Vacuum Polarization Effects in Proton-Proton Scattering*†

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The problem of vacuum polarization scattering of protons by protons is treated to first order in the vacuum polarization interaction. The phase shifts caused by the interaction are calculated, and the corresponding addition to the p-p scattering matrix is constructed. The phase shifts are calculated using Coulomb wave functions as the unperturbed wave functions. The corresponding addition to the p-p scattering matrix is then constructed, including exactly the Coulomb phase shift factors $\exp[2i(\sigma_L-\sigma_0)]$ appearing in the series representation of the scattering amplitudes. Other electromagnetic and relativistic modifications of the Coulomb scattering amplitudes are also examined in the limit of low energies. Numerical results are given for the vacuum polarization contributions to the p-p scattering cross section over the energy range 1.4–4.2 Mev. It is found that vacuum polarization scattering may easily be confused with nuclear P-wave scattering.

1. INTRODUCTION AND NOTATION

PROTON-PROTON scattering experiments provide a powerful tool for the study of the forces between nucleons. The theoretical interpretation of such experiments has been studied extensively.[1–4] The scattering data are normally analyzed in terms of the Coulomb scattering amplitudes,[5] modified by the phase shifts of the asymptotic two-body wave functions.

It appears probable that the main contributions to the p-p differential scattering cross section up to about 40 Mev arise from Coulomb forces plus nuclear interaction in the ^1S state.[4,6] While small P- and D-wave phase shifts have been observed at the higher energies, and some indications of P wave scattering have been found also below 10 Mev,[3] the low-energy cross sections were not sufficiently well known until recently[7] to allow a definitive analysis to be made. New experiments,[7] show definite deviations from pure S wave plus Coulomb scattering in the energy range 1.8–4.2 Mev. The effects were tentatively analyzed in terms of nuclear P wave scattering with mean phase shifts of around $-0.1°$ [8]; but false indications of nuclear scattering of this extent could arise from a variety of non-nuclear sources. If the precise status of nuclear scattering in high angular momentum states is to be determined at low energies, the consideration of such non-nuclear contributions to the cross section is needed.

One of the largest non-nuclear effects originates in the vacuum polarization interaction between the protons, as was pointed out by Foldy and Eriksen.[9] The phenomenon of vacuum polarization is well established theoretically.[10–13] The phenomenon results in a modification of the Coulomb potential between two protons by the addition of a second "vacuum polarization" potential. This added interaction was first discussed by Uehling.[12] A number of other authors have considered the problem, most recently Euwema and Wheeler,[14] who avoided the regularization difficulties common to earlier work through the use of a causality condition and by Wichmann and Kroll,[15] who calculated the vacuum polarization potential induced by a point charge Ze to all orders of $Z\alpha$, with α standing for the fine structure constant. In the present work, only the leading term in a series expansion of the potential in powers of α is important. This term, originally derived by Uehling,[12] has been tested in a variety of experiments. It contributes an important part of the total Lamb shift of the electronic energy levels of hydrogen, deuterium, and helium,[16] and results also in a detectable shift in the energy levels of light π- and μ-mesonic

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[1] Breit, Condon, and Present, Phys. Rev. **50**, 825 (1936).
[2] Breit, Thaxton, and Eisenbud, Phys. Rev. **55**, 1018 (1939).
[3] J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 77 (1950).
[4] Yovits, Smith, Hull, Bengston, and Breit, Phys. Rev. **85**, 540 (1952).
[5] N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Clarendon Press, Oxford, 1950), second edition. The problems of Coulomb scattering are discussed in Chap. III.
[6] A. M. Saperstein and L. Durand, III, Phys. Rev. **104**, 1102 (1956); Hull, Ehrman, Hatcher, and Durand, Phys. Rev. **103**, 1047 (1956); H. Feshbach and E. Lomon, Phys. Rev. **102**, 891 (1956).
[7] Worthington, McGruer, and Findley, Phys. Rev. **90**, 899 (1953).
[8] H. H. Hall and J. L. Powell, Phys. Rev. **90**, 912 (1953). Dr. M. H. Hull, Yale University, also analyzed the data of Worthington, McGruer, and Findley,[7] and obtained essentially the same results as Hall and Powell. The results of this second analysis have not been published.
[9] L. L. Foldy and E. Eriksen, Phys. Rev. **95**, 1048 (1954); **98**, 775 (1955).
[10] W. Heisenberg, Z. Physik **90**, 209 (1934).
[11] R. Serber, Phys. Rev. **48**, 49 (1935).
[12] E. A. Uehling, Phys. Rev. **48**, 55 (1935).
[13] W. Pauli and M. E. Rose, Phys. Rev. **49**, 462 (1936).
[14] R. N. Euwema and J. A. Wheeler, Phys. Rev. **103**, 803 (1956).
[15] E. H. Wichmann and N. M. Kroll, Phys. Rev. **96**, 232 (1954); **101**, 843 (1956).
[16] A summary of the theoretical results on the Lamb shift in hydrogen and deuterium is given by E. E. Salpeter, Phys. Rev. **89**, 92 (1953); the corresponding experiments are summarized by Triebwasser, Dayhoff, and Lamb, Phys. Rev. **89**, 98 (1953). The results on the Lamb shift in helium are given by Novick, Lipworth, and Yergin, Phys. Rev. **100**, 1153 (1955).

atoms.[17] The work of Foldy and Eriksen[9] indicates the presence of the theoretically expected vacuum polarization interaction between protons. Those authors considered the change in the S-wave scattering caused by the vacuum polarization forces, using a method based on the nearly linear variation with energy for short-range (nuclear) interactions of the function f first introduced by Breit, Condon, and Present.[1] The function K occurring in the Foldy-Eriksen analysis is a later variant[3] of the f function, related to it by "K"$=\frac{1}{2}f-0.15443$. Because of its long range, the vacuum polarization interaction gives a markedly nonlinear contribution to f at low scattering energies. It was shown by Foldy and Eriksen[9] that the variation from linearity of the values of f determined from the low-energy p-p data, is in qualitative agreement with this vacuum polarization effect, but the data are not sufficiently accurate to make precise quantitative comparisons possible. Subsequent considerations of P wave scattering by Eriksen, Foldy, and Rarita[18] indicated that roughly half of the observed contribution of the P states to the 1.8–4.2 Mev cross sections[7,8] could be ascribed to the vacuum polarization P-wave phase shift. The analysis of vacuum polarization effects in p-p scattering is seen to be still incomplete because the long range of the vacuum polarization potential may be expected to result in contributions to the scattering from states with angular momentum $L>1$ to a degree comparable with that from P states.

Other small additions to the cross section of relativistic and magnetic origin also enter a careful analysis of the p-p scattering data. Corrections for relativistic kinematics in the transformations connecting scattering angles and cross sections in the laboratory system with those in the center-of-mass system are well known, and while small, are not negligible even at low energies. Dynamic relativistic and magnetic effects in Coulomb scattering have been treated by Breit[19] to first order in η, the Coulomb scattering parameter. Magnetic scattering involving the anomalous part of the proton magnetic moment has been included in the parallel work of Garren[20]; but Breit[21] has recently pointed out that certain of the terms appearing in Garren's[20] relativistic scattering matrix arise from contact interactions between the protons, and cannot be trusted in magnitude. Thus it is necessary to examine with some care those relativistic and magnetic corrections to the ordinary Coulomb scattering of protons which persist at low energies. The radiative corrections important in the energy level shifts in atoms and in electron scattering phenomena are reduced for protons by powers of the ratio m/M of the electron to the proton mass,[22] and are negligibly small.

The concern of the present work is the consideration of those non-nuclear contributions to the proton-proton scattering cross section of electromagnetic and relativistic origin which may be confused at low energies with nuclear scattering. Particular attention will be given to the problem of vacuum polarization scattering. The analysis is intended primarily to clarify the status of nuclear scattering in orbital angular momentum states with $L>0$.

The following notations will be used throughout the paper.

E = kinetic energy of the incident proton in the laboratory system.

$k=(ME/2\hbar^2)^{\frac{1}{2}}=2\pi$ times the reciprocal of the relativistic wavelength in the center-of-mass system.[19]

$\eta=e^2/\hbar v$ = the relativistic[19] Coulomb scattering parameter, with v the laboratory velocity of the incident proton.

θ = scattering angle in the center-of-mass system.

$x=\cos\theta$.

$\sigma(\theta)$ = differential p-p scattering cross section in the center-of-mass system.

K_L = singlet p-p phase shift for the state with orbital angular momentum quantum number L.

$\delta^L{}_J$ = triplet p-p phase shift for the state with orbital angular momentum L, total angular momentum J in units $\hbar$.

δ_L = phase shift for orbital angular momentum L caused by the vacuum polarization interaction.

$\sigma_L=\arg\Gamma(L+1+i\eta)$ = Coulomb phase shift for orbital angular momentum L.

$e_{L,0}=\exp\{2i(\sigma_L-\sigma_0)\}$.

$\nu=(4m/M)(mc^2/E)$ = the parameter characterizing the vacuum polarization scattering. Here m is the electron mass and M is the proton mass.

$\mathfrak{a}_{\text{vp}}(x)$ = the vacuum polarization scattering amplitude correct to first order in the vacuum polarization interaction between protons, with $x=\cos\theta$.

$\mathfrak{a}_{\text{vp}}{}^{(0)}(x)$ = the lowest order of the scattering amplitude $\mathfrak{a}_{\text{vp}}(x)$ in an expansion in powers of Coulomb parameter η. This function arises in the treatment of vacuum polarization scattering in the first Born approximation.

$\mathfrak{a}_{\text{vp}}{}^{(i)}(x)$ = the vacuum polarization scattering amplitude of order η^i relative to $\mathfrak{a}_{\text{vp}}{}^{(0)}(x)$.

2. CALCULATION OF VACUUM POLARIZATION SCATTERING OF PROTONS BY PROTONS

(a) Vacuum Polarization Potential

In the present work, the additions to the p-p scattering matrix caused by the vacuum polarization interaction between the protons will be calculated from the vacuum polarization potential in terms of phase shifts.

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[17] V. L. Fitch and J. Rainwater, Phys. Rev. 92, 789 (1953); Koslov, Fitch, and Rainwater, Phys. Rev. 95, 625 (A) (1954); A. B. Mickelwait and H. C. Corben, Phys. Rev. 96, 1145 (1954).
[18] Eriksen, Foldy, and Rarita, Phys. Rev. 103, 781 (1956).
[19] G. Breit, Phys. Rev. 99, 1581 (1955).
[20] A. Garren, Phys. Rev. 96, 1709 (1954); 101, 419 (1956).
[21] G. Breit, Phys. Rev. 106, 314 (1957).
[22] W. Heitler, *Quantum Theory of Radiation* (Clarendon Press, Oxford, 1954), third edition, Chap. 6.

Only the leading (Uehling) term in the expansion of the potential in a series in powers of αZ will be considered. The next terms in the series as given by Wichmann and Kroll[15] are much too small to influence detectably the scattering of protons by protons since in this case $Z=1$. The Uehling potential[12] is conveniently represented by an integral given by Schwinger,[23]

$$V_{\mathrm{vp}}(r)=\frac{2\alpha}{3\pi}\frac{e^2}{r}\int_1^\infty e^{-2\kappa r\xi}\left(1+\frac{1}{2\xi^2}\right)\frac{(\xi^2-1)^{\frac{1}{2}}}{\xi^2}d\xi, \quad (1)$$

where $\kappa^{-1}=\hbar/mc$ is the Compton wavelength of the electron divided by 2π. As was shown by Pauli and Rose,[13] the potential can be expressed in terms of Bessel functions,

$$V_{\mathrm{vp}}(r)=(2\alpha/3\pi)(e^2/r)I(2\kappa r), \quad (2.1)$$

with $I(z)$ given by

$$I(z)=[1+(z^2/12)]K_0(z)-\tfrac{5}{6}[1+(z^2/10)]zK_1(z) +(3z/4)[1+(z^2/9)]\int_z^\infty K_0(t)dt. \quad (2.2)$$

The Bessel functions of imaginary argument of the second kind, $K_n(z)$, are used as defined by Whittaker and Watson.[24] For small values of $z=2\kappa r$, the power series expansion of $I(z)$

$$I(z)=-C-\tfrac{5}{6}+\ln(2/z)+3\pi z/8-3z^2/8+\pi z^3/24 -(z^4/64)[(7/3)-C+\ln(2/z)]+O(z^6) \quad (2.3)$$

is quite useful. Here C is Euler's constant $C=0.5772\cdots$. The general features of the vacuum polarization potential are at once apparent. For $2\kappa r\to 0$, the potential diverges as $r^{-1}\ln(1/\kappa r)$, while for $2\kappa r\gg 1$, it decreases nearly exponentially to zero. Except for the factor $I(2\kappa r)$, the potential is Coulombian; this factor acts as a cutoff function for $r\to\infty$, giving the potential a finite range of the order of the Compton wavelength of the electron. While short compared to the distances over which Coulomb forces are important in the *p-p* problem, this range is very long relative both to the range of nuclear forces and to the wavelength of relative motion of the protons, even at low energies. This characteristic of the potential is important in the scattering of high angular momentum states. The potential is sketched in Fig. 1.

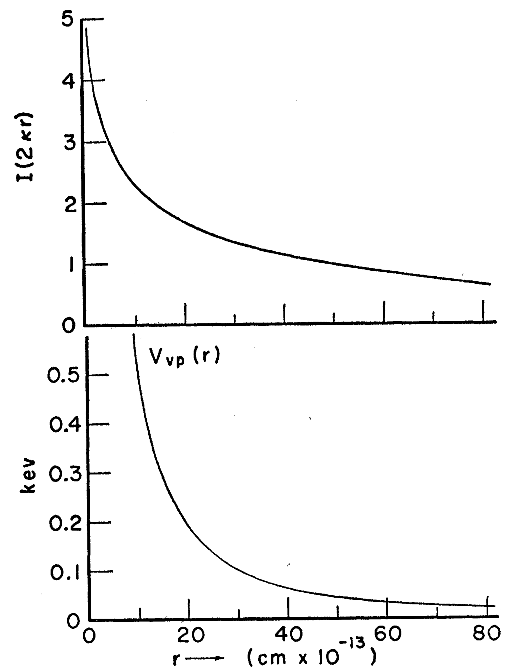

FIG. 1. The vacuum polarization potential $V_{\mathrm{vp}}(r)$. The upper curve is a plot of the cutoff function $I(2\kappa r)$ of Eq. (2) entering in the definition of the potential. The lower curve gives the potential in kilovolts; for comparison, the ordinary Coulomb potential between protons is 145 kev for a proton separation $r=10^{-12}$ cm.

(b) Vacuum Polarization Phase Shifts

At proton scattering energies less than about 10 Mev, the effects in the cross sections ascribed to the scattering of states with orbital angular momentum quantum numbers $L>0$, are quite small, leading to negative mean phase shifts no more than a few tenths of a degree in magnitude.[3,7,8] There is as yet no substantial evidence for any dependence at low energies of the triplet phase shifts on the total angular momentum quantum number J.[8] The vacuum polarization P-wave phase shift is of the same sign and order of magnitude as the observed phase shifts.[18] It is probably safe to assume that the correct nuclear phase shifts are not much different from those observed thus far. Even for nuclear phase shifts an order of magnitude larger, the distortion of the proton wave function is very slight, and produces a negligibly small change in the value of the vacuum polarization phase shift as compared with that calculated using wave functions undistorted by nuclear forces. Conversely, the presence of the vacuum polarization forces will produce only a small change in the nuclear phase shifts.[25] If such wave function distortion effects are neglected, the total phase shifts for each partial wave can be written as the sum of a vacuum polarization part δ_L, and a nuclear part $(\delta^L{}_J)_n$, $\delta^L{}_J\cong(\delta^L{}_J)_n+\delta_L$. That this is not the case for $L=0$ was pointed out by Foldy and Eriksen,[9] who have included in their calculations of the S state vacuum polarization scattering the effects of the large nuclear distortion of the proton wave function. These are quite appreciable since the S-wave phase shift K_0 is around 45° at 2 Mev.

[23] J. Schwinger, Phys. Rev. **75**, 651 (1949).

[24] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis* (Cambridge University Press, New York, 1952), fourth edition, pp. 372–374 and 384.

[25] Quantitative considerations on the nuclear distortion effect for the P-wave phase shifts are given in the author's thesis, Yale University (unpublished).

For phase shifts of the size observed or of the size caused by the vacuum polarization interaction, the only significant contributions to the p-p scattering cross section arise from interference terms involving the relatively large Coulomb or S-wave scattering amplitudes. Contributions quadratic in the phase shifts are negligibly small. Thus, it is sufficiently accurate to retain in the scattering matrix only terms linear in these phase shifts. It will be assumed, then, that the total phase shifts $\delta^L{}_J$ are sufficiently small for $L>0$ to justify one in replacing quantities $Q^L{}_J=(\sin\delta^L{}_J)\times\exp(i\delta^L{}_J)$ appearing in the formulation of the p-p scattering matrix given by Breit and Hull,[26] by $Q^L{}_J\simeq\delta^L{}_J\simeq(\delta^L{}_J)_n+\delta_L$. The higher powers of $\delta^L{}_J$ appearing in the exact expansion of $Q^L{}_J$ will be dropped. The vacuum polarization phase shifts δ_L do not depend on the value of the total angular momentum quantum number J, and do not therefore contribute in this linear approximation to the off-diagonal elements of the scattering matrix. Direct substitution of the phase shifts in the α_i representation of the unsymmetrized scattering matrix given by Breit and Hull[26] yields for the first-order vacuum polarization contributions

$$\Delta\alpha_1=\Delta\alpha_3=\Delta\alpha_4=0, \tag{3.1}$$

$$k(s_{00}-S^c)=\Delta\alpha_2=\Delta\alpha_5=\mathfrak{a}_{\text{vp}}, \tag{3.2}$$

where

$$\mathfrak{a}_{\text{vp}}(x)=\sum_{L=0}^{\infty}(2L+1)e_{L,0}\delta_L P_L(x). \tag{3.3}$$

Here $P_L(x)$ is the Legendre polynomial of order L, $x=\cos\theta$, and the Coulomb factors $e_{L,0}$ are as defined in the list of notation. In the present analysis, the Coulomb and S-wave scattering amplitudes will be treated exactly, with δ_0 removed from (3.3) and treated as a part of the large total S wave phase shift K_0. Equations (3) will then be valid so long as the total phase shifts for states with $L>0$ are small enough to deal with in the linear approximation. In this case the net effect of the vacuum polarization interaction will be to add to each diagonal element of the p-p scattering matrix the term $\mathfrak{a}_{\text{vp}}(x)-\delta_0$; and the calculational problem reduces to the evaluation of Eq. (3.3) and δ_0.

The phase shifts caused by the vacuum polarization potential will be calculated to first order in the interaction by means of the usual formula[1]

$$\delta_L=-(2/E)\int_0^\infty V(\rho)F_L{}^2(\rho)d\rho. \tag{4}$$

Here $F_L(\rho)$ is ρ times the unperturbed wave function for orbital angular momentum L, $\rho=kr$, $V(\rho)$ is the perturbing potential, and $E/2$ is to a sufficient approximation the scattering energy for two protons in the center-of-mass system. Since in the present case case nuclear distortion of the proton wave functions is to be neglected, $F_L(\rho)$ is an ordinary Coulomb wave function,[27]

$$F_L(\rho)=e^{-\pi\eta/2}\frac{|\Gamma(L+1+i\eta)|}{2\Gamma(2L+2)}(2\rho)^{L+1}e^{-i\rho}\times{}_1F_1(L+1-i\eta,2L+2;2i\rho), \tag{5.1}$$

where ${}_1F_1(a,c;z)$ is the confluent hypergeometric function,

$${}_1F_1(a,c;z)=1+\frac{a}{c}\frac{z}{1!}+\frac{a(a+1)}{c(c+1)}\frac{z^2}{2!}+\cdots. \tag{5.2}$$

Using the integral representation (1) of the vacuum polarization potential in (4) and interchanging the order of the integrations, a procedure which may be justified, one obtains

$$\delta_L=-\frac{4\alpha}{3\pi}\eta\int_0^1 d\xi\left(1+\frac{1}{2\xi^2}\right)\frac{(\xi^2-1)^{\frac{1}{2}}}{\xi^2}\int_0^\infty d\rho F_L{}^2(\rho)\times\rho^{-1}e^{-2\kappa\xi\rho/k}. \tag{6.1}$$

The second integral can be evaluated in terms of the ordinary hypergeometric function of Gauss. Using an integral representation for the confluent hypergeometric function,[28]

$${}_1F_1(a,c;z)=[\Gamma(c)/\Gamma(a)\Gamma(c-a)]\times\int_0^1 e^{zt}(1-t)^{c-a-1}t^{a-1}dt, \tag{6.2}$$

and performing the integration over ρ in Eq. (6.1), one obtains the general result

$$\int_0^\infty F_L{}^2(\rho)\rho^{-1}e^{-2\rho/z}d\rho=\tfrac{1}{2}M_L(1+2z^{-2}), \tag{6.3}$$

where

$$M_L(1+2z^{-2})=e^{-\pi n}|\Gamma(L+1+i\eta)|^{-2}\int_0^1 du\int_0^1 dt u^{L-i\eta}t^{L+i\eta}\times(1-u)^{L+i\eta}(1-t)^{L-i\eta}\int_0^\infty d\rho e^{-[(2/z)+2it-2iu]\rho}(2\rho)^{2L+1} \tag{6.4}$$

$$=\tfrac{1}{2}e^{-\pi\eta}\frac{\Gamma(2L+2)}{|\Gamma(L+1+i\eta)|^2}z^{2L+2}\int_0^1 du\int_0^1 dt u^{L-i\eta}\times t^{L+i\eta}(1-u)^{L+i\eta}(1-t)^{L-i\eta}(1+izt-izu)^{-2L-2}. \tag{6.5}$$

[26] G. Breit and M. H. Hull, Jr., Phys. Rev. **97**, 1047 (1955). If coupling between triplet states of the same J and different L is present, the extension of this formalism to the coupled case may be used, following the results of Breit, Ehrman, and Hull, Phys. Rev. **97**, 1051 (1955).

[27] The properties of the Coulomb wave functions have been studied extensively, notably by Breit and his collaborators. The essential properties of the functions and a bibliography thereon is given in *Tables of Coulomb Wave Functions*, National Bureau of

The last double integral may be compared with a known representation of the hypergeometric function[29]

$$\int_0^1 du\int_0^1 dt u^{a-1}t^{b-1}(1-u)^{c-a-1}(1-t)^{c-b-1}(1-ztu)^{-c}$$
$$=[\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)/\Gamma(c)\Gamma(c)]\times{}_2F_1(a,b;c;z), \quad (6.6)$$

where for $|z|<1$, ${}_2F_1(a,b;c;z)$ is defined by the usual power series in z. The changes in the variables of integration from t and u to x and y,

$$t=x/[1+iz-izx], \quad u=y/[1-iz+izy], \quad (6.7)$$

put (6.5) in the form of (6.6); one then obtains the required result for M_L,

$$M_L(1+2z^{-2})=\tfrac{1}{2}e^{-\pi\eta}\left(\frac{z^2}{1+z^2}\right)^{L+1}\left(\frac{1-iz}{1+iz}\right)^{i\eta}\times\frac{|\Gamma(L+1+i\eta)|^2}{\Gamma(2L+2)}\,{}_2F_1\left(L+1+i\eta,\; L+1-i\eta;\, 2L+2;\,\frac{z^2}{1+z^2}\right). \quad (6.8)$$

This result is closely related to the monopole integrals occurring in the quantum-mechanical treatment of the Coulomb excitation of nuclear energy levels.[30] The phase shifts δ_L caused by the vacuum polarization interaction become, to first order in the interaction,

$$\delta_L=-(\alpha\eta/3\pi)\int_0^1(1+\tfrac{1}{2}y)(1-y)^{\frac{1}{2}}y^{-1}\times M_L(1+\nu y^{-1})dy, \quad (7)$$

where the change of variable $y=\xi^{-2}$ has been made in Eq. (6.1) and ν is a parameter characteristic of vacuum polarization scattering,

$$\nu=2\kappa^2/k^2=4m^2c^2/ME.$$

The use of the Coulomb wave functions in (6.1) yields results considerably more complicated than those obtained with the free particle wave functions. The use of these wave functions leads to the well-known integrals[31]

$$(\pi/2)\int_0^\infty J_{L+\frac{1}{2}}{}^2(\rho)e^{-2\rho/z}d\rho=\tfrac{1}{2}Q_L(1+2z^{-2}). \quad (8.1)$$

Here Q_L is the Legendre function of the second kind. This second result may also be obtained directly from Eq. (6.9) by replacing η by zero. The expression for the phase shift δ_L in terms of Q_L neglects the distortion of the proton wave function by the Coulomb field, but gives, in fact, a surprisingly good approximation for δ_L above about 1 Mev,

$$\delta_L{}^{(0)}=-(\alpha\eta/3\pi)\int_0^1(1+\tfrac{1}{2}y)(1-y)^{\frac{1}{2}}y^{-1}Q_L(1+\nu y^{-1})dy. \quad (8.2)$$

This expression and Eq. (7) have been used in a numerical investigation of the properties of the vacuum polarization phase shifts. Equation (8.2) gives a first approximation for δ_L; the difference between Eqs. (7) and (8.2) then gives the correction to $\delta_L{}^{(0)}$ caused by the use of the correct Coulomb wave functions rather than free particle wave functions. The important case of (8.2) with $L=0$ was evaluated approximately as a series in powers of the small quantity $\nu/2$, yielding the result

$$\delta_0{}^{(0)}=-(\alpha\eta/6\pi)\{\tfrac{1}{2}[\ln(2/\nu)]^2+1.7615-0.2804\ln(2/\nu)+\cdots\}. \quad (8.3)$$

This representation of $\delta_0{}^{(0)}$ is quite accurate down to about 100 kev, the next term in the series being of order $\nu/2$. The special case of Eq. (8.2) with $L=1$ was given by Eriksen, Foldy, and Rarita.[18] For other values of L, the integral is most easily evaluated numerically. Values of $\delta_L{}^{(0)}$ have been calculated for $L\leq 5$ at a variety of energies and at 2 Mev the calculations were extended exactly to $L=10$ and by a semianalytic approximation for the L dependence of the phase shifts to $L=20$. The results for $L\leq 5$ are shown in Fig. 2. It is apparent from the slow decrease in the size of the phase shifts with increasing L, that many angular momentum states may be important in the vacuum polarization scattering. The variation of the δ_L with energy is also slow in the high L states. Both conditions result from the long range of the vacuum polarization forces as compared with the range of the centrifugal barrier and the proton wavelength.

The difference between Eqs. (7) and (8.2) was used to obtain by numerical integration the corrections to $\delta_L{}^{(0)}$ necessary because of the presence of the Coulomb field. The results are given in Table I. This table shows values of the correction in δ_L caused by Coulomb distortion of the wave function as a fraction of the phase shift $\delta_L{}^{(0)}$ obtained with neglect of the distortion. The variation with scattering energy and orbital angular momentum quantum number L of the $\delta_L{}^{(0)}$ is shown in Fig. 2. For $L=0$, the differences were calculated from Eqs. (8.3) and (8.3′); while for $L=1$, they were calculated using Eqs. (11) and (12) of Eriksen, Foldy, and Rarita.[18] The differences for $L>1$ were calculated from the integrals (7) and (8.2) for δ_L and $\delta_L{}^{(0)}$; these results may be in error by as much as 5–10%. In these calcu-

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[28] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Company, Inc., New York, 1953), p. 255.

[29] See reference 28, p. 78.

[30] The Coulomb excitation integrals are summarized by Alder, Bohr, Huus, Mottelson, and Winther, Revs. Modern Phys. **28**, 432 (1956).

[31] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, New York, 1952), second edition, p. 389.

TABLE I. Coulomb distortion effects on the vacuum polarization phase shifts as represented by values of $(\delta_L-\delta_L^{(0)})/\delta_L^{(0)}$.

L	1 Mev	2 Mev	4 Mev
0	−0.0717	−0.0393	
1	−0.0393	−0.0228	−0.0130
2	−0.0271	−0.0161	−0.00978
3	−0.0233	−0.0139	−0.00812
4	−0.0205	−0.0122	−0.00714
5	−0.0184	−0.0110	−0.00651

lations it was convenient to evaluate the functions M_L for $L=0$ and $L=1$ only and to obtain the functions for higher values of L by means of the following recurrence relation

$$[L+1+\eta^2(L+1)^{-1}]M_{L+1}(z)-(2L+1)\times[z+\eta^2L^{-1}(L+1)^{-1}]M_L(z)+[L+\eta^2L^{-1}]M_{L-1}(z)=0, \quad (9)$$

which reduces to that for the Legendre functions in the special case of $\eta=0$. It was, in fact, derived directly from the properties of the hypergeometric function appearing in M_L, with aid of the knowledge that for $\eta\to0$, $M_L\to Q_L$. A further complication in the evaluation of M_L is caused by the proximity of the argument of M_L to unity over most of the range of the integration in Eq. (7). The hypergeometric function in M_L, Eq. (6.8), assumes a logarithmic behavior for $z^2\to\infty$. This behavior can be taken into account explicitly by expanding the hypergeometric function about $z^2=\infty$. To do so it is convenient to use the formula[32]

$$ {}_2F_1(a,b;a+b;z)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\,{}_2F_1(a,b;1;1-z)\ln\frac{1}{1-z}-\frac{\Gamma(a+b)}{[\Gamma(a)\Gamma(b)]^2}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(n+1)\Gamma(n+1)}(1-z)^n\times[\psi(n+a)+\psi(n+b)-2\psi(n+1)], \quad (10.1)$$

where $\psi(x)$ is the logarithmic derivative of the gamma function,

$$\psi(x)=\Gamma'(x)/\Gamma(x). \quad (10.2)$$

Using these relations it is a simple matter to construct a series representation of $M_L(1+2z^{-2})$ which converges rapidly for $z^2\to\infty$, while the original expression, Eq. (6.9), suffices for calculations with $z^2\to0$.

$$M_L(1+2z^{-2})=\tfrac{1}{2}[\exp(-2\eta\cot^{-1}z)]\Bigg\{(1+z^{-2})^L\times{}_2F_1\left(-L-i\eta,-L+i\eta;1;\frac{1}{1+z^2}\right)\ln(1+z^2)-(1+z^{-2})^{-L-1}\sum_{n=0}^{\infty}\frac{\Gamma(L+n+1+i\eta)\Gamma(L+n+1-i\eta)}{\Gamma(L+1+i\eta)\Gamma(L+1-i\eta)}\times\frac{(1+z^2)^{-n}}{[\Gamma(n+1)]^2}\left[\sum_{m=0}^{L-1}\frac{2}{n+m+1}+\sum_{m=L+n+1}^{\infty}\frac{2\eta^2}{m(m^2+\eta^2)}\right]\Bigg\}. \quad (11)$$

The functions M_0 and M_1 were calculated at 1, 2, and 4 Mev using this series; the recurrence relation of Eq. (9) was then used to obtain the M_L for $L\geq1$ which was needed in the calculation of the Coulomb distortion corrections to the approximate phase shifts $\delta_L^{(0)}$. It is seen from the values listed in Table I that the fractional error incurred by approximating δ_L by $\delta_L^{(0)}$ is not large for the proton scattering energies of primary interest and, in fact, decreases fairly rapidly with increasing L and E. This suggests that calculations made using the $\delta_L^{(0)}$ may be fairly accurate; and, since the functions Q_L entering Eq. (8.2) are known in closed form, such

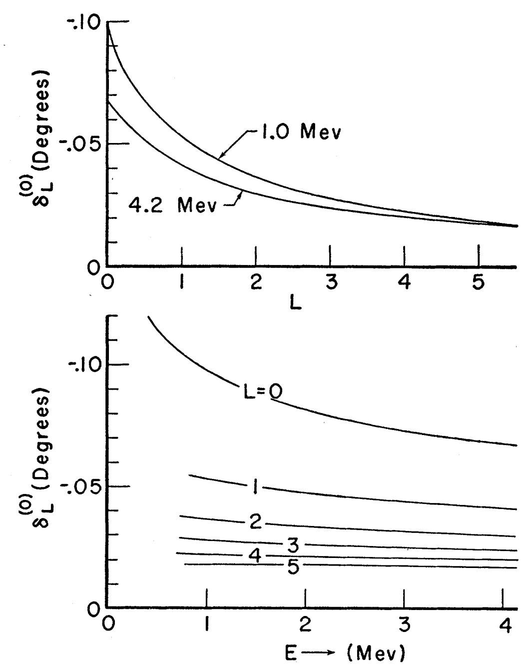

FIG. 2. The behavior as functions of the orbital angular momentum quantum number L and the scattering energy E of the vacuum polarization phase shifts $\delta_L^{(0)}$. Although L assumes only integral values, the upper curves are shown as continuous.

[32] This result was obtained from Barnes' representation of the hypergeometric function as a contour integral. The first part of the work paralleled the development given by E. T. Whittaker and G. N. Watson in *Modern Analysis* (The Macmillan Company, New York, 1946), Secs. 14.52 and 14.53, specializing the Barnes' representation to the case $c-a-b=0$. The result is easily checked for $a=b=L+1$, $z=2/(1+u)$, giving just the Legendre functions $Q_L(u)$ as required.

calculations are more easily made than those using the functions M_L and Eq. (7).

The integral of Eq. (7) has been evaluated approximately for $L=0$, using for M_0 the series of Eq. (11). Terms involving η^3 and higher powers of η were dropped, and terms in high powers of the small quantity $\nu/2$ were also neglected. Proceeding in this manner the phase shift δ_0 was approximated by

$$\delta_0=\delta_0^{(0)}+(\alpha\eta/3\pi)\{5.2\eta-3.83\eta(\nu/2)^{\frac{1}{2}}-0.29\eta^2 +[1.202\eta^2-2.36\eta(\nu/2)^{\frac{1}{2}}]\ln(2/\nu) +O[\eta^3,\eta^2(\nu/2)^{\frac{1}{2}},\eta(\nu/2)^{\frac{3}{2}}]\}. \quad (8.3')$$

This result for δ_0 should be accurate above 1 Mev, but terms in η^3, etc., may contribute significantly at lower energies.

(c) Vacuum Polarization Scattering Amplitude

The vacuum polarization scattering amplitude correct to first order in the interaction is given by the series of Eq. (3.3), in which the phase shifts δ_L appear weighted by the statistical factors $(2L+1)$. The phase shifts decrease slowly with increasing L (Fig. 2); and the real and imaginary parts of the quantities $(2L+1)e_{L,0}\delta_L$ reach their maximum values for fairly large values of L, as is seen in Fig. 3 ($L\simeq5$ and $L\simeq11$ for the real and imaginary parts of 2 Mev). Thus, despite the considerable amount of cancellation occurring at each angle because of the oscillations in sign of the Legendre polynomials of high order, phase shifts for states of high orbital angular momentum may contribute appreciably to the scattering amplitude. One may estimate roughly that value of L beyond which contributions to the scattering amplitude become very small by noting that for proton separations less than the classical turning point $kr\simeq L$, the wave function is strongly suppressed by the centrifugal barrier. Thus, if the potential becomes very small beyond a cutoff radius r_c, the maximum L for which the calculated phase shift will be significant is roughly $L\simeq kr_c$. The vacuum polarization potential decreases[12] for $2\kappa r\gg1$ as $(2\kappa r)^{-\frac{3}{2}}e^{-2\kappa r}$, and it may be assumed that the potential is negligibly small for $2\kappa r>1$. This rough estimate for the cutoff indicates that about 45 angular momentum states may be important in the scattering, a conclusion supported by numerical calculations of the scattering amplitude at 2 Mev using up to 21 phase shifts. The use of only the first few terms in the series for $\mathfrak{a}_{\text{vp}}$ Eq. (3.3), gave completely misleading results. Even using the first 21 terms in the series it was not possible to obtain definite values for $\mathfrak{a}_{\text{vp}}$ without resorting to the use of summation methods involving the averaging of successive partial sums of the series. It was apparent that only for much higher values of L would the contribution of the last term retained in the series be negligible relative to the sum up to that point.

Analytic summation of the series for $\mathfrak{a}_{\text{vp}}$ was used therefore. This summation proved possible in several

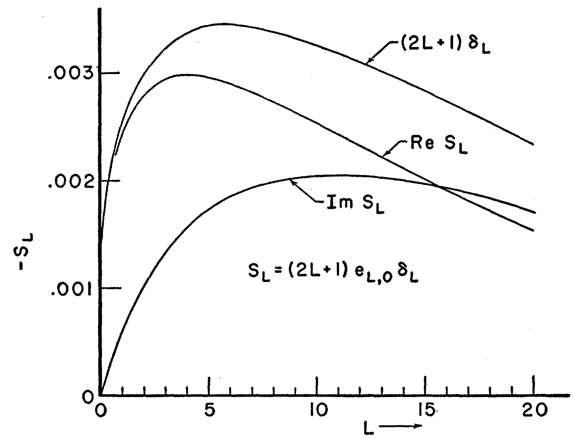

FIG. 3. The variation with L at 2 Mev of the quantity $S_L=(2L+1)e_{L,0}\delta_L$ entering in the calculation of the vacuum polarization scattering amplitude as in Eq. (3.3). The Coulomb factors $e_{L,0}$ are defined in the list of notation; $|e_{L,0}|=1$. The deviation of S_L from $(2L+1)\delta_L$ is an approximate measure of the influence of the $e_{L,0}$ on the vacuum polarization scattering.

approximations, the crudest of which is equivalent to the use of the first Born approximation for the scattering. This evaluation involves the use of the phase shifts $\delta_L^{(0)}$ calculated using free particle wave functions, and the approximation of the Coulomb factors $e_{L,0}$ by unity. The resulting series for $\mathfrak{a}_{\text{vp}}$ can be summed, or else the first Born approximation to the scattering amplitude employing undistorted plane waves may be evaluated by the usual method, yielding the representation

$$\mathfrak{a}_{\text{vp}}^{(0)}(x)=-(\alpha\eta/3\pi)\int_0^1(1+\tfrac{1}{2}y)(1-y)^{\frac{1}{2}} \times[(1-x)y+\nu]^{-1}dy. \quad (12.1)$$

Here the superscript on $\mathfrak{a}_{\text{vp}}^{(0)}(x)$ indicates that this is the lowest order approximation for $\mathfrak{a}_{\text{vp}}(x)$ with respect to the Coulomb field effects. The substitution $t=(1-y)^{\frac{1}{2}}$ reduces the equation to simple integrals, yielding the result given by Uehling,[12]

$$\mathfrak{a}_{\text{vp}}^{(0)}(x)=-(\alpha/3\pi)\eta F(x)/(1-x), \quad (12.2)$$

where

$$F(x)=-5/3+X+(1+X)^{\frac{1}{2}}(1-\tfrac{1}{2}X) \times\ln\left[\frac{(1+X)^{\frac{1}{2}}+1}{(1+X)^{\frac{1}{2}}-1}\right], \quad (12.3)$$

with

$$X=\nu/(1-x). \quad (12.3')$$

For all of the present p-p scattering experiments, the energies and angles are such that $X\ll1$. Thus, for practical purposes, $F(x)$ may be expanded in the series

$$F(x)=-5/3+\ln(4/X)+3X/2-(\tfrac{3}{8})X^2[\tfrac{1}{2}+\ln(4/X)] -(X^3/8)[\tfrac{2}{3}-\ln(4/X)]+\cdots. \quad (12.4)$$

The functions $\mathfrak{a}_{\text{vp}}^{(0)}(x)$ and $F(x)/(1-x)$ for $E=1855$ kev are plotted in Fig. 4. For comparison of the size

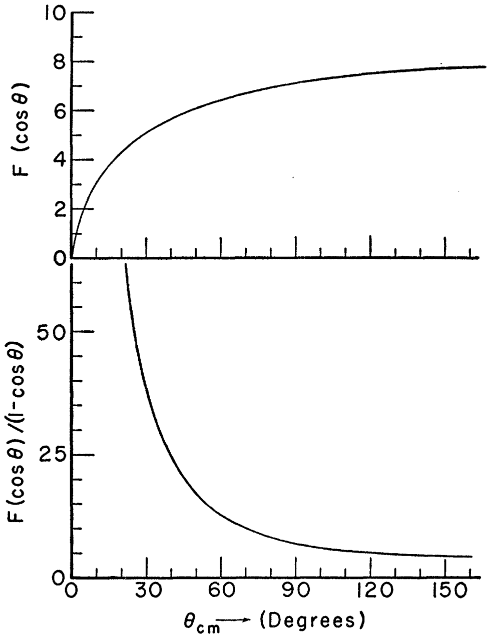

FIG. 4. The vacuum polarization scattering amplitude in the first Born approximation $\mathfrak{a}_{vp}^{(0)}$. Upper curve: the function $F(x)$ entering Eq. (12.2) which is used in the evaluation of $\mathfrak{a}_{vp}^{(0)}$. Here $x=\cos\theta$, where θ is the proton scattering angle in the center-of-mass system. Lower curve: the quantity $F(x)/(1-x)$; $\mathfrak{a}_{vp}^{(0)}(x)=(-\alpha\eta/3\pi)F(x)/(1-x)$.

of the vacuum polarization effects with the Coulomb and nuclear S wave parts of the p-p scattering cross section, the fractional change in the cross section caused by including in the scattering matrix $\mathfrak{a}_{vp}^{(0)}(x)$ is plotted in Fig. 5 for $E=1855$ kev and $K_0=44.2°$. The vacuum polarization S-wave phase shift $\delta_0^{(0)}$ has been subtracted from $\mathfrak{a}_{vp}^{(0)}$ leaving just the contributions from angular momentum states with $L>0$. Also shown in Fig. 5 is the fractional change in the cross section caused by a small negative mean P wave phase shift, $\bar{\delta}^p=-0.025°$. This is roughly half the size of the vacuum polarization P-wave phase shift at this energy. In the region of scattering angles around $\theta=35°$, the vacuum polarization and P-wave contributions to the cross section are similar in form; despite the differences occurring at larger and smaller scattering angles, there is enough over-all similarity between the effects to make confusion of vacuum polarization with true nuclear P wave scattering possible.

That the first-order undistorted plane wave approximation for $\mathfrak{a}_{vp}$ is not necessarily accurate may be inferred from the curves of Fig. 3, where the quantity $(2L+1)\delta_L$ entering the series for $\mathfrak{a}_{vp}^{(0)}$ is compared with the real and imaginary parts of $(2L+1)e_{L,0}\delta_L$. Even for quite small values of L, the quantity $e_{L,0}$ is significantly different from unity at the energies of interest, and the influence on the scattering of the phase shifts for large L is considerably modified. A next approximation to $\mathfrak{a}_{vp}$ may be obtained by using the free particle phase shifts $\delta_L^{(0)}$ combined with the exact values of the Coulomb factors $e_{L,0}$ in Eq. (3.3). This approximation is considerably better; the use of the phase shifts $\delta_L^{(0)}$ introduces a much smaller error into $\mathfrak{a}_{vp}$ than does the approximation of the $e_{L,0}$ by unity. Details on this approximation are given in the author's dissertation.[33]

The exact summation of the series (3.3) for $\mathfrak{a}_{vp}(x)$ will now be undertaken. Substitution of the representation of Eq. (7) for δ_L in Eq. (3.3) yields the result

$$\mathfrak{a}_{vp}(x)=-(\alpha\eta/3\pi)\sum_{L=0}^{\infty}(2L+1)e_{L,0}\int_0^1(1+\tfrac{1}{2}y)\times(1-y)^{\frac{1}{2}}y^{-1}M_L(1+\nu y^{-1})P_L(x)dy. \quad (13.1)$$

The order in which the summation and the integration are performed may be reversed. Thus, introducing the explicit form of M_L from Eq. (6.9), and writing the Coulomb factors $e_{L,0}$ as

$$e_{L,0}=\frac{\Gamma(L+1+i\eta)\Gamma(1-i\eta)}{\Gamma(L+1-i\eta)\Gamma(1+i\eta)}, \quad (13.2)$$

one obtains

$$\mathfrak{a}_{vp}(x)=-(\alpha\eta/3\pi)\frac{\Gamma(1-i\eta)}{\Gamma(1+i\eta)}\int_0^1(1+\tfrac{1}{2}y)(1-y)^{\frac{1}{2}}y^{-1}\times\{\exp[-2\eta\tan^{-1}(\nu/2y)^{\frac{1}{2}}]\}\sum_{L=0}^{\infty}(2L+1)\times P_L(x)\left(\frac{2y}{2y+\nu}\right)^{L+1}\frac{[\Gamma(L+1+i\eta)]^2}{2\Gamma(2L+2)}\times{}_2F_1\left(L+1+i\eta,L+1-i\eta;2L+2;\frac{2y}{2y+\nu}\right)dy. \quad (13.3)$$

It is convenient to consider the series occurring in this representation with z replacing $2y/(2y+\nu)$; the hypergeometric function ${}_2F_1$ may be expressed in terms of a confluent hypergeometric function ${}_1F_1$ by an integral relation easily derived from the representation (6.2) for ${}_1F_1$, *viz.*,

$$\Gamma(a)\,{}_2F_1(a,b;c;x/y)=y^a\int_0^\infty e^{-yt}t^{a-1}\,{}_1F_1(b,c;xt)dt, \quad |x/y|<1, \quad (14.1)$$

whence

$$z^{L+1+i\eta}\Gamma(L+1+i\eta)\,{}_2F_1(L+1+i\eta,L+1-i\eta;2L+2;z)=\int_0^\infty e^{-t/z}t^{L+i\eta}\,{}_1F_1(L+1-i\eta,2L+2;t)dt. \quad (14.2)$$

Using this expression in (13.3), and again interchanging the order of the summation and the integrations, one

[33] L. Durand, dissertation, Yale University, 1957 (unpublished).

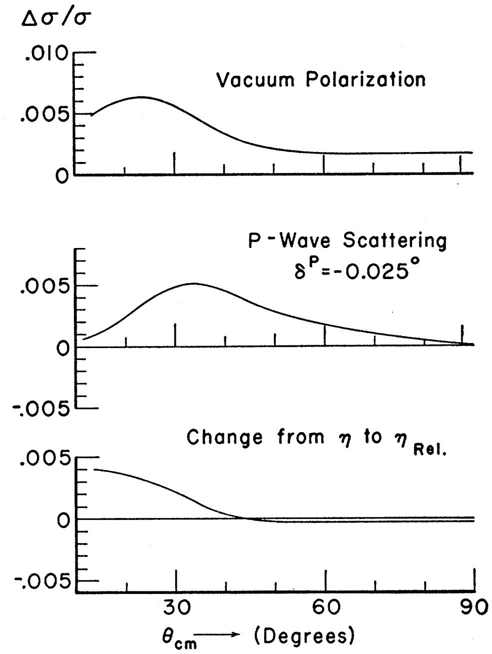

FIG. 5. The fractional change in the theoretical p-p scattering cross section caused by various effects discussed in the text. The parameters determining the initial cross section σ_{th} are $E=1855$ kev and $K_0=44.25°$. This value of K_0 gives a good fit to the 1855-kev experimental data.[7] Top: the change caused by including in the p-p scattering matrix the vacuum polarization scattering amplitude $\mathfrak{a}_{\text{vp}}^{(0)}$ with $\delta_0^{(0)}$ omitted. Middle: the fractional change in the cross section caused by a mean P-wave phase shift $\delta^P=-0.025°$. Bottom: the fractional change caused by using the relativistic rather than the nonrelativistic value of the Coulomb parameter η.

obtains for the series alone

$$\sum_{L=0}^{\infty}(2L+1)z^{L+1}P_L(x)[\Gamma(L+1+i\eta)]^2 \times {}_2F_1(L+1+i\eta,\, L+1-i\eta;\, 2L+2;\, z)/\Gamma(2L+2)$$
$$=z^{-i\eta}\int_0^\infty dt\,t^{i\eta}e^{-t/z}\sum_{L=0}^{\infty}(2L+1)t^LP_L(x)\Gamma(L+1+i\eta) \times {}_1F_1(L+1-i\eta,\, 2L+2;\, t)/\Gamma(2L+2). \quad (15.1)$$

This series may be related to the expansion of a Coulomb scattered wave given by Gordon,[34]

$$\Gamma(1+i\eta)e^{iux/2}\,{}_1F_1(-i\eta,\, 1;\, \tfrac{1}{2}iu(1-x))$$
$$=\sum_{L=0}^{\infty}(2L+1)(iu)^LP_L(x)\frac{\Gamma(L+1+i\eta)}{\Gamma(2L+2)}e^{-iu/2} \times {}_1F_1(L+1-i\eta,\, 2L+2;\, iu), \quad (15.2)$$

where for Coulomb scattering $u=2kr$, and the right hand side of the equation is usually expressed in terms of the Coulomb wave functions of Eq. (5.1). The series of Eq. (15.1) has precisely this form. Identifying t with the variable iu appearing in (15.2), Eq. (15.1) may be reduced to the form

$$z^{-i\eta}\Gamma(1+i\eta)\int_0^\infty t^{i\eta}e^{-t/z+t(1+x)/2} \times {}_1F_1[-i\eta,\, 1;\, \tfrac{1}{2}t(1-x)]dt. \quad (15.3)$$

This is a convergent integral provided $\mathrm{Re}(1/z)>1$, a condition satisfied in the present application. The integration may be performed using the result of Eq. (14.1) yielding finally

$$\sum_{L=0}^{\infty}(2L+1)z^{L+1}P_L(x)[\Gamma(L+1+i\eta)]^2 \times {}_2F_1(L+1+i\eta,\, L+1-i\eta;\, 2L+2;\, z)/\Gamma(2L+2)$$
$$=[\Gamma(1+i\eta)]^2z^{-i\eta}[(1/z)-(1+x)/2]^{-1-i\eta} \times {}_2F_1\left(-i\eta,\, 1+i\eta;\, 1;\, \frac{(1-x)z}{2-(1+x)z}\right). \quad (15.4)$$

Substituting in Eq. (13.3) for $\mathfrak{a}_{\text{vp}}$ with $z=2y/(2y+\nu)$, one obtains an integral representation for $\mathfrak{a}_{\text{vp}}$ exact in its inclusion of Coulomb field effects on the first-order scattering amplitude,

$$\mathfrak{a}_{\text{vp}}(x)=-(\alpha\eta/3\pi)|\Gamma(1+i\eta)|^2\int_0^1(1+\tfrac{1}{2}y)(1-y)^{\frac{1}{2}} \times[(1-x)y+\nu]^{-1}\left[\frac{2y+\nu}{(1-x)y+\nu}\right]^{i\eta}$$
$$\times\exp[-2\eta\tan^{-1}(\nu/2y)^{\frac{1}{2}}] \times {}_2F_1\left(-i\eta,\, 1+i\eta;\, 1;\, \frac{y}{y+X}\right)dy. \quad (16)$$

If η is set equal to zero everywhere except in the factor $-(\alpha\eta/3\pi)$ multiplying the entire quantity, Eq. (16) reduces exactly to the free wave approximation for the scattering amplitude as in Eq. (12.1).

A relatively simple and accurate approximation for $\mathfrak{a}_{\text{vp}}$ may be obtained by expanding the exact result of Eq. (16) in powers of the small quantity $\eta=e^2/\hbar v$ and retaining only the first few terms. At the lowest energy at which proton-proton scattering experiments have been performed, 200 kev, η has the value 0.35; η decreases with increasing energy, varying from 0.13 to 0.08 over the range of the most accurate experiments, 1.8–4.2 Mev. Hence if the energy is not too low an expansion of $\mathfrak{a}_{\text{vp}}$ in powers of η may be expected to converge fairly rapidly. The parameter ν is also small, $\nu\simeq 1/180$ at 200 kev, $\nu\simeq 1/1800$ at 2 Mev. Thus the argument of the hypergeometric function in (16), $u=(1-x)y/[(1-x)y+\nu]=y/(y+X)$ is close to unity over most of the interval in y contributing significantly to the integral. It is more convenient to expand the hypergeometric function in powers of $1-u$, a change

[34] W. Gordon, Z. Physik 48, 180 (1928).

accomplished by the transformation of Eq. (10.1),

$$|\Gamma(1+i\eta)|^2\,{}_2F_1[-i\eta,1+i\eta;1;y/(y+X)] = -i\eta\{\ln[(y+X)/X]\}\,{}_2F_1[-i\eta,1+i\eta;1;X/(y+X)] - \frac{i\eta}{\Gamma(-i\eta)\Gamma(1+i\eta)} \times\sum_{n=0}^{\infty}\frac{\Gamma(n-i\eta)\Gamma(n+1+i\eta)}{\Gamma(n+1)\Gamma(n+1)}\left(\frac{X}{y+X}\right)^n \times\left[\frac{1}{n-i\eta}-\sum_{m=n+1}^{\infty}\frac{2\eta^2}{m(m^2+\eta^2)}\right], \quad (17.1)$$

with a further expansion of the remaining quantities in powers of η, one obtains the required result given here correct to second order in η,

$$|\Gamma(1+i\eta)|^2\,{}_2F_1[-i\eta,1+i\eta;1;y/(y+X)] = 1-i\eta\ln\left(\frac{y+X}{X}\right)+\eta^2\left[\ln\left(\frac{y+X}{X}\right)\right]\left[\ln\left(\frac{y}{y+X}\right)\right] -\eta^2\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{X}{y+X}\right]^n+O(i\eta^3). \quad (17.2)$$

The entire integrand of Eq. (16) may now be expanded in powers of η. The factor $\exp[-2\eta\tan^{-1}(\nu/2y)^{\frac{1}{2}}]$ involves also the small quantity $(\nu/2)^{\frac{1}{2}}$; consequently, only terms through order η will be retained in its expansion. The function $\mathfrak{a}_{\rm vp}$ may then be written in the form

$$\mathfrak{a}_{\rm vp}(x)=\mathfrak{a}_{\rm vp}^{(0)}(x)+\mathfrak{a}_{\rm vp}^{(1)}(x)+\mathfrak{a}_{\rm vp}^{(2)}(x)+\cdots, \quad (18.1)$$

where the terms added to the first Born approximation term $\mathfrak{a}_{\rm vp}^{(0)}$ are

$$\mathfrak{a}_{\rm vp}^{(1)}(x)=-(\alpha\eta/3\pi)\int_0^1(1+\tfrac{1}{2}y)(1-y)^{\frac{1}{2}} \times[(1-x)y+\nu]^{-1}\left\{-2\eta\tan^{-1}\left(\frac{\nu}{2y}\right)^{\frac{1}{2}} +i\eta\left[\ln\left(\frac{2y+\nu}{\nu}\right)-2\ln\left(\frac{y+X}{X}\right)\right]\right\}dy, \quad (18.2)$$

and

$$\mathfrak{a}_{\rm vp}^{(2)}(x)=-(\alpha\eta/3\pi)\int_0^1 dy(1+\tfrac{1}{2}y)(1-y)^{\frac{1}{2}} \times[(1-x)y+\nu]^{-1}\left\{\frac{\eta^2}{2}\left[\ln\left(\frac{2y+\nu}{(1-x)y+\nu}\right)\right] \times\left[3\ln\left(\frac{y+X}{X}\right)-\ln\left(\frac{2y+\nu}{\nu}\right)\right] +\eta^2\left[\ln\left(\frac{y+X}{X}\right)\right]\left[\ln\left(\frac{y}{y+X}\right)\right] -\eta^2\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{X}{y+X}\right]^n\right\}. \quad (18.3)$$

The integrals for $\mathfrak{a}_{\rm vp}^{(1)}$ and $\mathfrak{a}_{\rm vp}^{(2)}$ can apparently not be evaluated in closed form but $\mathfrak{a}_{\rm vp}^{(1)}$ may be obtained quite accurately using an approximation based on the closeness of the factor

$$(1+\tfrac{1}{2}y)(1-y)^{\frac{1}{2}}=1-\tfrac{3}{8}y^2-\tfrac{1}{8}y^3+\cdots$$

to unity. The factor is rewritten in the form

$$1+[(1+\tfrac{1}{2}y)(1-y)^{\frac{1}{2}}-1],$$

and the integrals are split accordingly into two parts. The parts involving the term "1" are integrable in relatively simple form, while those parts involving the term in square brackets are much smaller and may be approximated. Details of the integrations appear in the author's dissertation.[35] The result obtained for $\mathfrak{a}_{\rm vp}^{(1)}(x)$, valid for $\nu/(1-x)\ll 1$, is

$$\operatorname{Im}\mathfrak{a}_{\rm vp}^{(1)}(x)=\frac{\eta}{1-x}\left(\frac{\alpha\eta}{3\pi}\right)\left\{\frac{1}{2}\left[\ln\left(\frac{1}{X}\right)\right]^2 -\left[\ln\left(\frac{2}{1-x}\right)\right]\left[\ln\left(\frac{1}{X}\right)\right] +\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{1+x}{2}\right)^n+0.1166 -0.2804\left[\ln\left(\frac{2}{\nu}\right)-2\ln\left(\frac{2}{1-x}\right)\right]+O(X)\right\}, \quad (19.1)$$

and

$$\operatorname{Re}\mathfrak{a}_{\rm vp}^{(1)}(x)=\frac{4\eta}{1-x}\left(\frac{\alpha}{3\pi}\eta\right)\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} \times\left[\tan^{-1}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}-\tan^{-1}\left(\frac{\nu}{2}\frac{1+x}{1-x}\right)^{\frac{1}{2}}\right]. \quad (19.2)$$

This result is not very convenient for numerical work, but a simpler formula obtained by dropping the smallest terms suffices for energies above a few hundred kilovolts. This simpler result replaces the imaginary part of the above quantity by

$$\operatorname{Im}\mathfrak{a}_{\rm vp}^{(1)}(x)\simeq\frac{\eta}{1-x}\left(\frac{\alpha}{3\pi}\eta\right)\left[\ln\left(\frac{1}{X}\right)\right] \times\left[\tfrac{1}{2}\ln\left(\frac{2}{\nu}\right)-\tfrac{3}{2}\ln\left(\frac{2}{1-x}\right)\right]. \quad (19.2')$$

The real part of $\mathfrak{a}_{\rm vp}^{(1)}$ is quite small, being of the same magnitude at 2 Mev as the contributions to $\operatorname{Re}\mathfrak{a}_{\rm vp}$ from $\mathfrak{a}_{\rm vp}^{(2)}$. The term $\operatorname{Im}\mathfrak{a}_{\rm vp}^{(1)}$ arises entirely from the Coulomb factors $e_{L,0}$, while $\operatorname{Re}\mathfrak{a}_{\rm vp}^{(1)}$ arises from the alteration of the phase shifts caused by the Coulomb distortion of the proton wave functions. These con-

[35] Details of most of the necessary integrations are given in Sec. 4 and Appendix A of the author's dissertation, reference 33. The manipulations are for the most part elementary.

clusions are apparent if the various factors entering Eqs. (16) and (18.2) are traced back to their origins and the functions M_L and Q_L entering the formulas (7) and (8.2) for the distorted and undistorted phase shifts are compared.

The small second-order correction to $\mathfrak{a}_{\mathrm{vp}}^{(0)}$ has been evaluated roughly by replacing the factor $(1+\frac{1}{2}y)(1-y)^{\frac{1}{2}}$ in Eq. (18.3) by unity. The error thus incurred is only a few percent of the total value of the integral. The factor $\ln\{(2y+\nu)/[(1-x)y+\nu]\}$ was approximated by $\ln[2/(1-x)]$, an approximation which is permissible provided $X=\nu/(1-x)\ll 1$. The result thus obtained is

$$\mathfrak{a}_{\mathrm{vp}}^{(2)}(x)\simeq-\frac{\eta^2}{1-x}\left(\frac{\alpha}{3\pi}\eta\right)\left\{\left[\ln\left(\frac{2}{1-x}\right)\right]\left[\ln\left(\frac{1}{X}\right)\right]\times\left[\frac{1}{2}\ln\left(\frac{2}{\nu}\right)-\ln\left(\frac{2}{1-x}\right)\right]+\frac{1}{2}\left[\ln\left(\frac{2}{1-x}\right)\right]\times\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{1+x}{2}\right)^n-1.202\right\}. \quad (20)$$

These high-order contributions to $\mathfrak{a}_{\mathrm{vp}}$ are graphed in Fig. 6 as fractions of $\mathfrak{a}_{\mathrm{vp}}^{(0)}$ for an energy of 1855 kev. The imaginary part of $\mathfrak{a}_{\mathrm{vp}}^{(1)}$ is seen to be quite large, reaching about 50% of the value of $\mathfrak{a}_{\mathrm{vp}}^{(0)}$ at both small and large angles. The correction $(\mathrm{Re}\,\mathfrak{a}_{\mathrm{vp}}^{(1)}+\mathfrak{a}_{\mathrm{vp}}^{(2)}-\Delta\delta_0)$ is plotted as a unit, the functions $\mathrm{Re}\,\mathfrak{a}_{\mathrm{vp}}^{(1)}$ and $\mathfrak{a}_{\mathrm{vp}}^{(2)}$ being similar in size. Here $\Delta\delta_0$ is the difference $(\delta_0-\delta_0^{(0)})$, representing the effect of Coulomb distortion on the S wave vacuum polarization phase shift. It is convenient to remove this difference from $\mathfrak{a}_{\mathrm{vp}}$ here, and the remainder of δ_0, namely $\delta_0^{(0)}$, may be removed separately from $\mathfrak{a}_{\mathrm{vp}}^{(0)}$, δ_0 being treated as only a part of the total S wave phase shift K_0. It is seen that the ratio

$$(\mathrm{Re}\,\mathfrak{a}_{\mathrm{vp}}^{(1)}+\mathfrak{a}_{\mathrm{vp}}^{(2)}-\Delta\delta_0)/\mathfrak{a}_{\mathrm{vp}}^{(0)}$$

is fairly small especially for small angles where the vacuum polarization scattering is most important. This contribution to $\mathfrak{a}_{\mathrm{vp}}$ has been neglected at and above 1855 kev, although it has been included in the calculations at lower energies.

The vacuum polarization scattering amplitude to first order in the interaction is given including all Coulomb effects exactly by Eq. (16). It was found to be convenient to expand this exact result in a power series in the small quantity η; and terms through order η^2 are given approximately by Eqs. (19) and (20). These equations are sufficiently accurate for numerical work over the range of size of the parameters ν and η of the most interest, but are restricted by the condition that $\nu/(1-x)\ll 1$ ($\nu=1-x$ for $\theta=1.4°$ at 1855 kev).

III. RELATIVISTIC AND MAGNETIC SCATTERING CORRECTIONS

Relativistic and magnetic corrections in high-energy proton-proton scattering have been treated extensively by several authors, but little attention has been given such corrections at low energies.[21] The simplest relativistic effects are the well-known changes in the transformations connecting the cross sections and scattering angles measured in the laboratory system with those in the center-of-mass system.[36] These kinematic corrections, while small, are not negligible even at low energies. The change in the transformation of the scattering angles results, for example, in a diminution of the theoretical *p-p* cross section in the region of dominant Coulomb scattering by the factor $1-(E/Mc^2)$, a change of 0.43% at 4 Mev. Of a more subtle nature are the dynamic relativistic and magnetic contributions to the *p-p* scattering matrix discussed by Breit[19] and Garren.[20] These authors have considered the modification of the scattering amplitudes necessitated by a relativistic treatment of Coulomb and magnetic scattering correct to terms linear in e^2 and v^2/c^2. It was found that in the usual Coulomb scattering amplitudes, the non-relativistic value of the parameter η, $\eta=(e^2/\hbar)\times(M/2E)^{\frac{1}{2}}$, should be replaced by the value $\eta=e^2/\hbar v$, where v is the laboratory velocity of the incident proton calculated relativistically. The value in the center-of-mass system of the parameter k^2, $k^2=ME/(2\hbar^2)$, is correct relativistically. The relativistic treatment of Coulomb scattering results also in the appearance of

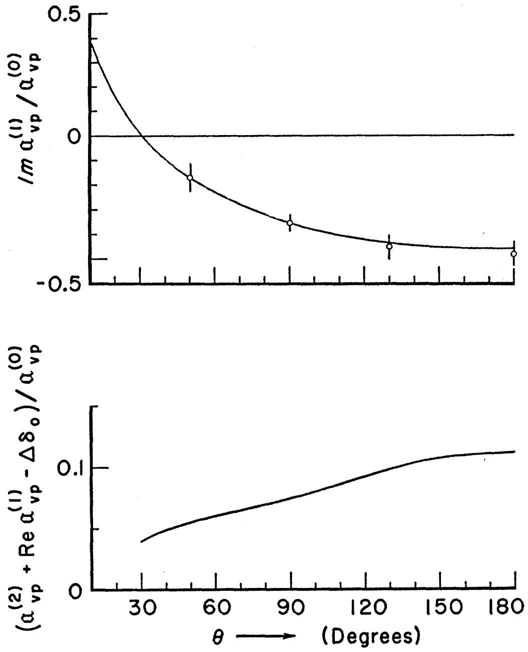

FIG. 6. The leading additions to $\mathfrak{a}_{\mathrm{vp}}^{(0)}$ in the expansion of the first order vacuum polarization scattering amplitude $\mathfrak{a}_{\mathrm{vp}}$ in powers of η. Top: $\mathrm{Im}\,\mathfrak{a}_{\mathrm{vp}}^{(1)}/\mathfrak{a}_{\mathrm{vp}}^{(0)}$. $\mathfrak{a}_{\mathrm{vp}}^{(0)}$ is shown in Fig. 4. The calculated points were obtained by numerical summation of the imaginary part of Eq. (3.3), using the first 21 phase shifts $\delta_L^{(0)}$. The bars indicate the uncertainty in the sum. Bottom: $(\mathrm{Re}\,\mathfrak{a}_{\mathrm{vp}}^{(1)}+\mathfrak{a}_{\mathrm{vp}}^{(2)}-\Delta\delta_0)/\mathfrak{a}_{\mathrm{vp}}^{(0)}$. Here $\Delta\delta_0$ is the difference $\delta_0-\delta_0^{(0)}$. All further additions to $\mathfrak{a}_{\mathrm{vp}}^{(0)}$ are of order at least η^3.

[36] O. Chamberlain and C. Wiegand, Phys. Rev. 79, 81 (1950).

additional spin-dependent forces between the protons, similar in origin to the $\mathbf{L}\cdot\mathbf{S}$ forces characteristic of the Dirac theory of an electron in a central field. The phase shifts caused by these added interactions between protons have been calculated correct to terms linear in e^2 and v^2/c^2 by Ebel and Hull,[37] while Breit[19] has calculated the corresponding scattering amplitudes in the first Born approximation after showing such a calculation to be equivalent to this order to a phase shift analysis. Garren[20] has included in his work the additions to the scattering matrix caused by the anomalous magnetic moment of the proton also calculated in the first Born approximation. The uncertainty in these results arising from the distortion of the proton wave functions by the Coulomb and nuclear forces has recently been considered by Breit,[21] but such distortion effects are relatively unimportant at low energies except in the S state. If for low scattering energies, the added matrix elements are expanded in powers of v^2/c^2, it is necessary to retain only those terms contributing to the cross section in the order v^2/c^2 or less; higher order terms in the cross section are insignificant at the energies of interest ($v^2/c^2<0.0053$ for $E<5$ Mev). In the absence of large triplet phase shifts, the off-diagonal matrix elements contribute to the cross-section terms of order v^4/c^4 or higher and will be neglected. Similarly, the parts of the diagonal matrix elements arising from relativistic effects contribute detectably to the cross section only through interference terms involving the relatively large Coulomb or nuclear S-wave scattering amplitudes. In the notation of Breit and Hull,[26] the magnetic and relativistic additions to the diagonal elements of the antisymmetrized scattering matrix are, to order v^2/c^2,

$$k\Delta S^a{}_{1,1}=-\tfrac{1}{2}k\Delta S^a{}_{0,0}=k\Delta S^a{}_{-1,-1}, \tag{21.1}$$

$$2^{-\frac{1}{2}}k\Delta S^a{}_{0,0}=-(\eta E/2Mc^2)(\mu_p+\mu_p{}^2), \tag{21.2}$$

$$2^{-\frac{1}{2}}k\Delta S^a{}_{1,1}=\eta E/8Mc^2(\mu_p{}^2+2\mu_p-1)\cos\theta. \tag{21.3}$$

Here μ_p is the anomalous part of the proton magnetic moment expressed in units of the nuclear magneton. The subscripts i and j on $\Delta S^a{}_{i,j}$ refer to the value in units of $\hbar$ of the total spin of the two protons in the scattered and the incident states respectively. Since $2\eta\ln\sin(\theta/2)$ is small in the angular region and at the energies at which the fractional changes in the cross section caused by each matrix element are of detectable magnitude, the not altogether certain factor

$$\exp[-2i\eta\ln\sin(\theta/2)]$$

by which the above elements of ΔS^a may have to be multiplied was approximated by unity in obtaining Eqs. (21). The contributions to the triplet cross section of order v^2/c^2 vanish in the absence of large triplet nuclear phase shifts; the $\Delta S^a{}_{i,i}$ then contribute to ${}^T\sigma$ only through an interference term with the triplet Coulomb scattering amplitude S^{ac}, entering in the combination

$$(\mathrm{Re}\,S^{ac})(\Delta S^a{}_{1,1}+\Delta S^a{}_{0,0}+\Delta S^a{}_{-1,-1})=0. \tag{22}$$

The singlet amplitude $\Delta S^a{}_{0,0}$ adds to the scattering matrix an isotropic term representing just an S-wave phase shift which should be treated as part of the total phase shift K_0. The source of this phase shift was pointed out by Breit[21] who showed that it originated from what are formally magnetic contact interactions between the protons and that the value obtained in a calculation of the phase shift is quite sensitive to the behavior of the wave function in the region in which the electric field of the proton deviates significantly from that of a point charge. This region is certainly within the range of the specifically nuclear interaction; thus the form of the wave function and the value of the phase shift are very uncertain. Since the magnetic forces are of a short-range character, they may be regarded as only one part of the total short-range S-state interaction between the protons, as has been pointed out by Breit.[38] From this point of view it is unnecessary to distinguish between the parts of K_0 arising from the magnetic and the nuclear interactions, but it is still convenient to separate out the part of K_0 caused by the long-range vacuum polarization forces.

It is concluded that the additions to the p-p scattering matrix caused by the relativistic and magnetic modifications of the Coulomb interaction enumerated by Breit[19] and Garren[20] may be dropped at low energies. The only significant modification of the low-energy Coulomb scattering is the change in the definition of the parameter η.[19] That this change should not be neglected in the analysis of the low-energy p-p scattering data is seen from the curve of Fig. 5 showing the fractional change in the theoretical 1855-kev cross section caused by changing η from its nonrelativistic to its relativistic value. The cross section was calculated assuming an S-wave phase shift K_0 of 44.25°, roughly the value fitting the 1855-kev experiments. The effect becomes larger at higher energies. The electromagnetic interactions leading in the case of electrons to the Lamb shift of atomic energy levels[16] and to corrections for radiation (bremsstrahlung) in the Coulomb scattering of electrons,[39] are, in the case of protons, diminished in magnitude by powers of the ratio of the electronic to the protonic mass.[22] Higher order vacuum polarization corrections[15] are likewise negligible. It is therefore thought that the electromagnetic interactions between low-energy protons are adequately described in the absence of large triplet phase shifts by the relativistic Coulomb and the first-order vacuum polarization scattering amplitudes alone.

[37] M. E. Ebel and M. H. Hull, Jr., Phys. Rev. 99, 1596 (1955).

[38] The author is grateful to Professor G. Breit for discussions regarding the origin and the uncertainties in this S-wave effect.

[39] J. Schwinger, Phys. Rev. 76, 790 (1949).

IV. VACUUM POLARIZATION CONTRIBUTIONS TO THE p-p SCATTERING CROSS SECTION

The vacuum polarization contributions to the low-energy p-p scattering cross sections have been calculated at energies spanning the range of the most accurate experiments.[7] It has been assumed that any nuclear phase shifts in states with $L>0$ are so small that only terms linear in the total phase shifts contribute significantly to the cross section. In this case the effect of the vacuum polarization interaction is to add to the diagonal elements of the p-p scattering matrix the term $\mathfrak{a}_{\rm vp}(x)$ in accordance with the discussion of Sec. 2, Eqs. (3.1), (3.2), (3.3). The vacuum polarization S wave phase shift δ_0 should be omitted from $\mathfrak{a}_{\rm vp}$, since it is only one part of the large total phase shift K_0; any vacuum polarization effects in S-wave scattering may then be treated as was done by Foldy and Eriksen.[9]

Tracing through the construction and antisymmetrization of the p-p scattering matrix as given by Breit and Hull,[26] one finds for the singlet and triplet scattering amplitudes correct to first order in the vacuum polarization interaction

$$^{S}a={}^{S}a_c+[\exp(iK_0)]\sin K_0+{}^{S}\mathfrak{a}_{\rm vp}-\delta_0, \quad (23.1)$$

$$^{T}a={}^{T}a_c+{}^{T}\mathfrak{a}_{\rm vp}. \quad (23.2)$$

Here $^{S}a_c$ and $^{T}a_c$ are the singlet and triplet Coulomb scattering amplitudes,

$$^{S,T}a_c=-(\eta/4)\{[\sin^{-2}(\theta/2)]\exp[-i\eta\ln\sin^2(\theta/2)] \pm[\cos^{-2}(\theta/2)]\exp[-i\eta\ln\cos^2(\theta/2)]\}, \quad (23.3)$$

where the plus and minus signs correspond to the singlet and triplet states, respectively. Similarly $^{S}\mathfrak{a}_{\rm vp}$ and $^{T}\mathfrak{a}_{\rm vp}$ are the antisymmetrized vacuum polarization scattering amplitudes,

$$^{S,T}\mathfrak{a}_{\rm vp}(\cos\theta)=\tfrac{1}{2}[\mathfrak{a}_{\rm vp}(\cos\theta)\pm\mathfrak{a}_{\rm vp}(-\cos\theta)]. \quad (23.4)$$

In the convention used here regarding the factors of two which enter in the process of antisymmetrizing the scattering amplitudes,[26] and in the absence of triplet p-p phase shifts, the differential scattering cross section is given by

$$k^2\sigma(\theta)=|{}^{S}a(\theta)|^2+3|{}^{T}a(\theta)|^2. \quad (24.1)$$

The cross section may be expanded into parts representing the dominant S-wave and Coulomb scattering, with additional terms involving the vacuum polarization scattering. Contributions to $\sigma(\theta)$ involving $|\mathfrak{a}_{\rm vp}|^2$ will be dropped, both because they are too small to detect, and because other terms involving the squares of the phase shifts δ_L have been neglected in the evaluation of $\mathfrak{a}_{\rm vp}$. If $\sigma_{\rm th}(\theta)$ is that part of $\sigma(\theta)$ involving only the Coulomb and S-wave scattering amplitudes, and if $\Delta\sigma_{\rm vp}$ is the contribution to $\sigma(\theta)$ linear in $\mathfrak{a}_{\rm vp}$ and δ_0, then

$$\sigma(\theta)\simeq\sigma_{\rm th}(\theta)+\Delta\sigma_{\rm vp}(\theta), \quad (24.2)$$

where

$$k^2\Delta\sigma_{\rm vp}=2\,{\rm Re}[({}^{S}\mathfrak{a}_{\rm vp}-\delta_0)({}^{S}a_c+[\exp(iK_0)]\sin K_0)^*] +6\,{\rm Re}[{}^{T}\mathfrak{a}_{\rm vp}\,{}^{T}a_c^*]. \quad (24.3)$$

There may be additional contributions to $\sigma(\theta)$ involving nuclear phase shifts for states with $L>0$, but the foregoing construction for $\Delta\sigma_{\rm vp}$ will be valid if the contributions to the cross section involving the square or higher powers of the total phase shifts are negligible.

The vacuum polarization contributions $\Delta\sigma_{\rm vp}$ to the p-p scattering cross sections were calculated for the energies and scattering angles of the Worthington, McGruer, and Findley experiments,[7] as well as for 1397 kev. The vacuum polarization scattering amplitude was used above 1400 kev in the approximation

$$\mathfrak{a}_{\rm vp}(x)-\delta_0\simeq\mathfrak{a}_{\rm vp}^{(0)}(x)-\delta_0^{(0)}+i\,{\rm Im}\,\mathfrak{a}_{\rm vp}^{(1)}, \quad (24.4)$$

with $\delta_0^{(0)}$ being as in Eq. (8.3), the small terms $\mathfrak{a}_{\rm vp}^{(2)}$, ${\rm Re}\,\mathfrak{a}_{\rm vp}^{(1)}$ and the difference $\delta_0-\delta_0^{(0)}$ being dropped. The error thus introduced is quite small, since the neglected terms contribute less than 6% of $\Delta\sigma_{\rm vp}$ at 1855 kev and decrease inversely as the energy for higher energies. However, the contribution to $\Delta\sigma_{\rm vp}$ from the term ${\rm Im}\,\mathfrak{a}_{\rm vp}^{(1)}$ was typically around one-third of the contribution from $\mathfrak{a}_{\rm vp}^{(0)}$; hence, the Coulomb field effects on the vacuum polarization scattering are not negligible. In the calculations, values of K_0 giving a best S-wave fit to the data were used; the Coulomb scattering amplitudes and values of $\sigma_{\rm th}(\theta)$ were obtained using an automatic computing program coded for an I.B.M. 650 digital computer by Shapiro and Pyatt.[40] The values obtained for $\Delta\sigma_{\rm vp}$ are given in Table II, while the ratio of $\Delta\sigma_{\rm vp}$ to the pure S wave plus Coulomb cross section $\sigma_{\rm th}$ is shown in Fig. 7. It should be noted that $\Delta\sigma_{\rm vp}$

TABLE II. Vacuum polarization contributions to the p-p scattering cross section excluding the S-wave effect.[a]

θ \ E (kev)	1397	1855	2425	3037	3527	3899	4203
Degrees	mb	mb	mb	mb	mb	mb	mb
12	99.9	77.1	37.1	24.7	18.8	15.8	13.8
14	56.1	33.2	20.5	13.7	10.4	8.72	7.59
16	33.6	19.9	12.2	8.13	6.18	5.16	4.49
20	13.9	8.25	4.99	3.33	2.53	2.11	...
24	6.66	3.94	2.40	1.59	...	...	...
25	...	...	...	...	1.11	0.943	0.818
30	2.68	1.60	0.993	0.663	0.508	0.425	0.369
35	1.48	0.909	0.574	0.392	0.304	0.257	0.226
40	0.939	0.605	0.399	0.280	0.220	0.188	0.167
50	0.569	0.404	0.285	0.212	0.171	0.149	0.134
60	0.489	0.366	0.269	0.203	0.167	0.147	0.134
70	0.479	0.367	0.271	0.209	0.171	0.150	0.136
80	0.484	0.373	0.278	0.212	0.174	0.154	0.140
90	0.487	0.375	0.280	0.213	0.176	0.155	0.140
K_0	39.1°	44.3°	48.4°	51.0°	52.5°	53.3°	53.8°

[a] The values of K_0 used in the calculations are given in the last row; possible nuclear scattering in states with $L>0$ was neglected.

[40] The author wishes to thank Dr. J. Shapiro and Mr. K. D. Pyatt for their cooperation at this stage of the work. The computing program to which the reference was made, was designed originally for the study of p-p scattering at high energies.

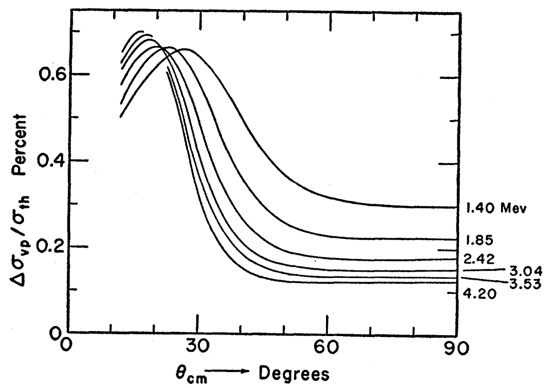

FIG. 7. The contribution $\Delta\sigma_{\text{vp}}$ to the theoretical p-p scattering cross sections arising from vacuum polarization scattering in orbital angular momentum states with $L>0$. The initial cross sections σ_{th} were calculated from good S wave fits to the data at the energies indicated.

includes only the effects of orbital angular momentum states with $L>0$. Thus, the nearly isotropic part of $\Delta\sigma_{\text{vp}}$ at large scattering angles ($50° \leq \theta \leq 90°$) represents not an S wave effect, but the combined effect of many partial waves. At smaller scattering angles, $\Delta\sigma_{\text{vp}}$ has somewhat the form of the change in the cross section caused by a small negative mean P-wave phase shift, as is seen from Figs. 5 and 7.

V. CONCLUSIONS

The influence on the proton-proton scattering cross sections of small electromagnetic and relativistic modifications of the Coulomb interaction between the protons has been considered, with particular attention to the largest such modification, the vacuum polarization interaction. The phase shifts caused by the vacuum polarization potential were calculated to first order in the interaction strength, including in the calculation the modification of the wave function resulting from the presence of the Coulomb field between the protons. The effects on the phase shifts of the distortion of the wave function by nuclear forces were neglected, because at the energies considered these effects are expected to be small in states with orbital angular momentum quantum numbers $L>0$. Experimental data suggest in fact that the P-wave phase shifts are a small fraction of a degree. If, on the other hand, these phase shifts should be larger than about 1° nuclear distortion of the wave functions should be considered.[25] The Coulomb distortion effects are not large above about 1 Mev, as is shown by the comparison in Table I of the vacuum polarization phase shifts calculated using free particle wave functions with those calculated using Coulomb wave functions; the Coulomb effect should, however, be important at lower energies. The vacuum polarization scattering amplitude was evaluated to first order in the interaction including exactly the Coulomb phase shift factors $e_{L,0}$; it was found that these factors materially modified the contributions to the amplitude of the high angular momentum states. The $e_{L,0}$ are, in fact, entirely responsible for the imaginary part of the scattering amplitude, which contributes up to one third of the vacuum polarization contribution to the p-p cross section in the energy range 1.4–4.2 Mev. Many angular momentum states contribute significantly to the scattering, resulting in appreciable modifications of previous work on vacuum polarization scattering based on S- and P-wave effects alone.[9,18,41] The discovery at low energies of large nuclear phase shifts in states with $L>0$ would necessitate a reconsideration of vacuum polarization scattering in which the vacuum polarization phase shifts for those states were removed from the scattering amplitude, and treated instead as part of the total observed phase shifts fitting the experimental data. Such a treatment would be similar to that given S-wave scattering by Foldy and Eriksen.[9]

Values of the vacuum polarization contribution to the proton-proton scattering cross section are given in Table II for the range of scattering angles and energies corresponding to the most accurate recent experiments. This addition to the cross section is of detectible magnitude [$\sim$0.7 percent maximum change in $\sigma(\theta)$], and is quite similar in its angular dependence to the change produced in a pure S wave plus Coulomb cross section by the addition of P-wave scattering. Thus it is believed that vacuum polarization scattering should definitely be considered in a precise analysis of low-energy p-p scattering data. It should be pointed out that the theoretical results of this paper are in principle applicable also to other cases, such as p-α, α-α, and p-d scattering, provided appropriate changes are made in the parameters η and ν; thus η becomes $Z_1Z_2e^2/\hbar v$, while ν becomes $(m/\mu)(mc^2/E')$, where μ is the reduced mass and E' is the scattering energy in the center-of-mass system. It is not meant to imply however that information on vacuum polarization will be obtainable by analyzing these more complicated cases.

Relativistic and magnetic contributions[20–21] to the p-p scattering matrix have also been considered in the limit of low scattering energies. The relativistic alteration in the Coulomb parameter η results in significant changes in the low-energy cross sections (Fig. 5); but the extra terms in the scattering matrix may be omitted. Other electromagnetic interactions between the protons are too small to influence detectably the scattering.

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[41] L. Durand, III, and M. de Wit (to be published).