not the case in a recent calculation by Jenkins,[5] who used a variational cellular method. It is difficult to determine the position of the minimum of the conduction bands, because the energies at Γ_{15} and at $X_1(2)$ are almost equal and the interpolation scheme is only approximately valid. Since, however, it has been shown[2] that the value of the energy at Γ_{15} decreases very little if plane waves of higher energy are taken into account,

it is to be expected that the minimum should lie near the point $\mathbf{K}=2\pi a^{-1}(1,0,0)$, as predicted by Herman.[6]

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[6] F. Herman, Phys. Rev. 95, 847 (1954).

Effect of Impurity Scattering on the Magnetoresistance of n-Type Germanium

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Observations in n-type germanium show a decrease in the anisotropy of the magnetoresistance with increased ionized-impurity content and with decreased temperature. Although similar to what is expected due to ionized-impurity scattering, the details of the variation do not agree with calculations assuming anisotropic impurity scattering. Deviations from the magnetoresistance symmetry conditions are observed for crystals with electron concentration $n>4\times10^{15}$ cm^{-3} at 77°K.

THE interpretation[1,2] of magnetoresistance observations[3] in n-type germanium has confirmed the (111) symmetry of the band structure deduced from cyclotron resonance experiments[4,5] at low temperature. However, there is apparent lack of agreement between the anisotropy observed[6–8] in the low-field galvanomagnetic effects and that expected from the cyclotron-resonance effective masses. It has also been observed that the magnetoresistance anisotropy seems to be a function of the concentration of impurities and the temperature at which the measurements were made. In this paper, experimental data pointing out this effect are presented, and possible explanations are discussed.

The magnetoresistance coefficients[3] b, c, and d are defined in Eq. (1). This expression describes the magnetoresistance for a cubic crystal in the low-field region, i.e., where the change in resistivity is proportional to the square of the magnetic field.

$$\frac{\Delta\rho}{\rho H^2}=b+c\frac{(\mathbf{I}\cdot\mathbf{H})^2}{I^2H^2}+d\frac{\sum_{i=1}^{3} I_i^2 H_i^2}{I^2H^2}. \qquad (1)$$

In this equation ρ is the resistivity, and $\mathbf{I}$ and $\mathbf{H}$ are

the current and magnetic field, with the components taken along the cubic axes. For a conduction-band structure like that in germanium, the coefficients satisfy the conditions:

$$b+c=0, \quad d>0. \qquad (2)$$

These coefficients are functions of the relaxation time τ and the shape of the energy surfaces, as represented by the ratio of longitudinal to transverse effective masses, K_m. If a specific form is assumed for the dependence of τ on energy, K_m can be calculated directly from the magnetoresistance. If, in addition to the magnetoresistance, the Hall coefficient R and the conductivity σ are known, K_m can be calculated, provided that τ is a function only of the energy of the carriers. Observations of the quantity

$$W\equiv\frac{b+c+d}{b+(R_0\sigma_0)^2} \qquad (3)$$

can be used to calculate[9] K from Eq. (4).

$$W=2\left(\frac{K^2-2K+1}{2K^2+5K+2}\right). \qquad (4)$$

Here K is the ratio of the mass anisotropy K_m to the relaxation-time anisotropy K_r. The latter is defined as the ratio of longitudinal to transverse scattering times in the same directions as the mass ellipsoids. If τ is a

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function of the energy only, K_τ is unity and K is equal to K_m.

The low-field data, analyzed in this way, yielded a considerably lower value for K than the 19.3 value[4] for K_m derived from cyclotron resonance observations. The Pearson and Suhl data[3] give 10–11; Benedek, Paul, and Brooks[6] calculated 14.5; Goldberg and Davis[7] report values close to 12, and the author has also reported[8] values between 12 and 13. On the other hand, the same parameter K calculated from measurements of high-field magnetoresistance by Furth and Waniek[10] and by Broudy and Venables[11] has values close to the cyclotron resonance value. Recent magnetoconductivity experiments by Goldberg[12] also give close to 20 for this ratio. There is thus a discrepancy between values of the same parameter K calculated from these somewhat different experiments. The observations to be described were
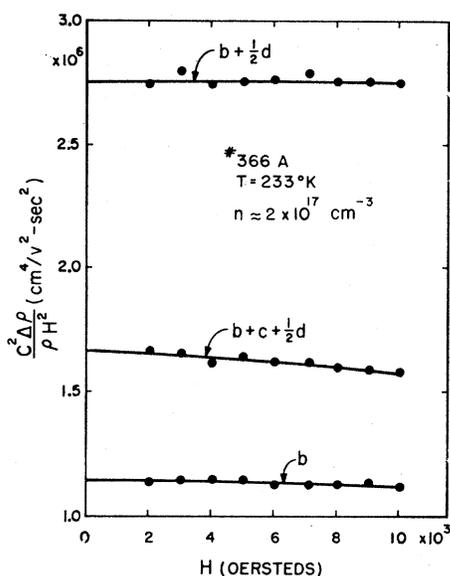

FIG. 1. The magnetic-field dependence of the magnetoresistance for a heavily doped crystal of n-type germanium.

made to investigate the effect of impurity scattering on the low-field magnetoresistance values K.

The measurements were of the conventional dc type, making use of bridge-shape samples.[3] The Hall effect and magnetoresistance effects were measured as a function of the magnetic field, in the range 500 to 10 000 oersteds, in order to search out the low-field region, where the Hall voltage is proportional to the magnetic field and the magnetoresistance proportional to the square of the field. Sufficient observations were made to calculate the three coefficients b, c, and d.

In Figs. 1 and 2, typical results for the magnetoresistance as a function of magnetic field are plotted.

[10] H. P. Furth and R. W. Waniek, Phys. Rev. **104**, 343 (1956).
[11] R. M. Broudy and J. D. Venables, Phys. Rev. **103**, 1129 (1956).
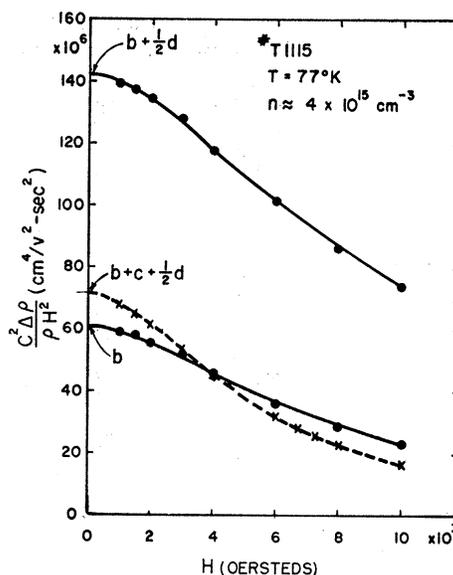[12] C. Goldberg, Bull. Am. Phys. Soc. Ser. II, **2**, 65 (1957).

FIG. 2. The magnetic-field dependence of the magnetoresistance in a crystal with intermediate doping.

The crystal described in Fig. 1 had a concentration of electrons of 2×10^{17} cm^{-3}. The relative change in resistivity has been divided by $(H/c)^2$, where c (here only) is the velocity of light, so that small deviations from a horizontal line are deviations from the low-field region. The magnetoresistance plotted here is in the low-field region over most of the region of magnetic field investigated. In Fig. 2 the same type of results are presented for a purer specimen (with an electron concentration $n=4\times10^{15}$ cm^{-3}) at liquid-nitrogen temperatures. Although the observations do not show a horizontal region above 1000 oersteds, they show signs of levelling off below that field value, and the curves have been extrapolated to zero field on the basis of the changes in curvature evinced in the 1–3000 oersted region. The difference in magnetic field dependence of these two figures is due to the great difference in mobility, as is evidenced by the fact that the magnetoresistance of
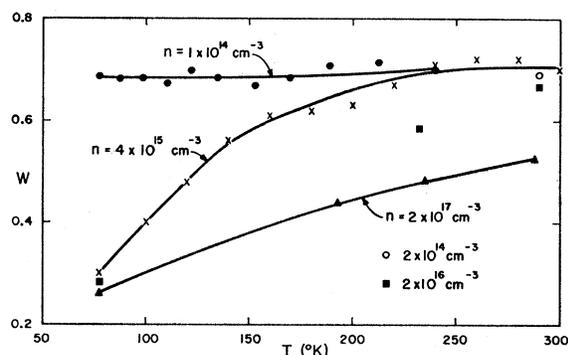

FIG. 3. Experimental values of the function W as a function of temperature. The curve labelled $n=1\times10^{14}$ cm^{-3} is given in reference 7.

TABLE I. Magnetoresistance of n-type germanium.

Crystal	Electron concentration (cm⁻³)	Temp. (°K)	Hall mobility (cm²/v-sec)	Magnetoresistance coefficients		
				b	c (10^6 cm⁴/v² —sec²)	d
674	2×10^{14}	289	4250	8.6	−8.0	18.0
1115	4×10^{15}	298	3540	4.70	−4.82	12.3
		200	6370	12.9	−13.0	33.8
		150	9500	22.1	−21.8	65.8
		100	15 800	44.4	−45.2	117.6
		77	21 050	61.3	−72.4	164.8
1084	1×10^{16}	294	3135	3.12	−3.04	8.54
784	2×10^{16}	289	2940	2.8	−2.6	7.5
		232	4140	4.25	−4.20	12.4
		193	4950	5.5		17.8
		77	10 650	15.5	−19.1	39.8
366	2×10^{17}	287	2030	0.87	−0.85	2.58
		234	2370	1.14	−1.14	3.28
		193	2685	1.38	−1.41	3.78
		77	3040	1.92	−2.23	3.22
1336	5×10^{18}	293	525	0.057	0.109	0.025
		77	600	0.076	0.177	0.006

the purer specimen was 50 times larger than that of the impure one. Curves of these types were taken for a number of specimens, ranging in electron concentration (and impurity content) from $n=10^{14}$ to 5×10^{18} cm⁻³. Some of the observations are presented in Table I. The values for the experimental ratio W for all except the last of these are plotted in Fig. 3, as a function of the temperature. W is determined by the coefficients b, c, and d and the Hall mobility $R\sigma$. If K_τ is assumed to be unity, W should only be a function of the energy-surface anisotropy.

The top curve, labelled $n=1\times10^{14}$ cm⁻³, shows the observations of Goldberg and Davis,[7] in which no change in the factor W was seen, on going down to liquid-nitrogen temperatures. The data reported here for electron concentrations $n=2\times10^{14}$ and 4×10^{15} cm⁻³ above 250°K agree with these measurements, which gave a K of about 12. However, for 4×10^{15} there is a systematic change with temperature to lower values of W, which would yield apparently lower values of K_m (the ratio of effective masses). For 2×10^{16} cm⁻³
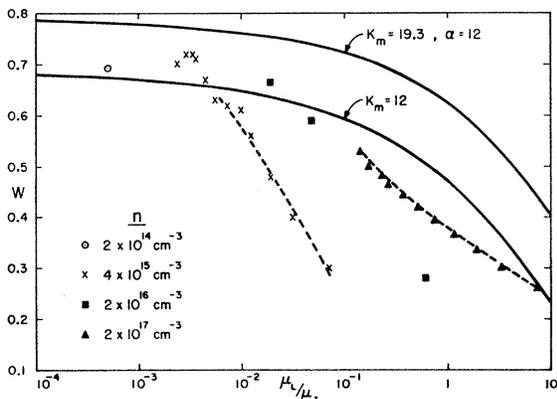

FIG. 4. The function W as a function of μ_L/μ_I. The solid curves are calculated; for the experimental points μ_L/μ_I is calculated from the measured mobilities.

the same behavior occurs, with a somewhat lower value even at room temperature, and for 2×10^{17} cm⁻³ similar behavior, with a still smaller value at room temperature. It should be pointed out that for all these curves, the (111) symmetry conditions were also investigated. They were obeyed fairly well for all except the three points at 77°K for which $W\approx0.3$, and also perhaps in the temperature range just above this for the sample with 2×10^{17} cm⁻³ concentration.

It was suggested by Ham[13] that for an energy-band surface like that of the electrons in germanium, scattering by ionized impurities would be strongly anisotropic, i.e., depend on the angle of scattering, so that the resulting scattering relaxation time must be represented by a tensor. He calculated that this tensor would be diagonal in the same representation as the effective-mass tensor, with a K_τ of about 12, if the energy surfaces have the anisotropy given by cyclotron resonance. This holds for the impurity scattering if there is also appreciable lattice scattering present. The magneto-resistance and Hall effect for mixed lattice and impurity scattering have been calculated, assuming the scattering to be represented by the relaxation times shown below:

$$\tau_L = A\,\epsilon^{-0.66}, \tag{5}$$

$$\tau_I = B\epsilon^{1.5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \tag{6}$$

$$\frac{1}{\tau} = \frac{1}{\tau_L} + \frac{1}{\tau_I}. \tag{7}$$

τ_L represents the lattice scattering, in which the deviations from a $T^{-1.5}$ mobility dependence have been attributed to the dependence on electron energy ϵ; τ_I represents ionized-impurity scattering, with α a

[13] F. Ham, Phys. Rev. 100, 1251 (A) (1955).

parameter giving the anisotropy of this scattering. Of course, the τ for combined scattering now has *different* energy dependences for the diagonal terms. A is a function of the temperature, and is taken from the measured values of the Hall mobility for the purest specimens.

New expressions for the Hall mobility and W were calculated in terms of a group of current integrals $J_{t^n l^m}$.

$$R_0\sigma_0 = 0.0983\mu_L\left(\frac{2K_mJ_{tl}+K_m^2J_{t^2}}{2K_mJ_t+J_l}\right), \tag{8}$$

$$W = 2\left(\frac{K_m^2J_{t^3}-2K_mJ_{t^2l}+J_{tl^2}}{2K_m^2J_{t^3}+5K_mJ_{t^2l}+2J_{tl^2}}\right), \tag{9}$$

$$J_{t^n l^m} = \alpha^m\int_0^\infty \frac{x^{\frac{3}{2}}e^{-x}dx}{[0.0707x^{0.66}+(\mu_L/\mu_I)x^{-1.5}]^n[0.8484x^{0.66}+(\mu_L/\mu_I)x^{-1.5}]^m}, \tag{10}$$

These include as a parameter the relative strength of the two types of scattering, given by μ_L/μ_I. In this expression for the J's, the K_τ for impurity scattering (the parameter α) has been taken[13] as 12. These integrals were evaluated numerically and from them μ_L/μ_I (not directly measurable) was calculated as a function of μ/μ_L (directly measurable). The function W was calculated and is shown in Fig. 4, plotted as a function of μ_L/μ_I. Pure lattice scattering is represented by $\mu_L/\mu_I=0$. The two curves represent results assuming $K_m=19.3$, and $K_m=12$, respectively. In both cases, the curves show some deviation from their values for pure lattice scattering for $\mu_L/\mu_I>10^{-3}$, and decrease fairly rapidly for $\mu_L/\mu_I>10^{-1}$. The measured points, previously shown for different concentrations as a function of temperature, are also plotted here. For each crystal (same n-value), the value at the smallest μ_L/μ_I is at room temperature. It can be seen that there is some correlation at *room temperature*, with a value of K_m of 12–14, but that otherwise there is no fit. It must be concluded that the observations cannot be explained *solely* on the basis of an impurity scattering mechanism of the type assumed.

At 77°K, it was noted that from $n=4\times10^{15}$ cm^{-3} to 2×10^{17} cm^{-3}, the magnetoresistance did not obey the symmetry conditions mentioned earlier for (111) energy surfaces. These conditions are that $b=-c$, and $d>0$. Although d remained positive, b became smaller than $-c$, i.e., the ratio $-c/b$ increased to values between 1.16 and 1.23 for these crystals. For a crystal of $n=5\times10^{18}$ cm^{-3}, b, c, and d were all positive at 300°K as well as at 77°K, i.e., there was no sign of the energy-band symmetry. These deviations from the conditions are also unexplained. Ham suggested[13] that very strong scattering by ionized impurities would give a magnetoresistance which did not obey the symmetry conditions. However, these deviations appear for crystals in which the electrons are still strongly scattered by the lattice.

In conclusion, changes in the magnetoresistance anisotropy have been observed for crystals with mixed impurity scattering and lattice scattering as compared to those with almost all lattice scattering. These observations are not completely explained by a theory which ascribes these changes to a simple type of anisotropic impurity scattering. Three possible reasons for the discrepancy can be suggested: (1) the theory used for the low-field magnetoresistance is at fault; (2) there is a different "type" of scattering entering, with an anisotropic temperature dependence; (3) the impurity scattering has an anisotropy which is strongly concentration dependent and/or temperature dependent. That such a dependence on concentration does occur seems reasonable, since the scattering will change from peaked forward (large anisotropy) at low concentrations, to more large-angle scattering (with less anisotropy) at higher concentrations. However, it is questionable that such a dependence can be sufficiently strong to explain the observations. It was also observed that the symmetry conditions fail somewhat worse (by 15–25%) at 77°K for $n>4\times10^{15}$ cm^{-3} than they do at higher temperatures. For $n=5\times10^{18}$ cm^{-3}, they are not at all observable.

It should be noted that there still remains the puzzle of the difference between the values 12 for K derived from low-field magnetoresistance and the larger values obtained from other measurements. Since the other observations also give K, and not K_m, it would appear that a value of K_τ different from 1 (for lattice scattering) would not fit the other observations.[10-12] The calculated slow variation of W at low impurity concentrations precludes an impurity-scattering explanation for this, as well.

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