nucleon number rather than strangeness quantum number.[23]

There seem to be good reasons why it is very difficult to discover experimentally this family of weakly interacting heavy bosons, at present and also in the near future, because members of this family have only very weak interactions with all other families and have very short lifetimes.

[23] The author owes his thanks to Dr. C. N. Yang for suggesting that the B meson and the X meson might have nucleon number one and zero, respectively.

ACKNOWLEDGMENT

I would like to express my gratitude to Professor Robert Oppenheimer for his kind hospitality at the Institute for Advanced Study.

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Solutions of the Static Theory Integral Equations for Pion-Nucleon Scattering in the One-Meson Approximation*

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Numerical solutions of the one-meson approximation of the Low equations for elastic pion-nucleon scattering in the fixed-nucleon, extended-source theory are obtained with a Gaussian cutoff function. The validity of the method requires that the scattering amplitudes have no zeros in the complex plane other than at $z=\pm 1$. The functions obtained for the (3,3) and (1,1) states satisfy the Low equations within the accuracy of the method, but the (1,3) and (3,1) states are only approximately given. This difficulty with the (1,3) and (3,1) states is correlated with the development of a zero in the corresponding scattering amplitude well before physically interesting values of the parameters (coupling constant and cutoff) are reached.

A best fit to the (3,3)-state data up to 170-Mev pion laboratory energy requires a coupling constant, f^2, less than 0.08. The solution is consistent with the (3,1)-state data, but gives a (1,1)-state phase shift of larger magnitude than experiment appears to permit.

It is found that the cutoff function does not prevent strong interactions at very high energies. Their occurrence appears to be a property of the static model. The contributions of such interactions to various static-theory calculations is briefly discussed. It is shown that in the p-wave part of the relativistic dispersion relations, where there is no cutoff function, if use is made of the low-energy data, then the very high-energy contributions of the static theory are replaced by recoil terms of order μ/M.

1. INTRODUCTION

THE formulation of the p-wave part of the pion-nucleon interaction proposed by Chew and Low[1] has been applied to a number of problems, including elastic pion-nucleon scattering,[1] photoproduction of mesons,[2] pion production in meson-nucleon collisions,[3] and calculation of the electromagnetic properties of nucleons.[4] Because of the degree of success that approximate methods have given in this theory, particularly in predicting the pion-nucleon resonance in the (3,3) state [isotopic spin $\frac{3}{2}$, spin $\frac{3}{2}$], it is of interest to obtain a solution of the Low equations for the scattering amplitudes. These equations are coupled to an infinite set of equations for other processes, and in principle the entire set should be solved simultaneously. However, in the one-meson approximation the scattering equations reduce to a set of three coupled equations which involve only the three scattering functions. We describe here results obtained in an attempt to find that solution of these three equations which is analytic in the coupling constant.[5]

The method of solution consists of an iterative procedure applied to the integral equations for the functions inverse to the scattering amplitudes.[1] These equations are valid if the scattering amplitudes have no zeros in the complex plane, other than thost at $z=\pm 1$, which is the case for sufficiently small values of the coupling constant. It has been conjectured[6] that this condition is maintained for values of the coupling

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[1] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[2] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1579 (1956).

[3] Saul Barshay, Phys. Rev. **103**, 1102 (1956); N. Fukuda and J. S. Kovacs, Phys. Rev. **104**, 1784 (1956); J. Franklin, Phys. Rev. **105**, 1101 (1957).

[4] H. Miyazawa, Phys. Rev. **101**, 1564 (1956); F. Zachariasen, Phys. Rev. **102**, 295 (1956); S. Fubini, Nuovo cimento **3**, 1425 (1956); S. Treiman and R. G. Sachs, Phys. Rev. **103**, 435 (1956); G. Salzman, Phys. Rev. **105**, 1076 (1957).

[5] These equations possess a large number of solutions, only one of which is analytic in the coupling constant. This was shown for both the charged and neutral scalar-meson theories by Castillejo, Dalitz, and Dyson, Phys. Rev. **101**, 453 (1956), who first called attention to the nonuniqueness of the solution. Klein has shown, Phys. Rev. **104**, 1136 (1956), that by taking appropriate linear combinations of the three scattering functions a set of equations is obtained which is equivalent to the original three equations, and whose solutions are generalized Wigner R functions. By this means the proof of nonuniqueness can be extended, as stated by Klein, to the symmetric pseudoscalar p-wave meson theory.

[6] G. F. Chew, *Encyclopedia of Physics* (Springer-Verlag, Berlin, to be published), second edition, Vol. 43.

constant large enough to provide physically interesting solutions, that is, solutions with the correct (3,3)-state resonance.

The neglect of nucleon motion in the static model implies of course that this theory must be regarded as provisional, and cannot be expected to agree with experiment for energies beyond the (3,3) resonance region. The low-energy part of the solution is compared with the experimental data in Sec. 6. Neglect of nucleon energy assumes that strong interactions do not occur at very high energies. The behavior of the high-energy part of the solution, although not of experimental interest, is thus important in determining the self-consistency of the static model, as is discussed in Secs. 5 and 7. The effect of nucleon recoil in modifying the static theory results is briefly examined in Sec. 8.

2. METHOD OF SOLUTION

Following the notation of Chew and Low, we introduce functions $h_\alpha(z)$, with $\alpha=1$, 2, and 3 for the (1,1), (1,3) and (3,1), and (3,3) states respectively. In the one-meson approximation, and for real values ω of z, $h_\alpha(z)$ satisfies the equation

$$h_\alpha(z)=\frac{\lambda_\alpha}{z}+\frac{1}{\pi}\int_1^\infty d\omega' p'^3v^2(p')\left\{\frac{|h_\alpha(\omega')|^2}{\omega'-z}+\sum_{\beta=1}^{3}A_{\alpha\beta}\frac{|h_\beta(\omega')|^2}{\omega'+z}\right\}, \quad (1)$$

where $z\to\omega+i\epsilon$ for $\omega>1$, and $z\to\omega-i\epsilon$ for $\omega<-1$,

$$(\lambda_\alpha)=\tfrac{2}{3}f^2\begin{pmatrix}-4\\-1\\+2\end{pmatrix}, \quad (2a)$$

$$(A_{\alpha\beta})=\tfrac{1}{9}\begin{pmatrix}1&-8&16\\-2&7&4\\4&4&1\end{pmatrix}; \quad (2b)$$

$\omega=(1+p^2)^{\frac{1}{2}}$ is the pion energy; $\omega'=(1+p'^2)^{\frac{1}{2}}$; $\hbar$, c, and the pion rest mass are set equal to 1; f^2 is the unrationalized, renormalized coupling constant; $v^2(p)$ is the cutoff function; $h_\alpha(\omega)$ is related to the phase shift $\delta_\alpha(\omega)$ by

$$h_\alpha(\omega)=\frac{\exp[i\delta_\alpha(\omega)]\sin\delta_\alpha(\omega)}{p^3v^2(p)} \quad \text{for} \quad \omega\geq 1. \quad (3)$$

Several properties of $h_\alpha(\omega)$ that are immediate consequences of Eqs. (1) and (2) are

$$\mathrm{Im}h_\alpha(\omega)=0 \quad \text{for} \quad -1\leq\omega\leq 1, \quad (4a)$$

$$\mathrm{Im}h_\alpha(\omega)=p^3v^2(p)|h_\alpha(\omega)|^2\geq 0 \quad \text{for} \quad \omega>1, \quad (4b)$$

$$h_\alpha(-\omega)=\sum A_{\alpha\beta}h_\beta(\omega) \quad \text{for all } \omega, \quad (4c)$$

$$\left[\frac{\omega h_\alpha(\omega)}{\lambda_\alpha}\right]_{\omega\approx 0}=1+\omega r_\alpha,$$

where the effective range is

$$r_\alpha=\frac{1}{\lambda_\alpha\pi}\sum(\delta_{\alpha\beta}+A_{\alpha\beta})\int_1^\infty d\omega'\frac{\mathrm{Im}h_\beta(\omega')}{\omega'}, \quad (4d)$$

and

$$\left[\frac{\omega h_\alpha(\omega)}{\lambda_\alpha}\right]_{\omega=\infty}=1+\frac{1}{3\pi f^2}\int_1^\infty d\omega'\,\mathrm{Im}[h_1(\omega')+h_2(\omega')-2h_3(\omega')]. \quad (4e)$$

The function $g_\alpha(z)$ is defined by

$$g_\alpha(z)=\frac{\lambda_\alpha}{zh_\alpha(z)}. \quad (5)$$

For small enough values of the coupling constant and cutoff, $h_\alpha(z)$ will have no zeros and no poles other than that at $z=0$. In this case $g_\alpha(z)$ satisfies the algebraic (crossing) relation

$$\frac{1}{g_\alpha(-z)}=\sum B_{\alpha\beta}\frac{1}{g_\beta(z)}, \quad (6a)$$

where

$$(B_{\alpha\beta})=\tfrac{1}{9}\begin{pmatrix}-1&2&8\\8&-7&8\\8&2&-1\end{pmatrix}, \quad (6b)$$

and, for real values ω of z, satisfies the integral equation

$$g_\alpha(z)=1-\frac{z}{\pi}\int_1^\infty d\omega'\frac{p'^3v^2(p')}{\omega'^2}\left\{\frac{\lambda_\alpha}{\omega'-z}+\frac{H_\alpha(\omega')}{\omega'+z}\right\}, \quad (7)$$

where $z\to\omega+i\epsilon$ for $\omega>1$, $z\to\omega-i\epsilon$ for $\omega<-1$, and

$$H_\alpha(\omega)=-|g_\alpha(-\omega)|^2\sum\frac{B_{\alpha\beta}\lambda_\beta}{|g_\beta(\omega)|^2}. \quad (8)$$

From Eqs. (3), (5), and (7) it follows that

$$\mathrm{Re}g_\alpha(\omega)=\lambda_\alpha\frac{p^3v^2(p)}{\omega}\cot\delta_\alpha(\omega) \quad \text{for} \quad \omega\geq 1, \quad (9a)$$

$$\begin{aligned}
&\mathrm{Im}g_\alpha(\omega)=0 && \text{for} \quad -1\leq\omega\leq 1,\\
&\mathrm{Im}g_\alpha(\omega)=-\lambda_\alpha p^3v^2(p)/\omega && \text{for} \quad \omega>1,\\
&\mathrm{Im}g_\alpha(-\omega)=H_\alpha(\omega)p^3v^2(p)/\omega && \text{for} \quad \omega>1,\\
&[g_\alpha(\omega)]_{\omega\approx 0}=1-\omega r_\alpha,
\end{aligned} \quad (9b)$$

where the effective range, as defined by Chew and Low, is

$$r_\alpha=\frac{1}{\pi}\int_1^\infty d\omega'\frac{p'^3v^2(p')}{\omega'^3}[\lambda_\alpha+H_\alpha(\omega')], \quad (9c)$$

and

$$g_\alpha(\infty)=1+\frac{1}{\pi}\int_1^\infty d\omega'\frac{p'^3v^2(p')}{\omega'^2}[\lambda_\alpha-H_\alpha(\omega')]. \quad (9d)$$

In place of Eqs. (7) and (8), the iteration procedure is applied to Eqs. (7) and

$$H_\alpha(\omega) = -\left|\sum_\gamma \frac{B_{\alpha\gamma}}{g_\gamma(\omega)}\right|^{-2} \sum \frac{B_{\alpha\beta}\lambda_\beta}{|g_\beta(\omega)|^2}, \tag{10}$$

which equation follows immediately from Eqs. (6) and (8). With $g_\alpha(-\omega)$ explicitly eliminated, only positive values of ω need be considered in Eq. (7). For $f^2=0$, $g_\alpha(\omega)=1$. This is taken as a trial solution. From Eq. (10) we then have $H_\alpha{}^0(\omega)=-\sum B_{\alpha\beta}\lambda_\beta$. $H_\alpha{}^0(\omega)$ is then used in Eq. (7), from which $\mathrm{Re}g_\alpha{}^1(\omega)$ is obtained. Equations (9b) and (10) then provide $H_\alpha{}^1(\omega)$ from $g_\alpha{}^1(\omega)$, and the process is repeated until the difference between $g_\alpha{}^n(\omega)$ and $g_\alpha{}^{n-1}(\omega)$ is negligible.

This procedure, in which only positive values of ω occur, has the advantage that $H_\alpha{}^n(\omega)$ does not occur in a principal value integral. This feature contributes to very rapid convergence. It has a disadvantage that is discussed in the next section.

The Gaussian cutoff function,

$$v^2(p) = \exp(-p^2/P^2),$$

is used, where P is the cutoff. The integration range, $1\leq\omega'\leq 15\frac{7}{8}$, is used for the integrals, which are numerically evaluated at each net point, $\omega=0, \frac{1}{8}, \frac{1}{4}, \cdots, 15\frac{7}{8}$, in the iteration range $1\leq\omega\leq 15\frac{7}{8}$. Simpson's approximation, with intervals $\Delta\omega'=\frac{1}{8}=d$, is used, except for the two intervals adjacent to the net point $\omega'=\omega$ in the principal-value integral. This contribution,

$$\int_{\omega-d}^{\omega+d} d\omega' \frac{f(\omega')}{\omega'-\omega},$$

is obtained by making a Taylor expansion of $f(\omega')$ about ω. For the smooth function $f(\omega')$, the first two terms of the expansion give a satisfactory approximation to the integral. The integration range is extended to $16\frac{7}{8}$ for the principal value integral, so that even for values of ω close to $15\frac{7}{8}$ this integral is accurately given. The accuracy of the numerical calculations is limited by Simpson's approximation and by the neglect of contributions to the integrals from beyond the integration range. These contributions are discussed in Sec. 4.

This iteration scheme proves to be strongly convergent, requiring in general only four or five iterations. The functions to which it converges, $g_\alpha(\omega)$, are shown in Fig. 1 for $f^2=0.08$ and for several cutoff values.

3. DISCUSSION OF PROCEDURE

Solutions of Eqs. (7) and (10) need not satisfy the crossing condition, Eq. (6). This actually occurs for the solution of physical interest, obtained with $f^2=0.08$ and $P=7$, as is discussed in Sec. 5. However, if $g_\alpha(\omega)$ is an analytic solution of Eqs. (7) and (10) which satisfies the crossing condition and has no zeros in the complex plane, then $[\lambda_\alpha/(\omega g_\alpha(\omega))]$ is a solution of Eq. (1). It is therefore not unreasonable to expect to obtain solutions of Eq. (1) by the iterative procedure employed, for small enough parameter values. This expectation is supported by the following two facts: First, Eqs. (7) and (10) and Eqs. (7) and (8) lead to the same $g_\alpha{}^1(\omega)$ for $\omega\geq 1$, if the same trial function, $g_\alpha{}^0(\omega)=1$, is used; and second, $g_\alpha{}^1(\omega)$ is very close to the converged function, $g_\alpha(\omega)$, obtained by iteration of Eqs. (7) and (10). This condition is satisfied by the functions of Figs. 1(a) and 1(b).

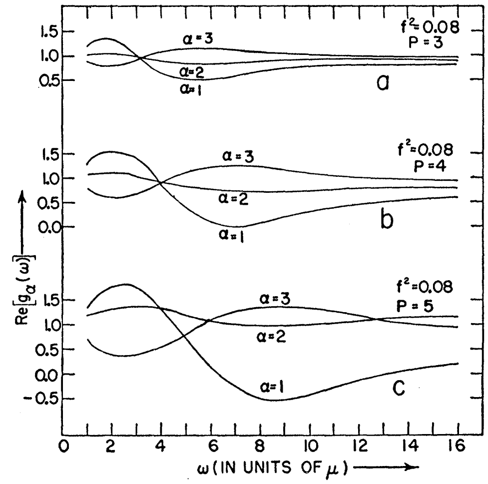

FIG. 1. Iterative solutions of Eqs. (7) and (10) obtained with several different cutoff values. There is a radical change in the behavior of the $\alpha=2$ state function induced by changing the cutoff, P, from 4 to 5.

The character of the functions changes as the cutoff is increased from 4 to 5 in the sense that the asymptotic value of $g_2(\omega)$ is no longer equal to that of the other two states. This change in the behavior of $g_2(\omega)$, shown in Fig. 1, is maintained as the cutoff is increased. From Eqs. (4e) and (5) it follows that if $g_\alpha(\infty)$ is not independent of α, then $[\lambda_\alpha/(\omega g_\alpha(\omega))]$ cannot satisfy Eq. (1). In these cases we also find that the asymptotic values of $g_\alpha(\omega)$ do not satisfy the crossing condition.

Along with this change in the asymptotic behavior of $g_2(\omega)$, we observe the appearance of a zero of $[\lambda_2/(\omega g_2(\omega))]$ on the negative real axis. More explicitly, we find that $\mathrm{Re}[\lambda_2/(\omega g_2(\omega))]$ has a zero at $\omega\sim -5$ both for $P=4$ and $P=5$. However, for $P=4$, $\mathrm{Im}[\lambda_2/(\omega g_2(\omega))]$ has a zero slightly above $\omega=-5$, while for $P=5$ this zero has moved down to $\omega=-7$. This indicates that there is a critical cutoff value, between 4 and 5, at which $[\lambda_2/(\omega g_2(\omega))]$ has a zero on the negative real axis at $\omega\sim -5$.

It appears very likely, on the basis of the evidence discussed so far in this section, that the functions $[\lambda_\alpha/(\omega g_\alpha(\omega))]$ for cutoffs below the critical value are solutions of Eq. (1), although the functions obtained for these values have not been so tested. If this is the case, then there are no zeros of $[\lambda_\alpha/(\omega g_\alpha(\omega))]$ for cutoffs

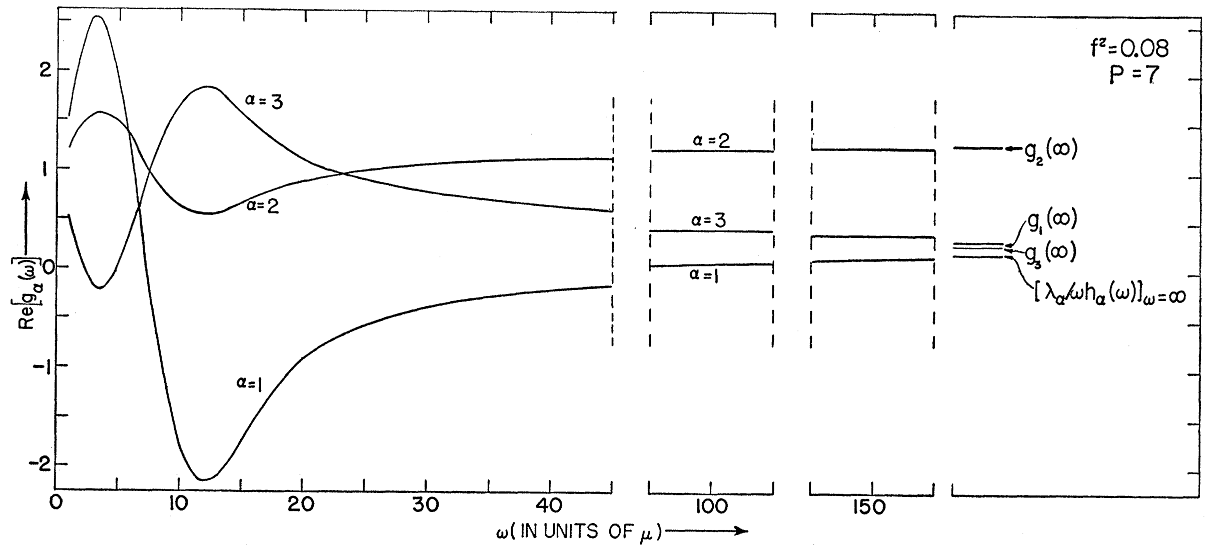

FIG. 2. The iterative solution of Eqs. (7) and (10) obtained with $f^2=0.08$ and a cutoff $P=7$, as extrapolated to very high energies, with high-energy corrections included.

below the critical one. Furthermore, if the zero of $h_2(z)$ continues to be present above the critical cutoff, then, since this requires that Eq. (7) be modified in the $\alpha=2$ state, it is sufficient to explain the abrupt change in the character of the functions that are obtained as the cutoff is increased from 4 to 5.

Despite this difficulty, which appears to be present only in the $\alpha=2$ state, it is still possible to obtain reasonably good functions for the (1,1) and (3,3) states above the critical cutoff. This is because of the small extent to which the $\alpha=2$ state couples into the equations for the (1,1) and (3,3) states, as is seen by examination of Eqs. (2a), (6b), (7), and (10). The presence of the principal-value integral in Eq. (7) also tends to suppress the relative importance of $g_2(\omega)$. In Sec. 5, where we examine the degree to which Eq. (1) is satisfied by the functions obtained for $P=7$, it will be seen that the functions for the (1,1) and (3,3) states are in fact given very well.

4. BEHAVIOR AT VERY HIGH ENERGY

Solutions of physical interest are obtained with values of f^2 in the range 0.08–0.10, and with $f^2P\sim0.6$. A typical one, that for $f^2=0.08$ and $P=7$, is considered. This solution consists of the functions shown in Fig. 2 in the range $1\leq\omega\leq15\frac{7}{8}$. It follows from Eq. (9a) that a resonance in $\delta_\alpha(\omega)$, $\sin^2\delta_\alpha(\omega)=1$, occurs when $\mathrm{Re}g_\alpha(\omega)=0$. In addition to the (3,3) resonance, $\delta_{33}(2.16)$, there are two other resonances in the iteration range, $\delta_{33}(\sim5)$ and $\delta_{11}(\sim7)$, and a second δ_{11} resonance at a very high energy $\omega=\omega_a$.

In order to determine the extent to which this solution satisfies Eq. (1), we must first obtain $\mathrm{Re}g_\alpha(\omega)$ for values of ω beyond the iteration range. There are non-negligible contributions to $h_\alpha(\omega)$ from the $\delta_{11}(\omega_a)$ resonance, where $\mathrm{Re}g_1(\omega_a)=0$ and $\mathrm{Im}g_1(\omega)\sim v^2(p_a)$. These come from an extremely small interval ϵ about ω_a, and, for values of ω outside of this interval, are given by

$$\begin{aligned}\Delta\,\mathrm{Re}h_\alpha(\omega)&=-\frac{1}{\pi}\int_{\omega_a-\epsilon}^{\omega_a+\epsilon}d\omega' p'^3v^2(p')\,|h_1(\omega')|^2\\&\quad\times\left(\frac{\delta_{\alpha1}}{\omega'-\omega}+\frac{A_{\alpha1}}{\omega'+\omega}\right)\\&=-\frac{\lambda_1}{\pi}\int_{\omega_a-\epsilon}^{\omega_a+\epsilon}d\omega'\frac{1}{\omega'}\left(\frac{\delta_{\alpha1}}{\omega'-\omega}+\frac{A_{\alpha1}}{\omega'+\omega}\right)\\&\quad\times\frac{\mathrm{Im}g_1(\omega')}{[\mathrm{Re}g_1(\omega')]^2+[\mathrm{Im}g_1(\omega')]^2}\\&\approx-\lambda_1\left(\frac{\delta_{\alpha1}}{\omega_a-\omega}+\frac{A_{\alpha1}}{\omega_a+\omega}\right)\frac{1}{\omega_a d(\omega_a)},\end{aligned}\tag{11}$$

where

$$d(\omega_a)=\left[\frac{d}{d\omega'}\mathrm{Re}g_1(\omega')\right]_{\omega'=\omega_a},$$

and use is made of the fact that for $\omega'\sim\omega_a\gg P$, $\mathrm{Im}g_1(\omega')$ is extremely small. This resonance gives no contribution in Eq. (7) because the factor $H_\alpha(\omega')$ in the integrand is completely well behaved at $\omega'=\omega_a$ and goes as $\lambda_1/B_{\alpha1}$.

There is, however, a contribution to $\mathrm{Re}g_1(\omega)$, which comes from a very high energy $\omega'=\omega_b>\omega_a$, due to the factor $|\sum B_{1\beta}/g_\beta(\omega')|^{-2}$ of $H_1(\omega')$. To see this, let

$$1/\mathfrak{g}_1(-\omega)=\sum B_{1\beta}/g_\beta(\omega)\quad\text{for}\quad\omega\geq1,\tag{12}$$

where $\mathfrak{g}_1(-\omega)$ is used to distinguish this function from the $g_1(-\omega)$ defined by Eq. (7). At ω_b,

$$\mathrm{Re}[1/\mathfrak{g}_1(-\omega_b)]=0,\quad\text{and}\quad\mathrm{Im}[1/\mathfrak{g}_1(-\omega_b)]\sim v^2(p_b).$$

The fact that $\mathrm{Re}[1/\mathfrak{g}_1(-\omega)]$ vanishes at a value of $\omega>\omega_a$ follows from the equation

$$\mathrm{Re}\frac{1}{\mathfrak{g}_1(-\omega)}\approx\frac{1}{9}\left\{\frac{-1}{\mathrm{Re}g_1(\omega)}+\frac{2}{\mathrm{Re}g_2(\omega)}+\frac{8}{\mathrm{Re}g_3(\omega)}\right\},$$

where the approximation, $\mathrm{Re}g_\alpha(\omega)\gg|\mathrm{Im}g_\alpha(\omega)|$, is extremely good for values of ω even slightly greater than ω_a. Since $\mathrm{Re}g_1(\omega)$ increases from 0 at ω_a to $\sim\mathrm{Re}g_3(\infty)$ at ∞, the function $\mathrm{Re}[1/\mathfrak{g}_1(-\omega)]$, which increases from negative to positive values, must have a zero beyond ω_a. Again, the contribution to $\mathrm{Re}g_1(\omega)$ comes from an extremely small interval ϵ about ω_b, and is given by

$$\begin{aligned}\Delta\,\mathrm{Re}g_1(\omega)&=-\frac{\omega}{\pi}\int_{\omega_b-\epsilon}^{\omega_b+\epsilon}d\omega'\frac{p'^3v^2(p')H_1(\omega')}{\omega'^2(\omega'+\omega)}\\&=-\frac{\omega}{\pi}\int_{\omega_b-\epsilon}^{\omega_b+\epsilon}d\omega'\frac{1}{\omega'(\omega'+\omega)}\\&\quad\times\frac{\mathrm{Im}[1/\mathfrak{g}_1(-\omega')]}{\{\mathrm{Re}[1/\mathfrak{g}_1(-\omega')]\}^2+\{\mathrm{Im}[1/\mathfrak{g}_1(-\omega')]\}^2}\\&\approx\frac{\omega}{\omega_b(\omega_b+\omega)D(\omega_b)},\end{aligned}\tag{13}$$

where

$$D(\omega_b)=\left[\frac{d}{d\omega'}\mathrm{Re}\frac{1}{\mathfrak{g}_1(-\omega')}\right]_{\omega'=\omega_b}.$$

The g_α functions are extrapolated to the asymptotic region by using the solutions obtained from the iteration procedure. The following values are obtained from these functions:

$$\begin{aligned}\omega_a&=\ 98,\quad d(\omega_a)=0.0018,\\\omega_b&=142,\quad D(\omega_b)=0.042.\end{aligned}\tag{14}$$

Even if $\Delta\,\mathrm{Re}g_1(\omega)$ is small, this may change the values of ω_a and ω_b considerably because at these high energies, the slopes $d[\mathrm{Re}g_\alpha(\omega)]/d\omega$ are very small. However, the values of $\Delta\,\mathrm{Re}h_\alpha(\omega)$ and $\Delta\,\mathrm{Re}g_1(\omega)$, as given by Eqs. (11), (13), and (14), although only approximate, are not likely to be significantly changed, because $d(\omega_a)$ and $D(\omega_b)$ are larger at lower values of ω_a and ω_b, respectively.

The correction $\Delta\,\mathrm{Re}g_1(\omega)$ is added to the $g_1(\omega)$ obtained by the iteration and extrapolation procedure. The corrected function is shown in Fig. 2. In the range $1\leq\omega\leq15\frac{7}{8}$ the correction is completely negligible. Even if this correction is included in the iteration process, the results in the iteration range are almost identical with those obtained here. However, at higher energies the correction becomes significant, and at $\omega=\infty$, it changes $\mathrm{Re}g_1(\infty)$ from 0.155 to 0.324. The main effect of this contribution is to alter the very high energy behavior, in particular, it reduces the energies ω_a and ω_b, and it increases the asymptotic value of $\mathrm{Re}g_1(\omega)$.

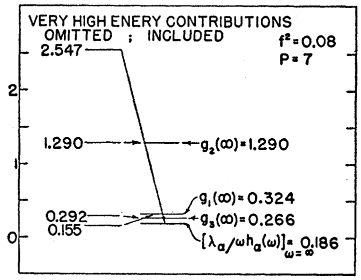

FIG. 3. The asymptotic values of $g_\alpha(\infty)$ obtained with $f^2=0.08$ and a cutoff $P=7$, calculated by Eq. (9d), and the common asymptotic value, $[\lambda_\alpha/(\omega h_\alpha(\omega))]_{\omega=\infty}$, calculated by Eq. (4e). On the left are the uncorrected values, obtained if the integrals are cut off at 16; on the right the high-energy corrections are included, as discussed in Sec. 4. The very high energy resonance in $h_1(\omega)$ must be taken into account to get the good agreement for the (1,1) and (3,3) states shown on the right-hand side.

A correction is also made to $g_3(\omega)$. At $\omega'=15\frac{7}{8}$ the integrand for the (3,3) state in Eq. (7) is not yet negligible. This "tail correction" is added to the $\mathrm{Re}g_3(\omega)$ obtained by the iteration and extrapolation procedure. The corrected function is shown in Fig. 2. This small negative correction is again found to be unimportant in the iteration range, and only becomes significant at very high energies. Its maximum effect is to change $\mathrm{Re}g_3(\infty)$ from 0.292 to 0.266, a 9% decrease.

5. AGREEMENT WITH THE LOW EQUATION

In this section the values of $[\lambda_\alpha/(\omega g_\alpha(\omega))]$ and $h_\alpha(\omega)$ obtained from them with Eq. (1) are compared. As mentioned in Sec. 3, $[\lambda_\alpha/(\omega g_\alpha(\omega))]$ cannot satisfy Eq. (1) exactly because $g_\alpha(\infty)$ is not independent of α, as is shown in Fig. 2. The near equality of $g_1(\infty)$ and $g_3(\infty)$ is consistent with the conjecture that the functions for the (1,1) and (3,3) states are good solutions of Eq. (1). In addition, comparison of $g_\alpha(-1)$ obtained from Eq. (7) and $\mathfrak{g}_\alpha(-1)$ obtained from the crossing condition, Eq. (12), shows that these values are in close agreement for the (1,1) and (3,3) states, as follows:

α	$g_\alpha(-1)$	$\mathfrak{g}_\alpha(-1)$	Difference
1	0.549	0.555	1%
2	0.734	0.614	18%
3	1.835	1.826	0.5%

It is also found that the values of $[\lambda_\alpha/(\omega g_\alpha(\omega))]$ and $h_\alpha(\omega)$ agree closely in the (1,1) and (3,3) states both at $\omega=\infty$ and $\omega=0$. The asymptotic value, $[\lambda_\alpha/(\omega h_\alpha(\omega))]_{\omega=\infty}$, is shown in Fig. 2. The extent of the agreement with $g_1(\infty)$ and $g_3(\infty)$ is remarkable in view of the large contribution, $\Delta\,\mathrm{Re}h_\alpha(\omega)$, in Eq. (1) from ω_a. The effect of the very high energy contributions, $\Delta\,\mathrm{Re}g_\alpha(\omega)$ and $\Delta\,\mathrm{Re}h_\alpha(\omega)$, in improving the agreement of the asymptotic values is shown in Fig. 3.

The values of the effective ranges obtained from Eqs. (9c) and (4d), denoted by r_α and r_α' respectively, are given in Table I. Again, there is good agreement in the (1,1) and (3,3) states. The values of r_α do not

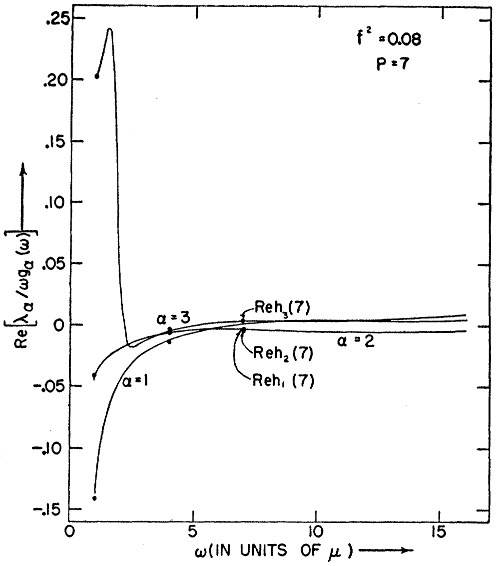

FIG. 4. The functions $\mathrm{Re}[\lambda_\alpha/(\omega g_\alpha(\omega))]$ obtained with $f^2=0.08$ and a cutoff $P=7$, shown as solid curves. The solid circles at $\omega=1$, 4, and 7 are the values obtained for $\mathrm{Re}h_\alpha(\omega)$ by Eq. (1), using $|[\lambda_\alpha/(\omega g_\alpha(\omega))]|^2$ for $|h_\alpha(\omega)|^2$ to evaluate the integrals. The contributions from the very high energy resonance in the (1,1) state are included.

satisfy the equation

$$4r_1+r_2+4r_3=0, \tag{15}$$

which follows from the crossing condition; however, the values of r_α' must automatically satisfy Eq. (15). If it is assumed that r_1 and r_3 are correct, then Eq. (15) gives $r_2=-0.304$, which agrees with r_2' to within 5%.

The values of the functions $\mathrm{Re}[\lambda_\alpha/(\omega g_\alpha(\omega))]$ and $\mathrm{Re}h_\alpha(\omega)$ are also compared at $\omega=1$, 4, and 7. In Fig. 4 the solid curves are the functions $\mathrm{Re}[\lambda_\alpha/(\omega g_\alpha(\omega))]$ and the solid circles are the values of $\mathrm{Re}h_\alpha(\omega)$. All the points, except the one labeled $\mathrm{Re}h_1(7)$, lie close to the appropriate curve. In Table II the contributions to $\mathrm{Re}h_\alpha(\omega)$

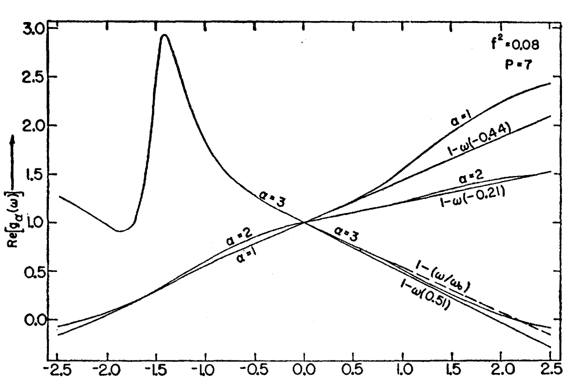

FIG. 5. The functions $\mathrm{Re}g_\alpha(\omega)$ are shown as curved solid lines. The straight solid lines shown for positive ω are the effective-range plots, $1-\omega r_\alpha$. The functions for negative ω are obtained from Eq. (6a). The dashed straight line is the linear plot $1-(\omega/\omega_0)$ with $\omega_0\approx2.16$.

are labeled A, Born term; B, integration range; and C, very high energy. At $\omega=1$ the Born term is the single largest contribution in each state; however, the other terms must be included to give the close quantitative agreement that is obtained.

At $\omega=4$ and $\omega=7$ the Born term, A, is of the same order of magnitude as B, but of opposite sign in each case. This cancellation between A and B further enhances the relative importance of the very high energy term, C. The continued close agreement in the (3,3) state, which at $\omega=7$ represents the almost complete cancellation of the three terms, each of much greater magnitude than the result, is strong confirmation that in this state the contribution from the integration range is given with considerable accuracy, and also that the very high energy contribution is rather well given.

The percentage difference is greatest in the (1,1) state at $\omega=7$. However, this is seen to be due to the importance of the very high energy contribution in this state, which is almost constant at these energies, and which accounts for the approximately energy-independent difference in this state, as given in the last column of Table II. The difference is $\sim20\%$ of the very high energy contribution. A 10% increase of $\Delta\,\mathrm{Re}h_\alpha(\omega)$ would give improved agreement in the (1,1) and (3,3) states; however, at this point the remaining differences are of the same order as the $\alpha=2$ state contributions, and further refinement would require a better solution than is now available.

6. LOW-ENERGY BEHAVIOR OF THE SOLUTION OF THE ONE-MESON APPROXIMATION

The solutions obtained for values of f^2 in the range 0.08–0.10 and for $f^2P\sim0.6$ are characterized by a resonance $\delta_{33}(\omega_0)$, where ω_0 is in the neighborhood of 2.16. For the (3,3) state, the energy dependence of $\mathrm{Re}g_3(\omega)$ is reasonably approximated by the linear expression suggested by Chew and Low,

$$\mathrm{Re}g_3(\omega)=\lambda_3\frac{p^3v^2(p)}{\omega}\cot\delta_{33}(\omega)\sim1-\frac{\omega}{\omega_0}, \tag{16}$$

TABLE I. The effective ranges r_α obtained from Eq. (9c) are compared with those, r_α', from Eq. (4d) in the second, third, and fourth columns. The remaining four columns show the importance, in calculating r_α', of including contributions other than that of the (3,3) state resonance region. We write $r_\alpha'=x_\alpha+y_\alpha+z_\alpha$. x_α is the contribution from the (3,3) state in the range $1\leq\omega'\leq3.5$, y_α contains all other contributions from the range $1\leq\omega'\leq15\frac{7}{8}$, and z_α is the contribution from the very high energy resonance in the (1,1) state at $\omega'=\omega_\alpha$. The ratio $(y_\alpha+z_\alpha)/x_\alpha$ is small in the $\alpha=2$ state because of the cancellation between y_2 and z_2.

α	r_α	r_α'	Difference	x_α	y_α	z_α	$\frac{y_\alpha+z_\alpha}{x_\alpha}$
1	−0.439	−0.418	4.9%	−0.292	−0.063	−0.063	0.43
2	−0.210	−0.319	41%	−0.292	−0.079	+0.052	0.09
3	+0.515	+0.500	3%	+0.365	+0.083	+0.052	0.37

in the range $1 \leq \omega \lesssim \omega_0$. This is shown by the dashed straight line in Fig. 5 for the particular function under discussion, that of Fig. 2.

The experimental points[7,8] shown in Fig. 6(a) are in very close agreement, for pion laboratory energies up to 170 Mev, with an energy dependence given by

$$(p^3/\omega^*) \cot\delta_{33}(\omega^*) = 9.00 - 4.17\omega^* \pm (0.38 - 0.21\omega^*), \quad (17)$$

where ω^* is the total pion energy plus the nucleon kinetic energy, in the center-of-mass system. In the no-recoil limit, $\omega^* \rightarrow \omega$. If the linear approximation of Eq. (16) is then fitted to the experimental values, with $v^2(p)$ taken as 1, the minimum and maximum values obtained for f^2 are 0.080 (with $\omega_0 = 2.14$) and 0.087 (with $\omega_0 = 2.17$).

From Fig. 5 it is seen that $\mathrm{Re}g_3(\omega)$ deviates from linear energy dependence in that $d^2[\mathrm{Re}g_3(\omega)]/d\omega^2$ is positive above the threshold, $\omega = 1$. This behavior is typical of the class of solutions under discussion. Comparison of the experimentally determined values of $(p^3/\omega^*) \cot\delta_{33}(\omega^*)$ with the function $(1/\lambda_3)\,\mathrm{Re}g_3(\omega)$ is shown in Fig. 6(a) for two coupling constants, $f^2 = 0.08$ (with $P = 7$) and $f^2 = 0.10$ (with $P = 6$), with ω taken equal to ω^*. Of the two values, the $f^2 = 0.08$ curve is in better agreement with the experimental points. A best-fit determination of the parameter values has not been made, but it is apparent that better agreement up to 170-Mev pion lab energy is obtained for a coupling constant smaller than 0.08. The Chew-Low plot thus gives larger values of f^2 than is obtained for a best fit solution of the (3,3)-state data up to 170 Mev.

The experimental point at 220 Mev is seen to lie below the theoretical curve. In general, as the energy increases beyond ω_0 the function $(1/\lambda_3)\,\mathrm{Re}g_3(\omega)$ lies increasingly above the straight line of Eq. (17), whereas the experimental points fall below it. The theoretical

TABLE II. Comparison of the function $\mathrm{Re}[\lambda_\alpha/(\omega g_\alpha(\omega))]$ and the functions $\mathrm{Re}h_\alpha(\omega)$ obtained from them by a single iteration of Eq. (1). In each state at each of the energies better agreement is obtained with the inclusion of the very high energy contribution, C, than without it, and in almost every case the magnitude of C is much larger than the remaining difference.

ω	α	Born term A	Integration range B	Very high energy C	$\mathrm{Re}h_\alpha(\omega)$ $A+B+C$	Real part of $[\lambda_\alpha/(\omega g_\alpha(\omega))]$ D	Difference $A+B+C-D$
	1	−0.2133	0.0582	0.0136	−0.1415	−0.1383	−0.0032
1	2	−0.0533	0.0150	−0.0027	−0.0410	−0.0439	0.0029
	3	0.1067	0.0904	0.0053	0.2024	0.2032	−0.0008
	1	−0.0533	0.0257	0.0139	−0.0137	−0.0117	−0.0020
4	2	−0.0133	0.0089	−0.0026	−0.0070	−0.0075	0.0005
	3	0.0267	−0.0359	0.0052	−0.0040	−0.0038	−0.0002
	1	−0.0305	0.0123	0.0143	−0.0039	−0.0006	−0.0033
7	2	−0.0076	0.0053	−0.0025	−0.0048	−0.0040	−0.0008
	3	0.0152	−0.0182	0.0051	0.0021	0.0027	−0.0006

[7] These values are taken from J. Orear, Nuovo cimento 4, 856 (1956).

[8] G. Puppi, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Nuclear Physics, 1956* (Interscience Publishers, Inc., New York, 1956), Sec. I., p. 18.

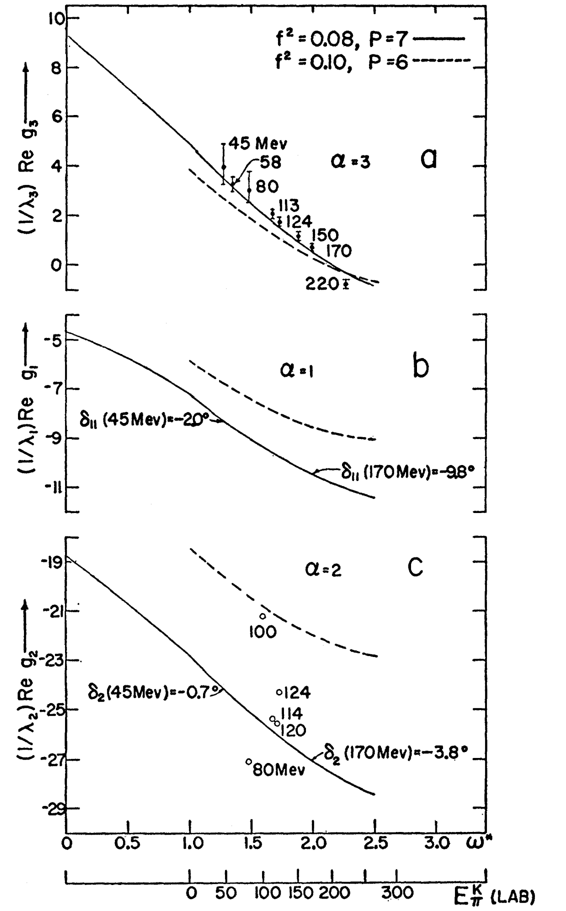

FIG. 6. The functions $(1/\lambda_\alpha)\,\mathrm{Re}g_\alpha(\omega)$ are shown as curved solid lines, plotted with ω taken equal to ω^*. The experimental points shown in Fig. 6(a) are taken from Orear,[7] and those in Fig. 6(c) from G. Puppi.[8]

curve of Fig. 6(a) indicates the broad resonance at $\delta_{33}(\omega_0)$ which is characteristic of static theory calculations,[9] in contrast to the narrow resonance determined experimentally.

The other p-wave phase shifts are not known well enough for a detailed comparison to be made, but they should be small in the range $1 \leq \omega \lesssim \omega_0$.[8,10] The points $(p^3/\omega^*) \cot\delta_{31}(\omega^*)$ obtained with Puppi's values[8] of $\delta_{31}(\omega^*)$ are shown in Fig. 6(c), without the quoted errors. They are not inconsistent with the curve $(1/\lambda_2)\,\mathrm{Re}g_2(\omega)$ for $f^2 = 0.08$ in view of the extreme uncertainties attached to them. The theoretical values of δ_{11} and δ_2 shown in Figs. 6(b) and 6(c) are obtained

[9] G. Chew, Phys. Rev. 95, 285 (1954); F. Salzman and J. N. Snyder, Phys. Rev. 95, 286 (1954); Friedman, Lee, and Christian, Phys. Rev. 100, 1494 (1955).

[10] H. L. Anderson *Proceedings of the Sixth Annual Rochester Conference on High-Energy Nuclear Physics, 1956* (Interscience Publishers, Inc., New York, 1956), Sec. I, p. 20. From the formula given here, δ_{11} (170 Mev) $= -3.3°$. A more recent fit to the pion-proton scattering data (private communication from Professor Anderson) by Metropolis and Anderson using an energy dependence of the Chew-Low type for the p waves also leads to small δ_{11} and $\delta_{13} (= \delta_{31})$ in the range $1 \leq \omega \lesssim \omega_0$, e.g., δ_{11} (170 Mev) $= 3.1°$.

from the equation,

$$(p^3/\omega^*)\cot\delta_\alpha(\omega^*)=(1/\lambda_\alpha)\,\mathrm{Re}g_\alpha(\omega),$$

with ω taken equal to ω^*. The value $-9.8°$ for δ_{11}(170 Mev) appears to be of too large magnitude, but may fall within experimental limits.[10,11] Decreasing the value of f^2 reduces the magnitudes of the small phase shifts.

The effective range approximation suggested for low energies by Chew and Low is expressed by

$$\mathrm{Re}g_\alpha(\omega)\sim 1-\omega r_\alpha, \tag{18}$$

where r_α is defined by Eq. (9c). The effective range curves for the $f^2=0.08$ solution are the solid straight lines shown in Fig. 5. Because the effective-range curve for the (1,1) state differs appreciably from $\mathrm{Re}g_1(\omega)$, the slopes in the physical region will not in general satisfy Eq. (15).

Equation (18) is obtained from Eq. (7) by completely neglecting ω in the integrals. An approximation that does not entirely ignore ω in the integrals is given by Chew.[6] It is based on the observation that the complete g_α equation, including inelastic processes, may be written

$$g_\alpha(z)=1-r_\alpha z+P_\alpha z^2-\mathcal{L}_\alpha(z),$$

$$\mathcal{L}_\alpha(z)=-\frac{z^3}{\pi}\int_1^\infty d\omega'\frac{1}{\omega'^3}\left\{\frac{\mathrm{Im}g_\alpha(\omega')}{\omega'-z}-\frac{\mathrm{Im}g_\alpha(-\omega')}{\omega'+z}\right\},$$

where in addition to the three effective ranges, r_α, there are three other constants, P_α, which are also to be determined from the experimental plots. The factor $1/\omega'^3$ in the integrand of $\mathcal{L}_\alpha(z)$ insures that the important contributions to the integral come from low values of ω', where the cross section is principally elastic. Then $\mathrm{Im}g_\alpha(\pm\omega')$ are approximated by the expressions given in Eqs. (9b), which are exact in the range $1\leq\omega'\leq 2$. Further approximations, also based on the presence of the $1/\omega'^3$ factor, are the replacement of $H_\alpha(\omega')$ by $H_\alpha(0)=H_\alpha{}^0=-\sum B_{\alpha\beta}\lambda_\beta$ and of $v^2(p')$ by 1. These substitutions give

$$\mathcal{L}_\alpha(z)\sim\frac{z^3}{\pi}\int_1^\infty d\omega'\frac{p'^3}{\omega'^4}\left\{\frac{\lambda_\alpha}{\omega'-z}+\frac{H_\alpha{}^0}{\omega'+z}\right\}.$$

The validity of this approximation depends on $H_\alpha(\omega')$ being close to $H_\alpha{}^0$ for low values of ω'. For the solution obtained we find that $H_3(\omega')$ is a sharply peaked function in the region $1\leq\omega'\leq 2$, as is indicated in Fig. 5 by the nonlinear behavior of $\mathrm{Re}g_3(\omega)$ for ω in the neighborhood of -1.5. Also $H_2{}^0=\frac{2}{3}f^2=-\lambda_2$, whereas the $H_2(\omega')$ obtained in the solution has the same sign as λ_2 and is of comparable magnitude.

[11] Ashkin, Blaser, Feiner, and Stern, Phys. Rev. **101**, 1149 (1956) give δ_{11}(170 Mev)$=0°$. H. Anderson and M. Glicksman, Phys. Rev. **100**, 268 (1955) give δ_{11}(165 Mev)$=-10.6°$.

7. LOW-ENERGY BEHAVIOR WITH INELASTIC PROCESSES

In Sec. 5 it was shown that improved agreement between $\mathrm{Re}[\lambda_\alpha/(\omega g_\alpha(\omega))]$ and $\mathrm{Re}h_\alpha(\omega)$ is achieved by inclusion of the very high energy contribution. One may still question whether this contribution, and in fact the entire contribution to the integrals from the energy range beyond the cutoff, is not just a feature of this particular solution. To show that this is a property of the static model for parameter values $f^2\sim 0.08$ and $P\sim 7$, it is convenient to write the equation for the (1,1) state as follows:

$$\mathrm{Re}h_1(\omega)=X(\omega)+Y(\omega)+Z(\omega),$$

$$X(\omega)=\frac{\lambda_1}{\omega}+\frac{16}{9}\frac{1}{4\pi^2}\int_1^{3.5}d\omega'\frac{\sigma_3{}^T(\omega')}{p'v^2(p')(\omega'+\omega)},$$

$$\begin{aligned}Y(\omega)=\mathrm{P}\frac{1}{4\pi^2}\int_1^{11}d\omega'\frac{\sigma_1{}^T(\omega')}{p'v^2(p')(\omega'-\omega)}&\\+\frac{1}{9}\frac{1}{4\pi^2}\int_1^{11}d\omega'\frac{\sigma_1{}^T(\omega')-8\sigma_2{}^T(\omega')}{p'v^2(p')(\omega'+\omega)}&\\+\frac{16}{9}\frac{1}{4\pi^2}\int_{3.5}^{11}d\omega'\frac{\sigma_3{}^T(\omega')}{p'v^2(p')(\omega'+\omega)},&\end{aligned}\tag{19}$$

$$\begin{aligned}Z(\omega)=\frac{1}{4\pi^2}\int_{11}^\infty d\omega'\frac{\sigma_1{}^T(\omega')}{p'v^2(p')(\omega'-\omega)}&\\+\frac{1}{9}\frac{1}{4\pi^2}\int_{11}^\infty d\omega'\frac{\sigma_1{}^T(\omega')-8\sigma_2{}^T(\omega')+16\sigma_3{}^T(\omega')}{p'v^2(p')(\omega'+\omega)},&\end{aligned}$$

where $\sigma_\alpha{}^T$ is the total cross section, inelastic included, for scattering in the α state and P means the principal value of the integral. In the elastic region,

$$\sigma_\alpha{}^e=\sigma_\alpha{}^T=(4\pi/p^2)\sin^2\delta_\alpha(\omega),$$

and $h_\alpha(\omega)$ is given by Eq. (3). $X(\omega)$ consists of the Born term and the contribution of the (3,3)-state resonance. $Y(\omega)$ consists of the other contributions that may be expected in the static theory, and $Z(\omega)$ consists of all higher energy contributions.

In general, $\sigma_\alpha{}^T=N_\alpha\sigma_\alpha{}^e$, where $N_\alpha\geq 1$. The following inequalities are then obtained from the unitarity condition:

$$\mathrm{Im}h_\alpha(\omega)=\frac{\sigma_\alpha{}^T(\omega)}{4\pi pv^2(p)}\leq\frac{1}{N_\alpha p^3v^2(p)}, \tag{20a}$$

$$|\mathrm{Re}h_\alpha(\omega)|\leq\frac{1}{2}\frac{1}{N_\alpha p^3v^2(p)}. \tag{20b}$$

The maximum cross section occurs for purely elastic, resonant scattering.

In order to satisfy Eq. (20b) it is necessary for the large negative λ_1/ω term to be largely canceled by the positive terms in Eq. (19). The following qualitative argument shows that the amount of this cancellation from $Z(\omega)$ attains a near minimum, and non-negligible value in the numerical solution.

First, the (3,3)-state resonance integral in $X(\omega)$ is well approximated by

$$\frac{1}{\pi}\int_1^{3.5} d\omega' \frac{\sin^2\delta_{33}(\omega')}{p'v^2(p')(\omega'+\omega)} \approx \frac{0.070}{2.0+\omega}, \tag{21}$$

where $\delta_{33}(\omega')\sim 90°$ for ω' in the range 2.1 to 3.5. Equation (19) may then be written as

$$\mathrm{Re}h_1(\omega)=\left\{-\frac{0.213}{\omega}+\frac{0.124}{2.0+\omega}+Y(\omega)+Z(\omega)\right\}, \tag{22}$$

where f^2 is taken as 0.08. From Eqs. (20b) and (22) it follows, since $N_\alpha \geq 1$, that

$$Y(\omega)+Z(\omega) \gtrsim 0.018 \quad \text{for} \quad 2.8\leq\omega\leq 5.0. \tag{23}$$

Second, consider the contributions to $Y(\omega)$. There is a broad resonance in $h_1(\omega')$ in the range $5.5 \lesssim \omega' \lesssim 9$. The resonance in $h_3(\omega')$ continues up to $\omega'\sim 7$, and the negative $h_2(\omega')$ term in $Y(\omega)$ is negligible. The only other significant contribution possible to $Y(\omega)$ is from the range $\omega' \lesssim 5$ in the principal-value (P.V.) integral. However, any appreciable change in the energy dependence of $h_1(\omega')$ in this range requires a larger $Z(\omega)$ in order to maintain inequality (23). To see this, we note that $Y(\omega)+Z(\omega)$ is very close to the minimum allowed by Eq. (23) for most of the interval, and $Z(\omega)\sim 0.014$. This value of $Z(\omega)$ requires that the P.V. integral be positive up to $\omega\sim 5$, even with the maximum allowed values for the other terms in $Y(\omega)$.

Any decrease in $Z(\omega)$ must be compensated for by an increase in the P.V. integral. Since the integrand of this integral is close to its maximum value for $\omega' \gtrsim 5$, an increase in the P.V. integral at $\omega\sim 5$ requires that the integrand be smaller in the range $1\leq\omega'\lesssim 5$. However, this in turn implies a decrease in the P.V. integral at $\omega\sim 1$, that is, less cancellation of λ_1/ω in the neighborhood of threshold, and consequently a larger magnitude for $\delta_{11}(\omega)$ in this region. Since even with $\delta_{11}(1.75)$ only $-9.5°$ the integrand is about as large as it gets for any ω', an increase of $|\delta_{11}(\omega')|$ in this region would increase the integrand there, and decrease the P.V. integral at $\omega=5$, which contradicts the original premise. $Z(\omega)$ then cannot be decreased everywhere, in particular at $\omega\sim 5$, where its value, 0.014, is of the same magnitude as that of the (3,3) resonance term, 0.018.

From Eqs. (19), (20b), and (22) it follows that an increase of either f^2 or $v^2(p)$ strengthens inequality (23), assuming the (3,3) resonance term remains unchanged, and in this case $Z(\omega)$ must be even larger than found above. However, it has been conjectured that the solution of the complete Low equation requires a smaller cutoff to give the (3,3) state resonance than is needed in the one-meson approximation. In this case it appears that the combined effect of $N_\alpha v^2(p)$ in inequalities (20a) and (20b) may give inequality (23) as strong as the one-meson approximation with the larger cutoff.

It is of interest to note that the phenomenological approach, in which it is assumed that

$$\int_1^{3.5} d\omega' \frac{\sigma_{33}{}^T(\omega')}{\omega' p' v^2(p')} \gg \int_1^{\infty} d\omega' \frac{\sigma_{11}{}^T(\omega')}{\omega' p' v^2(p')},$$

is not in agreement with the results obtained here, where the (1,1) state integral amounts to 54% of the (3,3) state resonance integral. The importance of including contributions other than that of the (3,3) state resonance in a calculation of the effective ranges is shown in Table I, where the other contributions from the integration range are labeled y_α, and the contributions from the very high-energy resonance, $\delta_{11}(\omega_a)$, are labelled z_α. It is seen that y_α and z_α are non-negligible compared to the (3,3) state resonance contribution, and must be included for the close agreement between r_α' and r_α.

In calculating the electromagnetic properties of nucleons, integrals similar to these occur. Inclusion of the (1,1) state contribution increases the isotopic vector part of the anomalous magnetic moment. Its inclusion enhances the already too large isotopic scalar part of the anomalous moment, and the charge density. Such contributions are extremely unreliable in the static theory because they come from energies at which nucleon recoil, higher partial waves, and other interactions are important.

8. RELATIVISTIC CONTRIBUTIONS

As a first approximation of recoil effects on the p-wave phase shifts, we consider the p-wave part of the expansion in powers of $1/M$ of the relativistic dispersion relations for pion-nucleon scattering,[6] where $1/M$ is the ratio of pion to nucleon mass. The appropriate linear combinations of these equations for real values ω of z, are

$$h_\alpha(z)=\frac{\lambda_\alpha}{z}+\frac{3f^2}{M}\delta_{\alpha 1}+\frac{1}{\pi}\int_1^\infty d\omega'\left\{\mathrm{Im}h_\alpha(\omega')\left[\frac{1}{\omega'-z}+\frac{1}{M}\right] +\sum A_{\alpha\beta}\frac{\mathrm{Im}h_\beta(\omega')}{\omega'+z}\right\}, \tag{24}$$

where $z\rightarrow\omega+i\epsilon$ for $\omega>1$, ω and p are the total energy minus the nucleon rest energy and the pion momentum respectively, in the center of mass, and terms up to order $1/M$ are included. In the elastic region,

$$h_\alpha(\omega)=\exp[i\delta_\alpha(\omega)]\sin\delta_\alpha(\omega)/p^3. \tag{25}$$

In the limit $1/M\rightarrow 0$, Eq. (24) reduces to the static theory equation except for the absence of a cutoff factor. This rules out the possibility of very high-energy contributions of the kind discussed in Sec. 4, and also acts to reduce generally the contributions of the high-energy parts of the integrals.

To determine the extent to which the recoil terms can replace very high-energy contributions in the (1,1) state equation, we write in analogy to Eq. (19)

$$\mathrm{Re}h_1(\omega)=X_r(\omega)+\mathfrak{X}_r(\omega),$$

$$X_r(\omega)=\frac{\lambda_1}{\omega}+\frac{3f^2}{M}+\frac{16}{9}\frac{1}{4\pi^2}\int_1^{3.5}d\omega'\frac{\sigma_{33}{}^T(\omega')}{p'(\omega'+\omega)}. \tag{26}$$

The (3,3) resonance integral is evaluated using Anderson's[10] values for $\delta_{33}(\omega)$ up to $\omega=2.7$ and the value of $\sigma^T(\pi^+,p)$ given by Piccioni,[12] from 2.7 to 3.5. The result,

$$\frac{1}{4\pi^2}\int_1^{3.5}d\omega'\frac{\sigma_{33}{}^T(\omega')}{p'(\omega'+\omega)}\approx\frac{0.068}{2.0+\omega},$$

is very close to the result found from the static theory solution, Eq. (21). Equation (26) may then be written, for $f^2=0.08$, as

$$\mathrm{Re}h_1(\omega)=\frac{-0.213}{\omega}+0.036+\frac{0.122}{2.0+\omega}+\mathfrak{X}_r(\omega). \tag{27}$$

The constant recoil term is about 90% of the (3,3) resonance contribution at $\omega=1$, and is larger than the sum $Y(1)+Z(1)$ of Eq. (22), which in the static theory corresponds to $\mathfrak{X}_r(1)$.

In this case the inequality $|\mathrm{Re}h_1(\omega)|\leq 1/(2p^3)$ is satisfied by Eq. (27) with $\mathfrak{X}_r(\omega)$ set equal to zero up to $\omega\sim 5$, but for higher energies $\mathfrak{X}_r(\omega)$ must be negative. To illustrate the importance of the recoil term, we note that even if $\mathfrak{X}_r(\omega)$ is taken equal to zero in Eq. (27), one obtains $\delta_{11}(2)=-8.5°$, which is of smaller magnitude than given by the static theory. It is also found that in the (3,3)-state equation, the recoil term more than replaces the very high energy contribution of the static theory.

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[12] O. Piccioni, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Nuclear Physics, 1956* (Interscience Publishers, Inc., New York, 1956), Sec. IV, p. 8.