Oxygen-16 by the Method of Generator Coordinates*†

JAMES J. GRIFFIN‡
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey, and Institute for Theoretical Physics, Copenhagen, Denmark

(Received June 20, 1957)

Dilatational and collective quadrupole excitations of the nucleus O^{16} are treated by the method of generator coordinates described in the preceding paper. In this method, one starts from assumed two-body forces and calculates the stiffness, inertia, and frequency of the collective motion. Numerical results are presented for these quantities.

Experimental data relative to collective 0+ and 2+ states in O^{16} are summarized, and other theoretical treatments of this problem are discussed. It is noted that collective models all imply too large a matrix element for the decay of the 6.06-Mev (0+) state. One concludes that this state is not primarily a dilatational excitation. Similar considerations apply to the 6.91-Mev (2+) state.

I. INTRODUCTION

THE low-lying levels of O^{16} have recently been the subject of some detailed study, both experimental and theoretical. This paper describes calculations of dilatational and quadrupole surface oscillations in O^{16} by means of the method of generator coordinates, described in the preceding paper.[1] To illustrate and test this method by applying it to a specific case is the primary motive for this investigation. In addition, we sought to discover whether the 0^+ and 2^+ states at 6.06 and 6.91 Mev may be described in terms of collective excitations of the doubly closed shells of O^{16}.

The calculations and their results (Tables I and II) indicate that the method of generator coordinates is certainly a feasible way to obtain an explicit description of collective motions. The calculated excitation energies of the first excited collective (0^+ and 2^+) states seem too high compared to the observed 6.06 and 6.91 Mev to support the thesis that these observed states are simple collective motions. The observed rate of pair emission from the 6.06-Mev 0^+ state supports this judgment, as will be discussed in Sec. VI.

In Sec. II we outline the assumptions on which the calculations are based and present the results. The calculations themselves are detailed in Appendix A. Section III summarizes some relevant experimental data, and Sec. IV, the results of previous theoretical work on O^{16}. The results of the analyses reported here are discussed in the context of the current experimental and theoretical situations in Sec. V, and the implications in Sec. VI.

II. CALCULATION OF COLLECTIVE STATES IN O^{16}

The quadratic approximation[1] to the generator equation has been explicitly applied to dilatational and quadrupole surface vibrations in O^{16}. We chose a Hamiltonian of the form

$$H(\mathbf{x}_1,\cdots\mathbf{x}_A)=\sum_i^A\left(-\frac{\hbar^2}{2M}\right)\nabla_i^2+\frac{1}{2}\sum_{ij}^A V_{ij}+\frac{1}{2}\sum_{ij}^A\frac{e^2}{|\mathbf{r}_{ij}|}. \quad (1)$$

The nuclear two-body interactions (V_{ij}) were assumed to be of exchange type and to have a Gaussian dependence upon the separation of the nucleons. The strength of the two-body interaction was adjusted to give a minimum in energy at a specified nuclear radius.

The nucleonic wave function was assumed to be a Slater determinant of single-particle harmonic-oscillator wave functions,

$$\phi(\mathbf{x}_1,\cdots,\mathbf{x}_A;\alpha)=(A!)^{-\frac{1}{2}}\begin{vmatrix} U_1(\mathbf{x}_1;\alpha)\cdots U_A(\mathbf{x}_1;\alpha)\\ \vdots\\ U_1(\mathbf{x}_A;\alpha)\cdots U_A(\mathbf{x}_A;\alpha)\end{vmatrix}. \quad (2)$$

The coordinates $\mathbf{x}_i$ represent the space, spin, and isotopic spin coordinates, $(\mathbf{x}_i)=(x_i,y_i,z_i,\sigma_{z_i},\tau_{z_i})$. Thus, U_j is a product of a three-dimensional oscillator function, a spin function, and an isotopic-spin function. For O^{16}, one has four different space states, and each occurs with four different spin-isotopic-spin combinations.

The manner in which the nucleonic function depends on the deformation coordinate, α, specifies the kind of deformation being treated. For dilatational deformations, α defines simply an isotropic scale factor for the extension of the nucleonic wave function:

$$U_i(\mathbf{x}_j;\alpha)=U_i(x_je^{-\alpha},y_je^{-\alpha},z_je^{-\alpha},\sigma_{zj},\tau_{zj};0) \quad \text{(dilatational)}. \quad (3)$$

In the quadrupole mode, α defines an extension along one axis combined with volume-preserving contractions

---

* Based in part upon the thesis submitted by James J. Griffin to Princeton University in June, 1955 in partial fullfilment of the requirements for the degree of Ph.D. Preliminary reports of this work appeared in *Proceedings of the 1954 Glasgow Conference on Nuclear and Meson Physics*, edited by E. H. Bellamy and R. G. Moorhouse (Pergamon Press, London and New York, 1955), pp. 42 and 43; in J. A. Wheeler, Suppl., Nuovo cimento **2**, 908 (1955); and in J. J. Griffin, Phys. Rev. **99**, 648(A) (1955).

† Grateful acknowledgment is made of a Fulbright Fellowship held in Copenhagen, 1955–1956.

‡ Present address: Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

[1] J. J. Griffin and J. A. Wheeler, preceding paper [Phys. Rev. **108**, 311 (1957)].

TABLE I. Nuclear two-body interactions. $V_{ij}=-V_0\exp\{-(r_{ij}/R)^2[w+mP^m+bP^b+hP^h]\}$.

	w	m	b	h	$V_0(r_0=1.5)$ (Mev)	$V_0(r_0=1.2)$ (Mev)	R (10^{-13} cm)
Rosenfeld	−0.13	+0.93	0.46	−0.26	54.0	70.5	1.75
Simplified	0.00	+0.80	+0.20	0.00	64.2	100.8	1.75
Ferrell-Visscher[a]	0.30	0.53	−0.03	0.20	52.2 Mev at $r_0=1.28$		1.732

[a] See reference 5.

along the two perpendicular axes:

$$U_i(\mathbf{x}_j;\alpha)=U_i(x_je^{\alpha/2},\,y_je^{\alpha/2},z_je^{-\alpha},\sigma_j,\tau_j;\,0)\quad\text{(quadrupole)}.\tag{4}$$

With these quantities defined, one can calculate the kernels of the generator equation (see Appendix A and preceding paper[1]):

$$I(\alpha,\beta)=\int\phi^*(\mathbf{x}_1,\cdots,\mathbf{x}_A;\alpha)\times\phi(\mathbf{x}_1,\cdots,\mathbf{x}_A;\beta)d\mathbf{x}_1\cdots d\mathbf{x}_A,$$
$$K(\alpha,\beta)=\int\phi^*(\mathbf{x}_1,\cdots,\mathbf{x}_A;\alpha)H(\mathbf{x}_1,\cdots,\mathbf{x}_A)\times\phi(\mathbf{x}_1,\cdots,\mathbf{x}_A;\beta)d\mathbf{x}_1\cdots d\mathbf{x}_A.\tag{5}$$

The quadratic approximation is obtained when $I(\alpha,\beta)$ is replaced by a Gaussian in $\delta=(\alpha-\beta)$,

$$I(\alpha,\beta)\approx\exp(-s\delta^2),\tag{6}$$

and the ratio, K/I, is replaced by the leading terms of its expansion in powers of δ and $\gamma=(\alpha+\beta)/2$,

$$K(\alpha,\beta)\approx K(0,0)+\frac{1}{2}\delta^2\frac{\partial^2K(\alpha,\beta)}{\partial\delta^2}\bigg|_{\delta=\gamma=0}+\frac{1}{2}\gamma^2\frac{\partial^2K(\alpha,\beta)}{\partial\gamma^2}\bigg|_{\delta=\gamma=0}.\tag{7}$$

The properties of dilatational and quadrupole states have been calculated for five specific interactions (summarized in Table I). The spread of the undeformed nucleonic wave functions is obtained by equating the root-mean-square displacement of the particles in the undeformed state with that for a uniformly dense sphere of the specified radius:

$$\langle r^2\rangle=\tfrac{3}{5}r_0^2A^{\frac{2}{3}}=9/(4k_0^2)\tag{8}$$

for O^{16} wave functions whose exponential dependence is $\exp(-\frac{1}{2}k_0^2r^2)$. The calculations were done for $r_0=1.5$, 1.28, and 1.2×10^{-13} cm. The exchange mixtures used and the strengths required to give a minimum in the energy at these radii are exhibited in Table I. The mixtures are the well-known Rosenfeld[2] mixture, the "simplified" mixture, used by Inglis[3] in similar calculations, and for dilatations, the empirically derived mixture used by Ferrell and Visscher.[4,5] The calculations themselves are discussed in more detail in Appendix A.

Tables II and III summarize the results of these calculations. It has been shown in the preceding paper[1] that the solutions of the generator equation in this approximation have eigenvalues which are uniformly spaced at intervals

$$E_{v+1}-E_v=\hbar(\mathcal{K}/\mathfrak{M})^{\frac{1}{2}}=\hbar\Omega,\tag{9}$$

and that the ground state of collective motion has an energy lower than the nucleonic state $\phi(\mathbf{x}_1,\cdots,\mathbf{x}_A;\alpha=0)$ by an amount

$$\Delta\langle E_0\rangle=\frac{s\hbar^2}{\mathfrak{M}}+\frac{\mathcal{K}}{16s}-\frac{\hbar\Omega}{2}.\tag{10}$$

TABLE II. Results for dilatational vibrations. The quantities, $\mathcal{K}$, $\hbar^2/\mathfrak{M}$, $\hbar\Omega$, and $\Delta\langle E_0\rangle$, all in Mev, have been calculated from first principles from the nucleon-nucleon interactions of Table I, by use of the method of generator coordinates.[a] They represent, respectively, the force constant, the inertial parameter, the spacing between vibrational levels and the change in the system energy made by the variational method of generator coordinates. $4sa_0^2$ and a are dimensionless quantities which represent respectively the proximity of the calculation to the limit at which the variational wave function no longer lowers the energy ($4sa_0^2=1$), and the spread of the ground-state collective wave function [Eq. (11)]. In order to make the calculation with the Ferrell-Visscher interaction correspond precisely with the work of those authors, that portion of the kinetic energy of the shell-model wave function which corresponds to translation of the center of mass of the system (in amount: $\frac{3}{4}\hbar^2k_0^2/M$) was subtracted from the total nucleonic kinetic energy before the Hamiltonian kernel was evaluated. This correction has only a slight effect on the vibrational spacing, and was not included in the other calculations summarized above. See reference 27.

r_0	Exchange mixture	$\mathcal{K}$	$\hbar^2/\mathfrak{M}$	$\hbar\Omega$	$\Delta\langle E_0\rangle$	$4sa_0^2$	a
1.5×10^{-13}	Rosenfeld	409	0.303	11.1	−1.3	1.96	0.11
1.5×10^{-13}	Simplied	615	0.303	13.6	−0.8	1.60	0.09
1.28×10^{-13}	Ferrell-Visscher	608	0.416	15.9	−1.6	1.88	0.11
1.20×10^{-13}	Rosenfeld	1090	0.474	22.7	−1.0	1.57	0.09
1.20×10^{-13}	Simplified	1629	0.474	27.8	−0.1	1.23	0.06

[a] In the dilatational mode, $\hbar^2/\mathfrak{M}$ is given exactly by the liquid-drop value

[2] L. Rosenfeld, *Nuclear Forces* (Interscience Publishers, Inc., New York, 1949), Part III, Sec. 11.33.

[3] D. R. Inglis, Phys. Rev. **97**, 701 (1955).

[4] R. Ferrell and W. Visscher, Phys. Rev. **102**, 450 (1956).

[5] This interaction differs slightly from that used in reference 4. Dr. Visscher was kind enough to point out that a slight error had been made in deriving the parameters from the data cited therein, and to supply the present corrected version of the interaction.

TABLE III. Quadrupole vibrations. The units are the same as in Table II. "S-E" denotes the stiffness as based on the surface term in the semiempirical mass formula. "Irrot. M" denotes the inertia corresponding to irrotational fluid flow.[a]

r_0	Exchange mixture	$\mathcal{K}$	$\hbar^2/\mathfrak{M}$	$\hbar\Omega$	$\Delta\langle E_0\rangle$	$4sa_0^2$	a
1.5×10^{-13}	Rosenfeld	546	0.445	15.6	−0.0	1.03	0.03
1.5×10^{-13}	Simplified	576	0.454	16.6	−0.3	1.02	0.02
1.5×10^{-13}	S-E K; Irrot. M	72.3	0.606	6.6	...	...	...
1.2×10^{-13}	Rosenfeld[b]	996	0.735	26.5	X	0.98	X
1.2×10^{-13}	Simplified[b]	1120	0.735	28.2	X	0.92	X
1.2×10^{-13}	S-E K; Irrot. M	70.9	0.948	8.2	...	...	...

[a] See A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16, 13 (1953).
[b] These rows are illustrative only, since the variational method does not lower the energy in these cases; i.e., the lowest energy is obtained with no collective motion of the system. The method of generator coordinates therefore yields no energy lowering and an infinitely sharp collective function: $\Delta\langle E_0'\rangle=0$; $a=0$.

This is a measure of the improvement of the combined generator wave function over the nucleonic wave function.

It has also been noted in the preceding paper, that the variational procedure yields no energy lowering if the quantity $4sa_0^2$ is less than or equal to 1. This quantity therefore indicates whether or not one is close to the point where the method of generator coordinates ceases to be useful.

Finally, the validity of the quadratic approximation depends on the assumption that the Gaussian approximation to $I(\alpha,\beta)$ is valid and that the spread of the collective wave function in α is small, that is, that the higher order terms in K/I are actually negligible. A measure of this spread is the quantity, a, which appears in the ground state collective wave function,

$$f_0(\alpha)=N_0\exp[-\tfrac{1}{2}(\alpha/a)^2]. \tag{11}$$

It can be seen from Table II that a is sufficiently small in each case to insure that the approximation is valid, at least as a first approximation. Figure 1 displays the accuracy of the Gaussian approximation to the overlap integral for the dilatational case. It can be seen that the approximation is still good to within 10% when $I(\alpha,\beta)$ has fallen to a value 0.04.

III. RELEVANT EXPERIMENTAL DATA

The energies, spins, and parities of many of the low-lying states of O^{16} are well known from the work of Bittner and Moffat,[6] Hornyak and Sherr,[7] Wilkinson, Toppel, and Alburger,[8] and the compilation of Azjenberg and Lauritsen.[9]

Figure 1 is a level diagram of the known 0^+, 2^+, and 4^+ states in O^{16} (the collective states considered in this paper are of this kind), and of some predictions of the α-particle model for such states.

The transition rate from the 6.06-Mev 0^+ state to the ground state has been measured,[10] and found to imply a matrix element

$$\langle\Psi_0|\sum_p r_p^2|\Psi_1\rangle=3.8\times10^{-26}\text{ cm}^2 \tag{12}$$

for this transition, with an error of less than 10%. Ψ_0 is the ground state wave function and Ψ_1 the wave function for the 6.06-Mev state; the sum extends over all protons.

Also of potential interest in the present discussion are the results of experiments on the reaction $O^{16}(\gamma,p)N^{15}$ in the region where the intermediate state of the O^{16} nucleus has an excitation energy from 12.1 to 18.0 Mev[11,12] and the inverse reaction, $N^{15}(p,\gamma)O^{16}$.[13] The first two of these experiments seem to show a resonance in the reaction cross section at an excitation energy of about 14.7 Mev. From a theoretical study of angular distribution of the photoprotons and from the integrated cross section obtained in these experiments, Wilkinson[14] at first proposed that this resonance might be due to the first excited collective quadrupole state, and gave arguments against the alternative interpretation, that the resonance is due to a fluctuation in the single-particle level density. In the report on his own experimental study with Bloom,[13] which showed no evidence of the corresponding resonance in the inverse reaction, he withdraws this suggestion, and states that some (unspecified) misinterpretation must have been made in the previous experiment. He also cites the (unpublished) study of the $O^{16}(\gamma,p)N^{15}$ reaction by Johanasson and Forkman, as a basis for believing the 14.7-Mev resonance to be spurious. The implications of these conflicting results are therefore rather unclear at this stage.

IV. PREVIOUS THEORETICAL ANALYSES OF O^{16}

General

Dennison[15] has calculated the excited states of a system of four alpha particles coupled by harmonic-oscillator forces, and has shown that by fitting the four constants in this model to four of the low-lying states in O^{16}, one is able to account fairly well for the observed spectrum of O^{16} up to about 13.5 Mev. Kameny[16] has extended the work of Dennison to 16 Mev, and has calculated some of the implications of this model for transition probabilities. In both the identification

[6] J. W. Bittner and R. D. Moffat, Phys. Rev. **96**, 374 (1954).
[7] W. F. Hornyak and R. Sherr, Phys. Rev. **100**, 1409 (1955).
[8] Wilkinson, Toppel, and Alburger, Phys. Rev. **101**, 673 (1956).
[9] F. Azjenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1955).
[10] Devons, Goldring, and Lindsay, Proc. Phys. Soc. (London) **A67**, 134 (1954).
[11] B. M. Spicer, Phys. Rev. **99**, 33 (1955).
[12] Stephens, Mann, Patton, and Winhold, Phys. Rev. **98**, 839 (1955).
[13] D. H. Wilkinson and S. D. Bloom, Phys. Rev. **105**, 685 (1957).
[14] D. H. Wilkinson, Phys. Rev. **99**, 1347 (1955).
[15] D. M. Dennison, Phys. Rev. **57**, 454 (1940); **96**, 378 (1954).
[16] S. Kameny, Phys. Rev. **103**, 358 (1956).

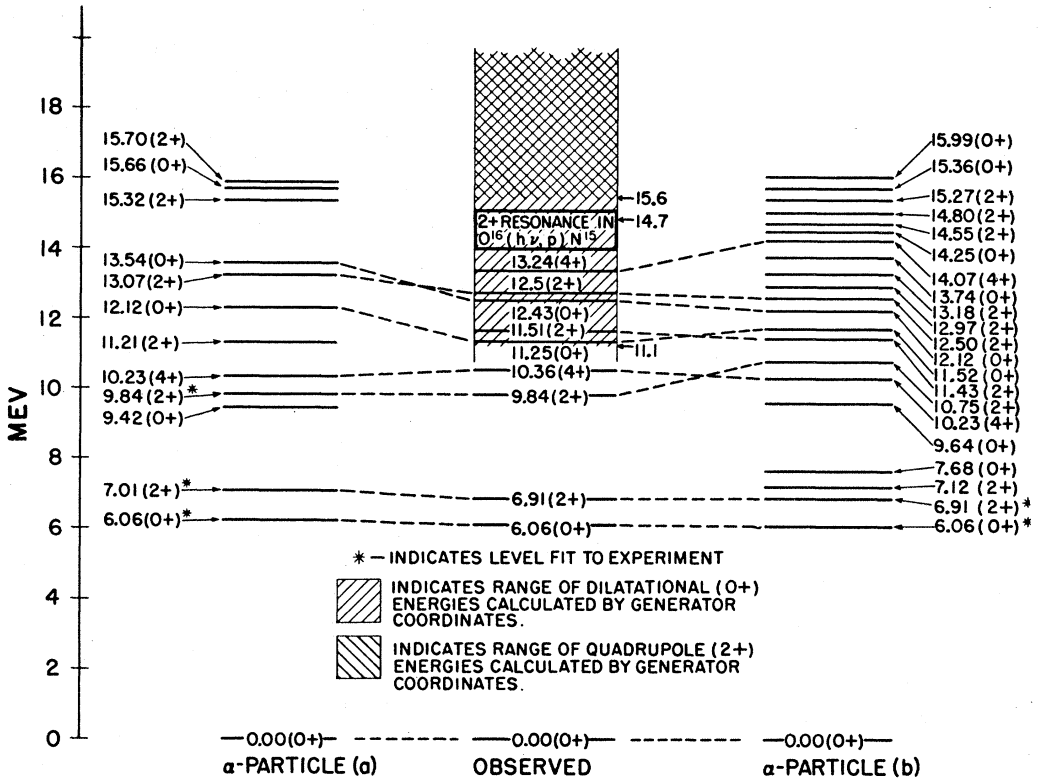

FIG. 1. 0^+, 2^+, and 4^+ levels in O^{16}. The center column gives the observed levels (energy, spin and parity), the resonance observed in $O^{16}(\gamma,p)N^{15}$, and the range of dilatational and quadrupole energies calculated in this paper (cross hatching). The first and third columns give the levels on two identification schemes in the α-particle model. The levels fit to the data to determine the constants in the α-particle model are indicated by an asterisk. The α-particle levels are connected to the observed levels which presumably correspond to them by dotted lines. The material for this figure was taken from references 6, 11, 12, 15, and 16.

schemes based on this alpha-particle model, the 0^+ state at 6.06 Mev is interpreted as a pure breathing mode. The matrix element for the

$$0^+(6.06\text{ Mev})\rightarrow 0^+(0.00\text{ Mev})$$

transition is given by Kameny: 14.7×10^{-26} cm^2. The energies of the 0^+ and 2^+ states based on the two proposed identifications are shown in Fig. 2.

Perring and Skyrme[17] begin with an alpha-particle model, but show that the wave functions for states of four alpha particles may be used to obtain approximate shell-model wave functions. In their treatment, they obtain for the breathing mode of the α model a shell-model wave function which is largely a mixture of $(1s)^{-1}2s$ and $(1p)^{-1}2p$ configurations, but also includes some admixture of two-particle excitations: $(1p)^{-2}(1d)^2$, $(1p)^{-2}(1d)(2s)$, and $(1p)^{-2}(2s)^2$. This wave function yields a matrix element (as defined in Table IV) for the transition $0^+(6.06)\rightarrow 0^+(0.00)$ equal to 11×10^{-26} cm^2. This work is also especially interesting as regards the more general question of the relationships between the shell model and the alpha-particle model.

[17] J. K. Perring and T. H. R. Skyrme, Proc. Phys. Soc. (London) **H69**, 600 (1956).

0^+ State

Visscher and Ferrell[18] point out that the generator wave function obtained in the preceding paper[1] for the first excited dilatational mode is approximately a shell model wave function composed of these same two configurations, $(1s)^{-1}(2s)$ and $(1p)^{-1}(2p)$. They obtain an energy of 9.1 Mev for the first excited dilatational state, which is significantly lower than the energy obtained from generator coordinates with identical two-body interactions (see Table II). In a later paper[19] they report that an energy of 6.65 Mev can be obtained for a dilatational state in which one allows different amplitudes of dilation for the s and p shells of O^{16}.

It would seem at first as though these results were in absolute conflict with the results reported here, since the Ferrell-Visscher wave function is an approximation to the generator wave function (variational), and would therefore be expected at best to yield a higher excitation energy. However, these authors do not actually calculate energies all the way through from first principles —that is, solely from assumed two-body forces and a trial wave function. Instead, they determine the diag-

[18] R. A. Ferrell and W. M. Visscher, Phys. Rev. **102**, 450 (1956).
[19] R. A. Ferrell and W. Visscher, Phys. Rev. **104**, 475 (1956).

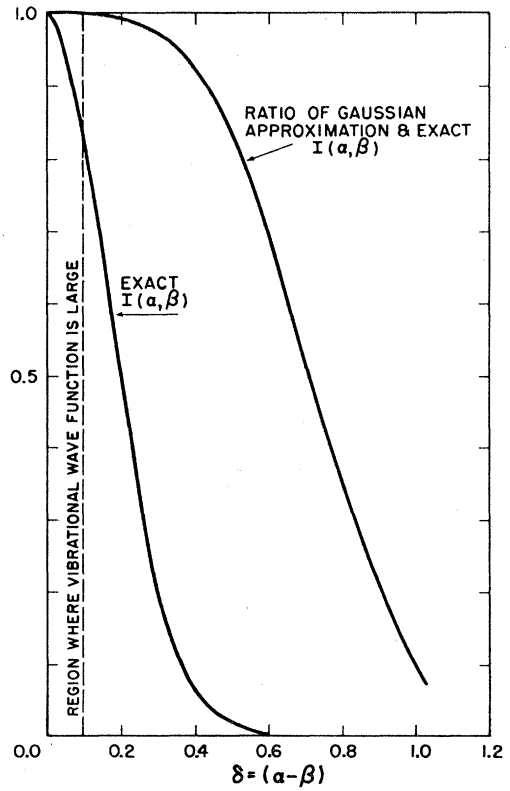

FIG. 2. The exact overlap kernel for dilatational deformations of the wave functions used in this paper, together with the ratio of the Gaussian approximation and this exact kernel. It can be seen that the Gaussian approximation is quite good, being in error by only 10% when the exact kernel has dropped to 0.04 of its maximum.

onal matrix element for the excited state from what they know empirically about energy levels in neighboring nuclei. Their calculation is therefore subject to the uncertainty associated with such a substitution, and the final excitation energy they obtain will be in error by the same energy as the error in this substitution.[20] Naturally, their method, containing a disposable constant, may well yield results in better agreement with observation than does an approach like the present one, which is uniquely specified as soon as the two-body forces have been chosen.

Their result that the energy of a dilatational state can be lowered by allowing the two shells to dilate with different amplitudes is undoubtedly valid in a more general context than any particular set of numerical results, and indicates one of the complications one might expect to occur in collective motions of this kind.

Ferrell and Visscher have also calculated the matrix element for pair emission from the 6.06-Mev 0^+ state, and find that it is too large to agree with the observations [Eq. (12)] by about a factor of two. Allowance for different amplitudes of vibration in the two shells does not seem to change this matrix element significantly.

Schiff[21] has calculated the matrix element for pair emission from the 6.06-Mev state in O^{16} and the 7.68-state in C^{12} (which probably corresponds to the O^{16} state) on two assumptions, (*a*) that the 7.68-Mev state in C^{12} is a 2-particle shell-model state, $(p_{\frac{3}{2}})^{-2}(p_{\frac{1}{2}})^2$ and (*b*) on the assumption that the 6.06-Mev state in O^{16} is a collective dilatational vibration (using both the liquid-drop and α-particle model). In case (*a*) the matrix element is too small, in case (*b*), too large. Elliott[22] points out that inclusion of $(2s)$ and $(2p)$ excitations to the extent of 50% would bring Schiff's shell-model calculation into agreement with the experimental value.

Redmond[23] has shown that a 0^+ state formed from shell-model configurations of the type $(1s)^{-1}(2s)$ gives a matrix element which matches the experimental value rather well. He gives no reason, however, for believing that the 0^+ state which is observed is an excitation only of $(1s)$ nucleons, rather than $(1p)$, or, more important, a linear combination of $(1s)^{-1}(2s)$ and $(1p)^{-1}(2p)$ excitations. These latter possibilities are those which occur in the wave functions used by Ferrell and Visscher and Perring and Skyrme.

In Table IV, the various theoretical values for the monopole matrix element are summarized together with the experimental value. *Note added in proof.*—J. Touchard, Compt. rend. **244**, 2499 (1957), also discusses the 0^+ state in terms of collective dilatational mode.

V. RESULTS OF THE ANALYSIS BY GENERATOR COORDINATES

We now discuss briefly the results presented in Tables II and III. It can be seen that the values obtained for the stiffness, $\mathcal{K}$, for the quadrupole vibrations are significantly higher than those obtained from the semi-empirical mass formula. This difference is qualitatively consonant with the well-known fact that vibrations of nuclei with closed shells are far stiffer than vibrations of nuclei halfway between closed shells. This behavior shows up, not only in theoretical analyses of shell effects on collective behavior,[24] but also in empirical surveys of level regularities.[25]

The calculated inertial parameter, $\mathfrak{M}$, on the other hand is quite close to the value implied by the assumption of irrotational flow. This, again, is in qualitative agreement with analyses of collective inertia,[24] for closed shell nuclei.

The energy spacing, $\hbar\Omega$, of the collective states seems in every case too high to allow identification of the 6.06-Mev and 6.91-Mev states of O^{16} with modes of

[20] This question has been discussed with Dr. Ferrell and Dr. Visscher, and they concur in this statement of the situation.

[21] L. Schiff, Phys. Rev. **98**, 1281 (1955).
[22] J. P. Elliott, Phys. Rev. **101**, 1212 (1956).
[23] P. J. Redmond, Phys. Rev. **101**, 751 (1956).
[24] S. Moskowski, Phys. Rev. **103**, 1328 (1956).
[25] Alder, Bohr, Huus, Mottelson, and Winther, Revs. Modern Phys. **28**, 432 (1956).

dilatational and quadrupole surface vibration. It also seems unlikely that any reasonable choice for the two-body interactions could alter this situation. However, it should be pointed out that the strength of the two-body interactions (for the exchange mixtures used here) required to give a radius $R=1.2\times A^{\frac{1}{3}}\times 10^{-13}$ cm are unrealistically large. This circumstance may well make the calculated rigidity of the nucleus also in these cases unrealistically large. It cannot be excluded that this effect is large enough to account for (1) the very high-energy spacings obtained for dilatational modes when $R=1.2\times 10^{-13}A^{\frac{1}{3}}$ cm and (2) the failure of the collective *quadrupole* vibrational wave function to lower the ground-state energy of the system.

VI. SUMMARY OF THE EXPERIMENTAL AND THEORETICAL SITUATION

Dilatational Modes

One would like an unequivocal answer to the question. Is the state at 6.06 Mev a dilatational state? This question can perhaps be discussed in two parts, referring to the two relevant available data, (*a*) the transition between it and the ground state and (*b*) its energy.

The transition matrix element is more amenable to discussion in general terms since it involves simply the wave functions one chooses to describe the states. In this framework, the implications of the various theoretical analyses displayed in Table IV are clear: Shell-model states involving two particles give too small a value, while collective dilatational states, and shell model states resembling them, give too large a value for this matrix element. The obvious inference supports Elliott's suggestion that a mixture of the two is what is required to explain the experimental fact.

The energies obtained for the first excited dilatational state (Table II) are not inconsistent with the hypothesis that the 6.06-Mev state is, in fact, a mixture of shell-model components corresponding to dilatational and two-particle excitations: These energies are sufficiently high to contradict the assumption that the collective motion is slow compared to the motion of the individual nucleons. Therefore, such states will not exist in pure form, but will instead mix rather strongly with nearby noncollective states. One can suppose that the state at 6.06 Mev is one of the resulting states.

Quadrupole Excitations

In the case of the 2^+ state at 6.91 Mev the lifetime has been measured.[26] The situation in this case seems the same as that of the 0^+ state: the lifetime is longer than that implied by collective models, and shorter than that obtained from two-particle excitations.

The energies obtained in this paper for quadrupole excitations are sufficiently large to suggest that the 2^+ state at 6.91 Mev is not a state of pure surface vibration. More generally, they imply that states of this kind will probably not appear in pure form, since the collective frequency is of the order of single-particle frequencies. Thus, the quadrupole state, if it occurred, would be mixed with other nearby 2^+ states, and would behave in the manner first proposed by Wilkinson to explain the observed results. Since this proposal has been withdrawn on the basis of later data, it is difficult to make any clear statement on experimental grounds.

[26] C. Swann and F. Metzger (unpublished).

TABLE IV. Calculated values of the matrix element for the $0^+(6.06\text{ Mev})\rightarrow 0^+(0.00\text{ Mev})$ transition in O^{16}. The numerical values are somewhat affected by the specific assumptions involved in each calculation but the qualitative increase in the matrix element as one increases the dilatational component in the wave function will not be changed by these details. Also, the quantitative mixture required to fit the experiment in the state proposed by Elliott will depend somewhat upon the particular linear combination of the configurations which is used. Nonetheless, it is clear that with enough admixture of two-particle excitations the correct matrix element can certainly be obtained. The same kind of situation prevails as regards the calculation of Redmond, whose result depends on the radius assumed for O^{16}. A footnote is used to indicate those assumptions which are capable of giving agreement with the experimental matrix element, provided the parameters of the wave functions are properly chosen, but which do not give this agreement uniquely.

Reference	$\langle M\rangle=\langle\Psi_0\vert\Sigma_p r_p^2\vert 1\rangle$	Model of 0^+ state
Devons *et al.*[a]	$(3.8\pm0.3)\times10^{-26}$ cm^2	Experimental
Schiff[b]	0.00	Pure two-excited-particle configurations
Schiff[c]	$\sim0.6\times10^{-26}$ cm^2	Two-particle configurations mixed by nucleon-nucleon forces
Redmond[d]	Experimental value[i]	$(1s)^{-1}(2s)$ state
Elliot[e]	Experimental value[i]	$\sim50\%$ $(1s)^{-1}(2s)$, $(1p)^{-1}(2p)$ $\sim50\%$ $(1p)^{-2}(1d)^2$, $(1p)^{-2}(2s)^2, (1p)^{-2}(1d)(2s)$
Ferrell and Visscher[f]	9×10^{-26} cm^2	Dilatational: linear combination of $(1s)^{-1}2s$ and $(1p)^{-1}(2p)$
Perring and Skyrme[g]	11×10^{-26} cm^2	Shell-model wave function obtained from α-model dilatational state
Kameny[h]	14.7×10^{-26} cm^2	α-particle model
	17×10^{-26} cm^2	α-particle model
Schiff[b]	19×10^{-26} cm^2	Liquid-drop model
Griffin	$(17\text{ to }22)\times10^{-26}$ cm^2	Generator coordinates

[a] See reference 10.
[b] See reference 21.
[c] Schiff actually calculates the matrix element for mixed configurations only in C^{12}, but it is unlikely that the result would be qualitatively different for O^{16}.
[d] See reference 23.
[e] See reference 22.
[f] See reference 18.
[g] See reference 17.
[h] See reference 16.
[i] Adjusted to agree with the experimental value of 3.8×10^{-26} cm^2 by suitable choice of constants in the wave function.

General

The reasonable inference regarding the question of collective states in O^{16}, seems to be that they probably do not exist in pure form. It is still possible that they

occur at significant admixtures into other states, which may exhibit, as a result, some effects most simply understood in collective terms.

As regards collective vibrations in other light nuclei, it is reasonable to believe that closed-shell nuclei are probably the most unfavorable to the existence of well-defined collective modes, since[24] these nuclei tend to be stiffer against deformation and have a lower collective inertia than their neighbors. Both effects operate to increase the energy of excited states, and therefore to invalidate the hypothesis that the collective motion is slow compared with the nucleonic motion. One would expect that adding a pair, or a few pairs, of particles to a closed shell nucleus will enhance the tendency towards low-lying states of collective vibration.

The author wishes to express his gratitude to Professor John Wheeler, who suggested this research, to Professor Niels Bohr for the hospitality of Universitetets Institut for Teoretisk Fysik, Copenhagen, and to the Fulbright Commission for a fellowship in Denmark during 1955–1956, and to acknowledge stimulating discussions with Professor Aage Bohr, Dr. Ben R. Mottelson, and Dr. William Visscher.

APPENDIX I. CALCULATION OF THE KERNELS IN THE GENERATOR EQUATION

A. Overlap Kernel, $I(\alpha,\beta)$

The overlap kernel is given by the expression

$$I(\alpha,\beta)=\int \phi^*(\mathbf{x}_1,\cdots,\mathbf{x}_A;\alpha) \times\phi(\mathbf{x}_1,\cdots,\mathbf{x}_A;\beta)d\mathbf{x}_1 d\mathbf{x}_2\cdots d\mathbf{x}_A, \quad (13)$$

where $\phi(\mathbf{x}_1,\cdots,\mathbf{x}_A)$ is an antisymmetrized product of one-particle oscillator states:

$$U_j(x_j;\alpha)=N_i H_{i_1}(k_x x_j)H_{i_2}(k_y x_j)H_{i_3}(k_z z_j) \times\exp[-\tfrac{1}{2}(k_x^2x^2+k_y^2y^2+k_z^2z^2)]\pi(\sigma_z)\nu(\tau_z). \quad (14)$$

Here i denotes a quintuplet of quantum numbers $(i_1, i_2, i_3, \sigma_z, \tau_z)$, and k depends on α according to the deformation considered:

$$\left.\begin{aligned} &k_y=k_x=k_z=k_0e^{-\alpha} \quad \text{for dilatations,}\\ &k_x=k_y=k_0e^{\alpha}\\ &k_z=k_0e^{-2\alpha} \end{aligned}\right\}\text{for quadrupole deformations.} \quad (15)$$

Direct integration yields the result

$$[\delta=\alpha-\beta;\ \gamma=(\alpha+\beta)/2]:$$
$$I(\alpha,\beta)=(\cosh\delta)^{-2s}. \quad (16)$$

The value of the constant is

$s=18$ for dilatational deformations,

$s=9$ for quadrupole deformations.

This expression for I is approximately equal to the Gaussian

$$I(\alpha,\beta)-\exp[-s\delta^2]. \quad (17)$$

The accuracy of this approximation for the dilatational case is illustrated in Fig. 2.

B. Hamiltonian Kernel, $K(\alpha,\beta)$

The Hamiltonian kernel is given by the expression

$$K(\alpha,\beta)=\int\phi^*(\mathbf{x}_1,\cdots\mathbf{x}_A;\alpha)\left[\frac{-\hbar^2}{2M}\sum_i^A\nabla_i^2+\frac{1}{2}\sum_{i\neq j}^A V_{ij} -\frac{1}{2}\sum_{i\neq j}^A\frac{e^2}{|r_{ij}|}\right]\phi(\mathbf{x}_1,\cdots\mathbf{x}_A;\beta)d\mathbf{x}_1\cdots d\mathbf{x}_A, \quad (18)$$

which we write as the sum of three parts

$$K(\alpha,\beta)=K_T(\alpha,\beta)+K_N(\alpha,\beta)+K_C(\alpha,\beta) \quad (19)$$

corresponding to the kinetic energy operator, nuclear interaction, and Coulomb interaction portions, respectively.

Direct integration of the kinetic part yields[27]

$$K_T(\alpha,\beta)=\begin{cases} I(\alpha,\beta)\dfrac{18\hbar^2k_0^2}{M}\dfrac{e^{-2\gamma}}{\cosh\delta} & \text{(dilatations)}\\ I(\alpha,\beta)\dfrac{6\hbar^2k_0^2}{M}\left[\dfrac{2e^{\gamma}}{\cosh\delta/2}+\dfrac{e^{-2\gamma}}{\cosh\delta}\right]. & \text{(quadrupole deformations).}\end{cases} \quad (20)$$

The nuclear interaction part of the Hamiltonian kernel involves evaluation of matrix elements of the two-body interaction

$$V_{ij}=-V_0\exp-\left(\frac{r_{ij}}{R}\right)^2\{w+mP^m+bP^b+hP^h\}. \quad (21)$$

P^m, P^b, and P^h are the space, spin, and space and spin exchange operators. After one has summed over all spin and isotopic-spin quantum numbers in the closed shells, the sum of these matrix elements reduces to[28]

[27] Dr. Visscher points out that one should subtract the kinetic energy corresponding to the center-of-mass motion, $\frac{3}{4}\hbar^2k_0^2/M$, from the total kinetic energy. If this is done, the 18 is replaced by 17.25 and the 6 by 5.75 in these expressions. In the calculations reported, this correction was made only for the case of the Ferrell-Visscher forces. See caption, Table II.

[28] This shows that results can depend on the exchange parameters only through combinations $d=4w-m+2(b-h)$, $e=4m-w+2(h-b)$. The letters d and e refer to the direct and exchange integrals of which these are the coefficients. For the three mixtures used in this paper the values of d and e are as follows: Rosenfeld, 0.01, 2.41; simplified, −0.04, +2.8; Ferrell-Visscher, 0.21, 2.27.

$$2\sum_{i,j}^{4}\int\int U_i^*(\mathbf{x}_1;\alpha)U_j^*(\mathbf{x}_2;\alpha)\left[-V_0\exp-\left(\frac{r_{12}}{R}\right)^2\right]$$
$$\times[(4w-m+2b-2h)$$
$$+(4m-w+2h-2b)P_{12}{}^m]$$
$$\times U_j(\mathbf{x}_2;\alpha)U_i(\mathbf{x}_1;\alpha)dx_1dx_2. \quad (22)$$

The final result for dilatations can be written mostly simply in terms of the combination, $\lambda=(k_0{}^2R^2)e^{-2\gamma}\cosh\delta$:

$$K_N(\alpha,\beta)=\frac{-V_0I(\alpha,\beta)^{\frac{3}{2}}}{(2+\lambda)^{7/2}}\{A+B\lambda+C\lambda^2\} \quad \text{(dilatations)}, \quad (23)$$

where

$$A=186(w+m),$$
$$B=288w+48m+96(b-h),$$
$$C=120w+48(b-h).$$

For quadrupole deformations, the result is slightly more complicated. Using λ and $\mu=(k_0{}^2R^2)e^{\gamma}\cosh(\frac{1}{2}\delta)$ for condensation, one obtains

$$K_n(\alpha,\beta)=-V_0I(\alpha,\beta)\mu\lambda^{\frac{1}{2}}\left\{\frac{A_1+B_1\mu+C_1\mu^2}{(2+\mu)^3(2+\lambda)^{\frac{3}{2}}}\right.$$
$$\left.+\frac{A_2+B_2\mu+B_3\lambda+C_2\lambda\mu}{(2+\mu)^2(2+\lambda)^{\frac{3}{2}}}+\frac{A_3+B_4\lambda+C_3\lambda^2}{(2+\mu)(2+\lambda)^{\frac{3}{2}}}\right\},$$
$$\text{(quadrupole)} \quad (24)$$

where

$$A_1=120(m+w),$$
$$B_1=168w+48m+48(b-h),$$
$$C_1=66w+6m+24(b-w),$$
$$A_2=48(m+w),$$
$$B_2=44w+4m+16(b-h),$$
$$B_3=64w-16m+32(b-h),$$
$$C_2=48w-12m+24(b-h),$$
$$A_3=18(w+m),$$
$$B_4=12(w+m),$$
$$C_3=6(w+m).$$

C. Finally, the Coulomb kernel, K_C, defined by

$$K_C(\alpha,\beta)=\frac{1}{2}\int\phi^*(\mathbf{x}_1,\cdots\mathbf{x}_A;\alpha)\left[\sum_{i\neq j}\frac{e^2}{|r_{ij}|}\right]$$
$$\times\phi(\mathbf{x}_1,\cdots\mathbf{x}_A;\beta)d\mathbf{x}_1\cdots d\mathbf{x}_A \quad (25)$$

was integrated exactly for dilatational deformations, with the result

$$K_C(\alpha,\beta)=I(\alpha,\beta)\frac{83e^2k_0}{2(2\pi)^{\frac{1}{2}}}\exp(-\gamma(\cosh\delta)^{\frac{1}{2}}$$
$$\text{(dilatations)}. \quad (26)$$

For the spheroidal deformations the expansion in δ and γ was carried out before integration, and the resulting coefficients were evaluated approximately.

Once the complete kernel, $K(\alpha,\beta)$, is obtained the quadratic approximation is made by expanding the ratio, $K(\alpha,\beta)/I(\alpha,\beta)$ to terms of order δ^2 and γ^2. The inertia, $\mathfrak{M}$, and stiffness, $\mathcal{K}$, are then determined from the coefficients of δ^2 and γ^2 as described in the preceding paper.[1] The results are given in Tables II and III.