calc. | Nucleus | B.E. Exp. | B.E. Calc. |
|---|---|---|---|---|---|
| $Pb^{209}$ | 3.87 | 3.88 | $Po^{211}$ | 4.55 | 4.50 |
| $Pb^{210}$ | 9.11 | 9.20 | $Po^{212}$ | 10.57 | 10.43 |
| $Pb^{211}$ | 12.89 | 12.97 | $Po^{213}$ | 14.89 | 14.82 |
| $Pb^{212}$ | 18.08 | 18.16 | $Po^{214}$ | 20.80 | 20.64 |
| $Pb^{214}$ | 16.98 | 16.90 | $Po^{215}$ | 24.89 | 24.91 |
| $Bi^{210}$ | 4.67 | 4.41 | $Po^{216}$ | 30.69 | 30.61 |
| $Bi^{211}$ | 9.78 | 9.82 | $Po^{218}$ | 40.39 | 40.36 |
| $Bi^{212}$ | 14.16 | 14.11 | $At^{214}$ | 15.98 | 15.97 |
| $Bi^{213}$ | 19.34 | 19.40 | $At^{215}$ | 21.89 | 21.88 |
| $Bi^{214}$ | 23.47 | 23.58 | $At^{216}$ | 26.48 | 26.68 |
| | | | $At^{217}$ | 32.46 | 32.47 |

$A_0=3.88$ $A_1=0.31$ $a=-0.06$ $D=1.48$ $\epsilon=0.22$

[16] H. Cramér, *The Elements of Probability Theory* (John Wiley and Sons, Inc., New York, 1955), p. 239.

We have shown above that a formula of the form (10) or even (11), with properly chosen parameters, can be made to fit experimental data over a wide range with a high accuracy. Similar results[1,2] were previously obtained for the light elements with an even more detailed expression which exhibited the dependence of binding energies on isotopic spin.

Although these expressions were derived for the shell model, their simplicity does not allow one to believe that they are peculiar to the shell model alone. Also, the fact that only relatively small changes in the parameters occur as long as one remains within a major shell, may indicate that the expression obtained is a result of the observed grouping of nucleons into shells, rather than being due to the detailed structure of the shell model. To investigate this point further, it is necessary to see to what extent can the parameters be derived from the shell-model wave functions and a given two-body interaction. Further work along these lines is being done here.

PHYSICAL REVIEW VOLUME 108, NUMBER 2 OCTOBER 15, 1957

# $\pi^-$-$p$ Elastic Scattering at 1.44 Bev*

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(Received June 17, 1957)

An investigation of $\pi^-+p$ elastic scattering, made in a liquid propane bubble chamber, is reported. Identification of events is made on the basis of kinematics. The problem of contamination by pion scattering from protons bound in carbon is considered in some detail; it is shown that the latter requires a correction of only $4\pm2.5\%$ of the total number of events. The angular distribution is presented. It shows a large diffraction peak at small angles and an approximately isotropic plateau over the backward hemisphere. The forward peak is fitted to a black-sphere diffraction pattern with a radius of $(1.08\pm0.06)\times10^{-13}$ cm. The total elastic cross section is found to be $\sigma_e=10.1\pm0.80$ mb.

## INTRODUCTION

We report here some results of the elastic scattering of 1.3-Bev (kinetic energy) negative pions obtained in an exposure of a propane bubble chamber, previously analyzed to study strange particle production.[1]

The study of $\pi p$ scattering in the Bev range has been in progress for some years now, using the hydrogen diffusion cloud chamber.[2–4] Our results are not qualitatively different, but are more extensive. From an experimental point of view, perhaps of greatest interest is the demonstration made in some detail in this paper, that the elastic hydrogen events may be differentiated quite clearly from other events found in the chamber. The pion beam is collimated, shielded, and magnetically analyzed as shown in Fig. 1. The resulting spread in beam energy deduced from trajectories plotted through the collimation system,[5] is $\pm1\%$. The absolute value of the pion beam momentum is $1.433\pm0.015$ Bev/$c$. This is determined, as explained in (1), from a study of two unstable-particle production events which were obtained in the same exposure.

The liquid propane bubble chamber has previously been described[6,7]; it is $6\frac{1}{8}$ in. in diameter and 4 in. in depth. The density of expanded propane is 0.429 g/cm$^3$; the partial density of hydrogen is 0.078 g/cm$^3$. There is no magnetic field.

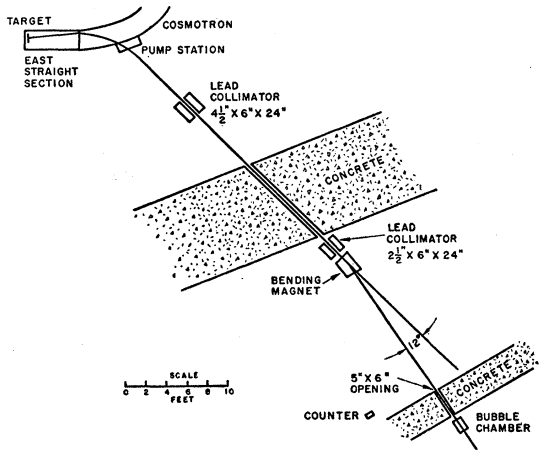


FIG. 1. Experimental setup showing $\pi$-beam trajectory collimators, bending magnet, and position of chamber.

* This research was supported by the Atomic Energy Commission and the Office of Naval Research.
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