Statistical Tensors for Oriented Nuclei

M. E. ROSE
Oak Ridge National Laboratory, Oak Ridge, Tennessee

(Received June 26, 1957)

Some properties of the statistical tensors, which govern the angular distribution of radiation emitted by oriented nuclei, are discussed. The discussion is limited to the case of axial symmetry in the spin-Hamiltonian which produces the orientation. An identification of the irreducible tensors, which define the statistical tensors, with those familiar in spin coupling is made. This also permits the matrix elements of the former tensors to be obtained quite easily. The temperature dependence of the statistical tensors for the case of small spin coupling is discussed.

I. INTRODUCTION

It is well known that the angular distribution of radiation emitted by oriented nuclei is governed by a set of quantities which provide a description of the orientation process in terms of tensor moments.[1] These quantities were called statistical tensors by Fano[2] who introduced them. For the case of axial symmetry Fano's definition of these statistical tensors becomes

$$G_\nu=\sum_m p(m)(-)^{j-m}C(jj\nu;m,-m);\quad \nu\leqslant 2j,\qquad (1)$$

where m is the projection quantum number for one of the $2j+1$ emitting substates, $p(m)$ is the corresponding population and $C(jj\nu;m,-m)$ is a vector addition coefficient. As emphasized by Fano, the factor

$$(-)^{j-m}C(jj\nu;m,-m)=[(2\nu+1)/(2j+1)]^{\frac{1}{2}}C(j\nu j;m,0)\qquad (2)$$

gives the m dependence of the diagonal elements of the density matrix characterizing the emitting state.[3]

The role played by these statistical tensors G_ν in the angular distribution of radiations emitted by oriented nuclei may be seen from the explicit form of the distribution function. It is sufficient, for our purposes, to consider emission of unpolarized, pure radiation of angular momentum L in a transition $j\rightarrow j'$. Then the intensity in a direction making an angle θ with the direction of orientation is, apart from a normalization factor,

$$I(\theta)=\sum_\nu G_\nu c_\nu(L)W(jjLL;\nu j')P_\nu(\cos\theta),\qquad (3)$$

where W is a Racah coefficient and the $c_\nu(L)$ are a set of parameters characterizing the radiation. For example, for emission of α particles[4]

$$c_\nu(L)=C(LL\nu;00),\qquad (3a)$$

and for emission of gamma rays[1,3]

$$c_\nu(L)=C(LL\nu;1,-1).\qquad (3b)$$

In these parity-preserving transitions, ν is an even integer and its maximum value is the smaller of $2j$ and $2L$. Of course, in these cases only the "effective alignment" of the emitting nucleus is instrumental in giving rise to an anisotropy. That is, the G_ν for ν even depend on deviations of $\langle m^\nu\rangle_{\text{Av}}$ from the isotropic value.[5] Thus, a weak polarization of the emitting state, which involves $G_1\sim\langle m\rangle_{\text{Av}}$, gives isotropy. On the other hand, in the parity nonpreserving β transition from a polarized nucleus, ν is odd and an isotropy in the angular distribution of β particles is observed.

In the following we wish to discuss some of the properties of the statistical tensors and, in particular, a connection will be established between them and the irreducible tensors characteristic of spin interactions.[6] Finally, some comments on the temperature dependence of these parameters will be made.

II. PROPERTIES OF THE STATISTICAL TENSORS

First of all, in the interest of accuracy, it should be noted that the definition given in (1) is not complete when the nuclear orientation arises from a coupling of the nuclear spin to another spin system. For example, when the nucleus is polarized or aligned by dipole (hyperfine) coupling with the electron spins, Eq. (1) should be replaced by[4]

$$G_\nu=\sum_{mm_e} p(m,m_e)(-)^{j-m}C(jj\nu;m,-m),\qquad (4)$$

where $p(m,m_e)$ refers to the population of the combined substate described by nuclear and electronic projection quantum numbers, m and m_e. The definition (4) is

---

[1] S. R. deGroot and H. A. Tolhoek, in *Beta- and Gamma-Ray Spectroscopy*, edited by K. Siegbahn (North-Holland Publishing Company, Amsterdam, 1955), Chap. 19, Part 3.

[2] U. Fano, National Bureau of Standards Report No. 1214 (unpublished).

[3] More generally, the nondiagonal elements involve

$$C(j\nu j';m,m'-m)$$

and an element of the representation of the rotation group in $2\nu+1$ dimensions. See, for example, Eq. (13) of L. C. Biedenharn and M. E. Rose, Revs. Modern Phys. **25**, 729 (1953).

[4] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley and Sons, Inc., New York, 1957), p. 176. See also reference 3.

[5] Simon, Rose, and Jauch, Phys. Rev. **84**, 1155 (1951).

[6] We make no attempt at completeness. In addition to the discussion given by Fano (reference 2), attention may be called to certain properties discussed in reference 4. For instance, suppose that $p(m)=p_0+q(m)$, where p_0 is independent of m and, therefore, contributes only to G_0 (or to the total intensity and not to the anisotropy) and consider $q(m)=-q(-m)$, as in the polarization produced by capture of slow polarized neutrons. Then the term $q(m)$ contributes only if ν is odd. Hence, gamma rays emitted subsequent to the capture are isotropic but are circularly polarized. See Biedenharn, Rose, and Arfken, Phys. Rev. **83**, 683 (1951).

cogent even though the z components of nuclear and electronic angular momenta are not separately diagonal because (4) can be written in a form which explicitly recognize the independence of G_ν on the representation. We use the fact that the diagonal matrix elements of any irreducible tensor of rank ν is

$$(jm|T_{\nu 0}|jm)=C(j\nu j;m0)(j\|T_\nu\|j), \tag{5}$$

where $(j\|T_\nu\|j)$ is an (irrelevant) reduced matrix element and the tensor component $T_{\nu 0}$ is entirely in the nuclear space, so that its matrix elements in the electron space are diagonal. It follows then that we can write

$$G_\nu=\frac{\mathrm{Tr}\;T_{\nu 0}\exp(-H/kT)}{\mathrm{Tr}\exp(-H/kT)}, \tag{6}$$

where H is the spin Hamiltonian responsible for the orientation.[5,7] The traces in (6) are over the combined nuclear-electron space and using the decoupled representation, one sees that the definition (1) is restored if we make the identification

$$p(m)=\sum_{m_e}(mm_e|\exp(-H/kT)|mm_e)\Big/\sum_{mm_e}(mm_e|\exp(-H/kT)|mm_e). \tag{7}$$

The problem to which we now turn our attention is the identification of the tensors $T_{\nu 0}$.

We can enumerate specific cases. Usually $\nu>4$ is not important in actual experimental cases. For $\nu=0$ we find immediately that

$$T_{00}=(2j+1)^{-\frac{1}{2}}. \tag{8}$$

For $\nu=2$ we use

$$C(j2j;m0)=\frac{3m^2-j(j+1)}{[j(j+1)(2j-1)(2j+3)]^{\frac{1}{2}}}$$

to obtain

$$T_{20}=\left[\frac{180(2j-2)!}{(2j+3)!}\right]^{\frac{1}{2}}(J_z^2-\tfrac{1}{3}\mathbf{J}^2). \tag{9}$$

The form of this result is to be expected since for a given tensor rank the irreducible tensors in spin space are unique, apart from a normalization factor. For $\nu=4$ we use the explicit form of $C(jj4;m,-m)$ obtained from Wigner's[8] result for the vector-addition coefficients to obtain

$$T_{40}=210\left[\frac{(2j-4)!}{(2j+5)!}\right]^{\frac{1}{2}}\left[J_z^4-\frac{1}{15}(3\mathbf{J}^4-\mathbf{J}^2)-\frac{1}{7}(6\mathbf{J}^2-5)(J_z^2-\tfrac{1}{3}\mathbf{J}^2)\right]. \tag{10}$$

Of course, $\mathbf{J}^{2n}$ can be replaced by $j^n(j+1)^n$. For completeness we also give some results for odd ν.

$$T_{10}=2\left[\frac{3(2j-1)!}{(2j+2)!}\right]^{\frac{1}{2}}J_z,$$

$$T_{30}=4\left[\frac{7(2j-3)!}{(2j+4)!}\right]^{\frac{1}{2}}J_z(5J_z^2-3\mathbf{J}^2+1).$$

One may verify quite easily that, in all cases given, the $T_{\nu 0}$ are components of irreducible tensors and (as a consequence) have zero trace for $\nu\neq 0$. For the smaller values of ν the connection of $T_{\nu 0}$ with the multipole operators in spin space is transparent. However, while it is quite certain that T_{40}, for example, is connected with the 2^4-pole spin operator this is not apparent from the form (10). It is now our purpose to establish the connection between the $T_{\nu 0}$ and the multipole operators in a general way. At the same time this will serve to facilitate the process giving the explicit form for the $T_{\nu 0}$.

We make use of the uniqueness property of tensors of given rank in the space of a given spin. A simple way to construct such tensors in general is by the method of polarized spherical harmonics. Thus,[4]

$$\mathcal{T}_{LM}(\mathbf{J})=(\mathbf{J}\cdot\nabla)^L\mathcal{Y}_{LM}(\mathbf{r}), \tag{11}$$

where $\mathcal{Y}_{LM}(\mathbf{r})$ is a solid harmonic (also an irreducible tensor of rank L). $\mathcal{T}_{LM}(\mathbf{J})$ is obviously an irreducible tensor of rank L and is entirely in the space of $\mathbf{J}$. These are the multipole operators referred to in the previous paragraph.[9] Any irreducible tensor of rank L has matrix elements

$$(jm|\mathcal{T}_{LM}|j'm')=C(j'Lj;m'M)\delta_{m,M+m'}(j\|\mathcal{T}_L\|j'), \tag{12}$$

and therefore any two tensors of this description, in a given space, can differ only by a factor, the ratio of the reduced matrix elements. Therefore, we set

$$T_{\nu 0}=A_\nu(j)\mathcal{T}_{\nu 0}(\mathbf{J}), \tag{13}$$

and only the constant $A_\nu(j)$ needs to be determined. To do this it is sufficient to observe that $\mathcal{Y}_{\nu 0}$ has a term proportional to z^ν and the process of polarization indicated in (11) reduces this to a term equal to $\nu!J_z^\nu$. Therefore, A_ν is fixed by comparing coefficients of J_z^ν on the two sides of Eq. (13).

On the left-hand side of (13), we recognize that the term in J_z^ν comes entirely from the term in

$$(-)^{j-m}C(jj\nu;m,-m)$$

which is proportional to m^ν. From Wigner's explicit expression for the vector addition coefficient, we find

$$(-)^{j-m}C(jj\nu;m,-m)=(\nu!)^2\left[\frac{(2\nu+1)(2j-\nu)!}{(2j+\nu+1)!}\right]^{\frac{1}{2}}S_\nu(j), \tag{14}$$

[7] A. Abragam and M. H. L. Pryce, Proc. Roy. Soc. (London) **A205**, 135 (1951).

[8] E. P. Wigner, *Gruppentheorie* (Friedrich Vieweg und Sohn, Braunschweig, 1931). See also Eq. (3.18) of reference 4.

[9] The role of these multipole operators in spin coupling has been discussed in reference 4, Chap. VIII.

where

$$S_\nu(j)=[(j+m)!(j-m)!]^{-1}\times\sum_\sigma \frac{(-)^\sigma}{(\sigma!)^2}\frac{(j+\nu+m-\sigma)!(j-m+\sigma)!}{[(\nu-\sigma)!]^2}. \quad (14a)$$

The coefficient of m^ν in $S_\nu(j)$ is seen to be[10]

$$S_\nu'=\sum_{\sigma=0}^{\nu}\frac{1}{(\sigma!)^2[(\nu-\sigma)!]^2}=2^\nu(2\nu-1)!!/(\nu!)^3, \quad (14b)$$

where $(2\nu-1)!!\equiv(2\nu-1)(2\nu-3)\cdots5\cdot3\cdot1$. Hence,

$$(-)^{j-m}C(jj\nu;m,-m)=\left[\frac{(2\nu+1)(2j-\nu)!}{(2j+\nu+1)!}\right]^{\frac{1}{2}}\frac{2^\nu(2\nu-1)!!}{\nu!}[m^\nu+\cdots], \quad (15)$$

where the $\cdots$ indicates terms with lower powers of m.

The coefficient of J_z^ν on the right-hand side of (12) is easily obtained. We have

$$\mathcal{Y}_{\nu0}=\left[\frac{2\nu+1}{4\pi}\right]^{\frac{1}{2}}\frac{(2\nu-1)!!}{\nu!}[z^\nu+\cdots], \quad (16)$$

where the $\cdots$ indicates terms with lower powers of z. Then

$$\mathcal{T}_{\nu0}(\mathbf{J})=\left[\frac{2\nu+1}{4\pi}\right]^{\frac{1}{2}}(2\nu-1)!![J_z^\nu+\cdots], \quad (17)$$

and now the $\cdots$ indicate terms with lower powers of J_z.

The results (14) and (16) lead immediately to

$$T_{\nu0}=\frac{2^\nu}{\nu!}\left[\frac{4\pi(2j-\nu)!}{(2j+\nu+1)!}\right]^{\frac{1}{2}}\mathcal{T}_{\nu0}(\mathbf{J}). \quad (18)$$

This result combined with the definition (11) is the desired relationship.

The result (18) can be used to obtain the matrix elements of the operators $\mathcal{T}_{\nu0}(\mathbf{J})$. From the equivalence of (1) and (6), we can write

$$(jm|T_{\nu0}|j'm)=\delta_{jj'}(-)^{j-m}C(jj\nu;m,-m)=\delta_{jj'}\left[\frac{2\nu+1}{2j+1}\right]^{\frac{1}{2}}C(j\nu j;m0). \quad (19)$$

[10] The sum in (14b) is readily evaluated by considering

$$\int^{2\pi}(1+e^{ix})^\nu(1+e^{-ix})^\nu dx=2\pi(\nu!)^2S_\nu'.$$

The integral is, of course, trivial.

Therefore[11]

$$(jm|\mathcal{T}_0(\mathbf{J})|j'm)=\delta_{jj'}\frac{\nu!}{2^\nu}\left[\frac{1}{4\pi}\frac{2\nu+1}{2j+1}\frac{(2j+\nu+1)!}{(2j-\nu)!}\right]^{\frac{1}{2}}C(j\nu j;m0). \quad (20)$$

The matrix elements of $\mathcal{T}_{\nu M}(\mathbf{J})$ are then obtained from Eq. (12).

From the explicit results for $T_{\nu0}(\nu\leqslant4)$ given above, it will be recognized that, apart from a scale factor, the polarization process defined in (11) replaces z and r^2 in $\mathcal{Y}_{\nu0}$ by J_z and $\mathbf{J}^2$, respectively, only for $\nu\leqslant2$. For $\nu\geqslant3$, additional terms arise from the noncommutation of the components of $\mathbf{J}$. For the nonaxially symmetric case, a similar remark applies to $\mathcal{T}_{\nu M}$ but attention must be given to the noncommutation referred to. This is automatically taken care of in the definition of Eq. (11).[4]

III. TEMPERATURE DEPENDENCE

We discuss the temperature dependence in the (frequently) practical case that the coupling energy is small compared to the thermal energy kT. Then, with $\alpha=-1/kT$, we expand (6) to terms of order α^2.

$$G_\nu\cong\{\alpha\ \mathrm{Tr}\ T_{\nu0}H+\tfrac{1}{2}\alpha^2\ \mathrm{Tr}\ T_{\nu0}H^2\}\{\mathrm{Tr}\ 1\}^{-1}. \quad (21)$$

Here we consider $\nu\neq0$ only, since G_0 is trivial. Then

$$\mathrm{Tr}\ T_{\nu0}=0.$$

Also we assume that

$$\mathrm{Tr}\ H=0.$$

which means that the energy levels are measured from their center of gravity.

Since H is rotationally invariant, it must have the form of a sum of contracted tensors. That is,

$$H=\sum_{LM}(-)^MT_{LM}(\mathbf{J})T_{L,-M}(\mathbf{X});\quad L\geqslant1 \quad (22)$$

where the $T_{LM}(\mathbf{J})$ are certain irreducible tensors (rank L) in the nuclear spin space and the $T_{LM}(\mathbf{X})$ are similar tensors in some other space; for example, $\mathbf{X}$ may describe the electron spin or an external field (electric or magnetic).

Considering the α (or $1/T$) term of (21), we see that a *necessary* condition that it shall not vanish is $L=\nu$. This follows from the fact that the product of two irreducible tensors of rank L and ν contains irreducible tensors of rank λ, where $|\nu-L|\leqslant\lambda\leqslant\nu+L$. Hence, for ν even, only the quadrupole coupling can contribute. The dipole coupling, as is well known, will not make any contribution to the $1/T$ term in an alignment. For the case of polarization, however, where $\nu=1$ is possible, the dipole coupling will, in general, contribute to the $1/T$ term.

[11] Reference 4, p. 147.

If the term in α^2 (or $1/T^2$) is considered, the necessary condition that it shall not vanish is that a triangular relation exist between ν, L and L' where L and L' are tensor indices appearing in (22). Thus $|L-L'|\leqslant\nu\leqslant L+L'$. If we consider $L=L'=1$ (dipole coupling), then the triangular condition is fulfilled for $\nu=2$ but not for $\nu>2$. The cross terms $L=1$, $L'=2$ and $L=2$, $L'=1$ (dipole-quadrupole cross terms) permit $\nu\leqslant 3$ and in alignment (or in α or γ emission) contribute only to the $\nu=2$ term in the angular distribution. The pure quadrupole term ($L=L'=2$) permits $\nu\leqslant 4$ and so may contribute to all terms of practical interest.[12]

ACKNOWLEDGMENTS

It is a pleasure to acknowledge rewarding conversations with Dr. L. D. Roberts, Dr. J. W. T. Dabbs, and Dr. T. A. Green.

[12] In terms of specific calculations of G_1 and G_2 most of these results were already familiar (see reference 5, for example). However, the general principles which are operative in producing these results had not been explicitly stated.

PHYSICAL REVIEW VOLUME 108, NUMBER 2 OCTOBER 15, 1957

Polarization Phenomena in the One-Quantum Annihilation of Positrons and the Photoelectric Effect*

KIRK W. McVOY
Brookhaven National Laboratory, Upton, New York
(Received June 5, 1957)

Cross sections are derived for the one-quantum annihilation of longitudinally-polarized positrons and for the photoelectric effect with longitudinally-polarized photons. Simple expressions, quantitatively reliable for light elements, are obtained by considering only K-shell electrons and describing the outgoing electron (incoming positron) by plane waves. In both cases, the incoming and outgoing particles have predominantly the *same* helicity if the free Dirac particle is relativistic.

In a note added in proof, these calculations are extended to the case of elliptically polarized radiation. For linearly polarized photons, the results are compared with those obtained by including lowest-order Coulomb corrections to the continuum wave function of the electron, as given by Sauter in 1931. The difference in the angular distribution is very marked, and indicates a sensitive dependence on the degree of screening of the Coulomb field.

INTRODUCTION

POSITRONS passing through matter emit radiation in flight through the processes of bremsstrahlung, two-quantum annihilation, and one-quantum annihilation with tightly-bound electrons. If the positrons are longitudinally polarized, the emitted photons will be as well, and in all three cases the higher-energy photon (if there is a choice) has predominantly the same helicity as the incoming positron; the degree of circular polarization of the radiation approaches 100% rapidly as the positron becomes relativistic.

The polarization of bremsstrahlung and two-quantum annihilation-radiation has been discussed previously[1,2]; we wish to present a simplified discussion of one-quantum annihilation and the related process, the photoelectric effect. In order to avoid such complications as those introduced by Coulomb wave functions, we shall base the derivation on the following simplifying assumptions:

(1) We assume that the outgoing electron (photoeffect) and incoming positron (annihilation) can be described with sufficient accuracy by plane waves.

(2) Since the cross sections are by far the largest for the most tightly bound electrons, we shall calculate them only for K-shell electrons. Screening is neglected, but otherwise we employ the correct relativistic wave functions for the bound electrons, in order to treat the spin effects properly.

In other words, we shall calculate the cross sections only to lowest order; this is valid for high-energy particles striking low-Z atoms, and will be at least qualitatively correct for heavier elements. ("High energy" merely means large compared to the K-shell binding energy.) We consider first the more straightforward photoelectric effect.

I. Photoelectric Effect

For the (free) outgoing electron, we define the spinors which describe states of complete longitudinal polarization by ($\hbar=c=1$)

$$(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m)u=Eu,$$
$$(\boldsymbol{\sigma}\cdot\mathbf{p}/p)u_R=+u_R,\ (\boldsymbol{\sigma}\cdot\mathbf{p}/p)u_L=-u_L. \quad (1)$$

We call the electron described by u_R a "right-electron," since its spin and momentum define a right-

* Work performed under the auspices of the U. S. Atomic Energy Commission.
[1] L. A. Page, Phys. Rev. 106, 394 (1957).
[2] K. W. McVoy, Phys. Rev. 106, 828 (1957). Note that reference 2 should read "Heitler, second edition" rather than "Heitler, third edition."