Energy Dependence of Cross Sections near Threshold: One Neutral and Two Charged Reaction Products*

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The energy dependence near threshold of cross sections for reactions leading to the escape of one neutral and two charged particles is determined. The method is an extension of one developed previously for uncharged particles, and utilizes only general properties of solutions of the time-independent three-particle Schrödinger equation valid outside a reaction zone of finite extent. Electron detachment from H^- by charged particle bombardment, and nuclear reactions of the type (n,np) are considered as examples.

I. INTRODUCTION

THE limiting energy dependence of reaction cross sections at threshold is of interest from a purely theoretical point of view because this has proved to be one of those rare features of many-body problems amenable to an exact theoretical determination. Such threshold laws for reactions leading to neutral products have been obtained by Guier and Hart,[1] and less general results for three uncharged particles have been obtained by others.[2,3] But when we consider reactions leading to charged products, the long-range Coulomb potentials introduce a new class of difficulties into the many-body problem. The simplest of these reactions, which appear to be those leading to one uncharged and only two charged products, are tractable, however. We shall consider this problem in sufficient detail to develop the limiting energy dependence at threshold of the cross section for reactions leading to three such products, no two of which are bound. (When two of them are bound, the result is well known, since that case is equivalent to a two-product reaction.)[4] We shall consider the region of validity only very briefly.

This theoretical result is also of some interest from an experimental point of view, since an accurate experimental determination of threshold energies for reactions requires extrapolation of the experimental yield-*vs*-energy curve. Examples of reactions to which this theory is directly applicable are the detachment of an electron from a singly charged negative ion by charged-particle impact, and (n,np) nuclear reactions.

The treatment which follows is applicable to reactions which can be considered to occur only when all reactants are within a finite distance of each other, i.e., when the contribution to the reaction decreases at least exponentially for large separations between any two reactants. This implies the existence of a *reaction* zone which may be considered finite, although it does *not* imply that the *interaction* vanishes outside this zone. That the reaction zone might be considered finite is perhaps self-evident when only short-range forces between the products are involved. But it also can exist when long-range forces (such as Coulomb) are involved, since the transfer of a finite (threshold) amount of energy between reactants is required for a reaction.

It should be recognized, however, that not every mathematical formulation of a reaction displays this finite reaction zone. For example, imagine a reaction consisting of the emission of charged balls from a charged sphere. Here we have clearly defined the sphere to be the reaction zone. If the wave function describing the outgoing flux is written in the form

$$\psi=\int G(\mathbf{r},\mathbf{r}')S(\mathbf{r}')d\tau',$$

where G is the Green's function including Coulomb potentials and where S is the source distribution within the sphere, then the finite extent of this source distribution appears explicitly. However, if we were to use the free-space Green's function without Coulomb potentials, G_0, then ψ would be given by

$$\psi=\int G_0(\mathbf{r},\mathbf{r}')\left[S(\mathbf{r}')+\frac{e^2}{|\mathbf{r}'|}\psi(\mathbf{r}')\right]d\tau'+\text{surface integrals},$$

and in such a representation the source distribution would *appear* to extend not only over the finite reaction zone, but over the entire infinite interaction zone. It is clear that, in general, the explicit appearance of a finite source distribution is insured in a Green's function formulation only if all potential terms which "distort" the wave front outside the reaction zone are included in the definition of the Green's function.

The actual extent of the reaction zone, apart from its being finite, is of importance only insofar as it affects the range of validity of the results. We shall not consider this aspect here, but merely note that in the absence of delayed breakup of a reaction product, the reaction zone can probably be considered comparable to the separation distance between reaction products at which the potential energy of interaction is equal to the threshold energy involved.

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[1] W. H. Guier and R. W. Hart, Phys. Rev. **106**, 297 (1957).
[2] G. Snow, Ph.D. dissertation, Princeton University, 1949 (unpublished).
[3] V. Mamasakhlisov, Zhur. Eksptl. i Teoret. Fiz. **25**, 36 (1953).
[4] E. Wigner, Phys. Rev. **73**, 1002 (1948).

Although a complete description of a reaction would require a detailed consideration of the internal structure (including spin) of the reactants and reaction product, it is shown (see also references 1–4) that the desired threshold behavior of reaction cross sections can be obtained by regarding the reaction products outside the reaction zone as structureless particles, and without any specification of the reactants. This property follows from the fact that the cross section can be expressed in terms of the scattered part of the wave function in the asymptotic region where the products are far from each other and the reaction zone. In this "detachment" region, the wave function can be regarded as a linear combination of asymptotic solutions of the Schrödinger equation, with each of these components describing a particular quantum state of three reaction products. The asymptotic solutions themselves are determined only by the long-range part of the interaction, but each is multiplied by an energy-dependent amplitude which, in general, is determined by the details of the short-range interaction. At threshold for a channel, however, the limiting *form* of the energy dependence of the corresponding amplitude actually becomes independent of the short-range interaction, and can be determined from the requirement that the wave function and its gradient be bounded and continuous for all energies.

Unfortunately, this limiting energy dependence is not obtainable from the usual type of asymptotic solution, which is not valid in the limit of zero momentum of the reaction products within any bounded region containing the reaction zone. Our major task, therefore, is to obtain an expression for the wave function which is valid in this limit (in order to apply at threshold the conditions of boundedness and continuity of the wave function), and which we can also evaluate explicitly in the usual asymptotic limit of large phase (in order to obtain the cross section from its representation as a flux integral).

II. WAVE FUNCTION

We begin with the three-particle Schrödinger equation, valid outside the reaction zone, for one neutral and two charged particles with total energy E_t,

$$\left\{\frac{1}{m_1}\nabla_1^2+\frac{1}{m_2}\nabla_2^2+\frac{1}{m_3}\nabla_3^2 +\frac{2}{\hbar^2}\left[E_t-\frac{Z_2Z_3e^2}{|\mathbf{r}_2-\mathbf{r}_3|}-Q(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\right]\right\}\Psi=0, \quad (1)$$

where Q contains all interparticle short-range interactions, which are required to decrease more rapidly than the inverse square of the interparticle separations.

In order to remove the ignorable center-of-mass coordinates, and at the same time separate variables, we define the new coordinates

$$\mathbf{u}=\left(\frac{m_2m_3}{(m_2+m_3)M_t}\right)^{\frac{1}{2}}(\mathbf{r}_2-\mathbf{r}_3),$$
$$\mathbf{v}=\left(\frac{m_1}{m_2+m_3}\right)^{\frac{1}{2}}\left[\frac{m_3(\mathbf{r}_1-\mathbf{r}_3)+m_2(\mathbf{r}_1-\mathbf{r}_2)}{M_t}\right], \quad (2)$$
$$\mathbf{w}=\frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2+m_3\mathbf{r}_3}{M_t},$$

where $M_t=m_1+m_2+m_3$. In the new coordinate system, the Schrödinger equation becomes

$$\left[\nabla_u^2+\nabla_v^2+\nabla_w^2+\frac{2M_t}{\hbar^2}\left\{E_t-\frac{Z_1Z_2e^2}{|\mathbf{u}|}\left(\frac{M_t}{m_2}+\frac{M_t}{m_3}\right)^{-\frac{1}{2}} -Q(\mathbf{u},\mathbf{v})\right\}\right]\Psi=0. \quad (3)$$

Transforming to the center-of-mass system ($\mathbf{w}=0$) leads to a Schrödinger equation characteristic of a two-particle system with a Coulomb force center at the origin which attracts only one of the particles, i.e.,

$$\left\{\nabla_u^2+\nabla_v^2+\left(k^2+\frac{2}{|\mathbf{u}|a}-q(\mathbf{u},\mathbf{v})\right)\right\}\Psi(\mathbf{u},\mathbf{v})=0, \quad (4)$$

where

$$k^2=2M_tE/\hbar^2, \quad q=(2M_t/\hbar^2)Q, \quad (4a)$$

with E representing the total energy in the center-of-mass system, and

$$\frac{1}{a}=\frac{-M_t}{\hbar^2}\left(\frac{M_t}{m_2}+\frac{M_t}{m_3}\right)^{-\frac{1}{2}}Z_2Z_3e^2. \quad (4b)$$

It is readily verified that solutions of Eq. (1) which correspond to a three-particle outward flux (detachment flux) are identified with solutions of Eq. (4) which correspond in the center-of-mass system to a two-"particle" outward flux in the new coordinates **u**, **v**.

One representation of the solution of Eq. (4) is in terms of the Green's function $\mathcal{G}$ which includes all potentials exterior to the reaction zone, and is given by

$$\Psi(\mathbf{u},\mathbf{v})=\Psi_i(\mathbf{u},\mathbf{v})+\int\mathcal{G}(\mathbf{u},\mathbf{v};\mathbf{u}',\mathbf{v}')\mathcal{S}_E(\mathbf{u}',\mathbf{v}')d\mathbf{u}'d\mathbf{v}'. \quad (4c)$$

Here, $\mathcal{S}_E$ represents the actual source for the scattered part of the wave function and must therefore vanish outside the bounded region in which the reaction actually takes place, i.e., outside the reaction zone. Ψ_i is some (usually readily obtainable) solution of the Schrödinger equation including such potential terms that, outside the reaction zone, Ψ_i has the incident wave (including the Coulomb potential) as its incoming part, but has no outgoing part corresponding to the reaction under study. $\mathcal{G}$ is the Green's function, satisfying the equation

$$\left[\nabla_u^2+\nabla_v^2+k^2+\frac{2}{a|\mathbf{u}|}-q_0(\mathbf{u},\mathbf{v})\right]\mathcal{G}(\mathbf{u},\mathbf{v};\mathbf{u}'\mathbf{v}')=\delta(\mathbf{u}-\mathbf{u}')\delta(\mathbf{v}-\mathbf{v}'), \quad (4d)$$

with the usual boundary condition that the singularity at $\mathbf{u}=\mathbf{u}'$, $\mathbf{v}=\mathbf{v}'$ correspond to a source, and where $q_0=q$ outside the reaction zone, and $q_0=0$ inside the reaction zone.

Unfortunately, this Green's function is, in general, very difficult to obtain, and we therefore consider an alternative representation of the wave function in terms of the Coulomb Green's function which does not include the short-range potentials; i.e., we represent Ψ by

$$\Psi(\mathbf{u},\mathbf{v})=\Psi_i(\mathbf{u},\mathbf{v})+\int G(\mathbf{u},\mathbf{v};\mathbf{u}',\mathbf{v}')S_E(\mathbf{u}',\mathbf{v}')d\mathbf{u}'d\mathbf{v}', \quad (5)$$

where G is the Coulomb Green's function satisfying the equation

$$\left(\nabla_u^2+\nabla_v^2+k^2+\frac{2}{a|\mathbf{u}|}\right)G(\mathbf{u},\mathbf{v};\mathbf{u}',\mathbf{v}')=\delta(\mathbf{u}-\mathbf{u}')\delta(\mathbf{v}-\mathbf{v}'). \quad (6)$$

In this representation, the (modified) source distribution, S_E, may, in general, extend beyond the vicinity of the reaction zone. In fact, one possible representation for S_E is given by

$$S_E=\mathcal{S}_E+q_0(\Psi-\Psi_i), \quad (6a)$$

which extends outside the reaction zone along the directions for which the short-range potential (q_0) does not vanish.

However, in the absence of delayed breakup of a reaction product, a representation with S_E confined to the vicinity of the reaction zone must exist for the wave function in the detachment region. [By detachment region we mean the region in $(\mathbf{u},\mathbf{v})$ space, or $(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ space where all particles are far from each other and the reaction zone.] This follows from the fact that in this region, both G and Ψ satisfy the same equation and have the same (outgoing or damped) asymptotic form, so that Ψ cannot have a virtual source distribution which is, in fact, unbounded. While it is difficult, in general, to find analytically a bounded representation for the source distribution (S_E), it will be sufficient for our purpose to show that the integral in Eq. (5) may be transformed into one having finite limits. The details are described in the appendix where it is shown that, in the absence of delayed breakup (e.g., metastable or virtual states) of a product, such a transformation is possible, and that the magnitude of these limits does not greatly exceed the reaction zone, unless a bound-state level exists in the neighborhood of threshold. Accordingly, we shall represent the wave function in the detachment region by Eq. (5), with the integral orev a source distribution S_E extending over a finite region in the neighborhood of the reaction zone.

For our purpose, it will be necessary to use only two properties of S_E: that S_E is bounded, and not identically zero, regardless of the energy. These properties, which arise from the conditions that the wave function and its gradient be bounded and continuous for any energy, are perhaps obvious ones, and the demonstration of their validity is quite simple. To show boundedness, we use Eq. (5), and the defining equation for the Green's function [see Eq. (6)] to obtain

$$\left(\nabla_u^2+\nabla_v^2+k^2+\frac{2}{a|\mathbf{u}|}\right)\Psi_s(\mathbf{u},\mathbf{v})=S_E(\mathbf{u},\mathbf{v}),$$

where $\Psi_s(\mathbf{u},\mathbf{v})$ is the scattered part of the wave function. Since this wave function and its gradient are required to be bounded and continuous for all energies, including threshold, the source function must also be bounded (except possibly for an integrable singularity at $\mathbf{u}=0$) for all energies. To demonstrate that the source function is not identically zero (even at threshold), we need merely note that this condition would imply that $\Psi=\Psi_i$ *identically*, which is clearly not possible.

It is perhaps worth indicating at this point why these are the only properties of the source distribution affecting the energy dependence of the cross section near threshold. We shall show in Sec. IV that when k is small, the Green's function G can be expressed in the asymptotic region as a sum of products of known partial waves in the particle coordinates $\mathbf{u}$, $\mathbf{v}$, multiplied by known functions of the source coordinates, $\mathbf{u}'$, $\mathbf{v}'$. The integral over the sources then determines the amplitude of each partial wave. The fact that the source distribution must approach some limiting function, bounded and not everywhere zero, as the energy approaches its threshold value, together with the fact that the integral extends over an unspecified but bounded region (to which S_E is confined), then permits determination of the limiting threshold energy dependence of these amplitudes without any further specification of the source distribution. In fact, *any approximations to the source distribution which preserves these two properties will yield the correct limiting energy dependence of the cross section near threshold*, although they need not agree at all in magnitude of the cross section.

There must always remain two possible exceptions to this general result. (1) It is possible to imagine an accidental combination of incident wave and internal structure for which some partial wave could not be excited at all, making a source distribution orthogonal to the appropriate term of the Green's function. (2) It is conceivable that an accidental resonance could occur in the internal structure of the particles at threshold.

In this case we could no longer neglect the internal structure even in the asymptotic region.

III. GREEN'S FUNCTION

The" Coulomb" Green's function G_E is defined as the solution of Schrödinger's equation, modified by an inhomogeneous term representing a pure source,

$$\left\{\nabla_{\mathbf{u}}^2+\nabla_{\mathbf{v}}^2+\left(k^2+\frac{2}{a|\mathbf{u}|}\right)\right\}G_E=\delta(\mathbf{u}-\mathbf{u}')\delta(\mathbf{v}-\mathbf{v}'). \quad (6b)$$

It does not satisfy the boundary conditions implicit in any real reaction, but plays a role entirely analogous to that of the free-space Green's function.

We shall now obtain the desired Green's function by a straightforward but rather involved application of one of the usual methods,[5] and merely indicate the major steps in the development.

Equation (6b) is immediately separable in the corresponding angle coordinates. This separation is readily performed by expanding the delta functions in terms of spherical harmonics. Let θ and φ be the spherical polar angles associated with $\mathbf{u}$, and Θ, Φ be the corresponding angles associated with $\mathbf{v}$. Then

$$\delta(\mathbf{u}-\mathbf{u}')=\frac{\delta(u-u')}{uu'}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_{l,m}(\theta,\varphi)Y_{l,m}^*(\theta',\varphi'), \quad (7a)$$

$$\delta(\mathbf{v}-\mathbf{v}')=\frac{\delta(v-v')}{vv'}\sum_{L=0}^{\infty}\sum_{M=-L}^{L}Y_{L,M}(\Theta,\Phi)Y_{L,M}^*(\Theta',\Phi'), \quad (7b)$$

where $Y_{l,m}(\theta,\varphi)$ are the normalized spherical harmonics.[6]

Substitution into Eq. (6b) yields

$$G_E=\sum_{l,m,L,M}g_{l,L}(u,v;u',v';k,a)\gamma_{l,L,m,M}(\theta,\varphi,\Theta,\Phi)\times\gamma_{l,L,m,M}^*(\theta',\varphi',\Theta',\Phi'), \quad (8a)$$

where

$$\gamma_{l,L,m,M}(\theta,\varphi,\Theta,\Phi)=Y_{l,m}(\theta,\varphi)Y_{L,M}(\Theta,\Phi), \quad (8b)$$

and where $g_{l,L}$ is defined by

$$\left[\frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2\frac{\partial}{\partial u}\right)+\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial}{\partial v}\right)+k^2+\frac{2}{au}-\frac{l(l+1)}{u^2}-\frac{L(L+1)}{v^2}\right]g_{l,L}=\frac{\delta(u-u')}{uu'}\frac{\delta(v-v')}{vv'}, \quad (9)$$

with the added condition that u', v' correspond to a source point. Equation (9) is now to be solved for $g_{l,L}$.

The radiation condition, namely that the solution represent an outgoing flux for large u, v requires an asymptotic form for $g_{l,L}$ having as a leading term[7]

$$\frac{\exp\{i[k(u^2+v^2)^{\frac{1}{2}}+\text{phase}]\}}{(u^2+v^2)^{5/4}}.$$

Therefore, although Eq. (9) is itself separable in u and v, the boundary condition is not, and the desired solution must be made up of a superposition of the solutions of the separated equation. Following reference 5, for example, we represent such a superposition as

$$g_{l,L}=\int_{-\infty}^{\infty}d\kappa\int_{-\infty}^{\infty}dK\times\frac{\psi_{K,l}^{(1)}(u)\psi_{K,l}^{(1)}(u')\psi_{\kappa,L}^{(2)}(v)\psi_{\kappa,L}^{(2)}(v')}{k^2-K^2-\kappa^2}, \quad (10)$$

where the paths are to be so chosen that the u', v' correspond to a source, and where the ψ functions are the orthonormal solutions of

$$\left\{\frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2\frac{\partial}{\partial u}\right)+K^2+\frac{2}{au}-\frac{l(l+1)}{u^2}\right\}\psi_{K,l}^{(1)}(u)=0, \quad (11a)$$

$$\left\{\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial}{\partial v}\right)+\kappa^2-\frac{L(L+1)}{v^2}\right\}\psi_{\kappa,L}^{(2)}(v)=0. \quad (11b)$$

The delta function character of Eq. (9) is readily verified by substituting Eq. (10) into the left-hand side of Eq. (9), carrying out the indicated differentiations, and then using the closure property, namely, that

$$\int_{-\infty}^{\infty}\psi_{k,l}^{(i)}(r)\psi_{k,l}^{(i)*}(r')dk=\frac{\delta(r-r')}{rr'}, \quad i=1\text{ or }2. \quad (11c)$$

For our case, $\psi^{(1)}$ is the radial part of the usual Coulomb continuum[8] wave function given by[9]

$$\psi_{K,l}^{(1)}(u)=\frac{\pi^{-\frac{1}{2}}K}{(2l+1)!}\exp\left(\frac{\pi}{2}\frac{1}{a|K|}\right)\left|\Gamma\left(1+l+\frac{1}{iKa}\right)\right|\times e^{-iKu}(2Ku)^l\,{}_1F_1\left(1+l-\frac{1}{iKa},2l+2,2iKu\right). \quad (12a)$$

[5] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), p. 820 ff.

[6] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 73.

[7] P. M. Morse and H. Feshbach, reference 5, p. 1732, Eq. (12.3.91).

[8] It might seem at first glance that including only continuum solutions in the construction of the Green's function is not sufficiently general. However, the above $\psi^{(1)}$ is a solution of the hydrogenic Schrödinger equation for *any* K, including the imaginary values corresponding to negative energy states which arise for the case of Coulomb attraction. Therefore, the superposition, as represented by the contour integral of Eq. (10), takes these states into account, and they could be made to appear explicitly by considering the poles of the gamma function occuring in $\psi^{(1)}$ [see Eq. (12a)].

[9] For Coulomb attraction, see, for example, H. A. Bethe and E. E. Saltpeter, *Encyclopedia of Physics* (Springer-Verlag, Berlin, 1957), Vol. XXXV, p. 107 ff. For Coulomb repulsion, see W. Gordon, Z. Physik 48, 180 (1928). Note that our wave functions differ from theirs by a factor $2^{-\frac{1}{2}}$, because the integral in Eq. (11c) extends from $-\infty$ to $+\infty$, instead of from 0 to ∞. This does not affect the usual normalization when we take into account the possibility that K (and κ) may be negative as well as positive.

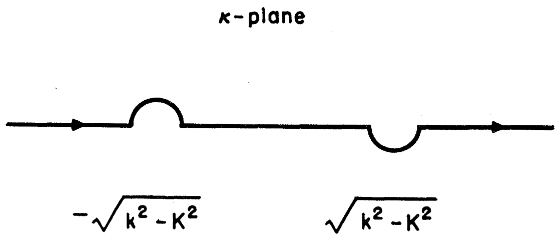

FIG. 1. Contour for the κ integration.

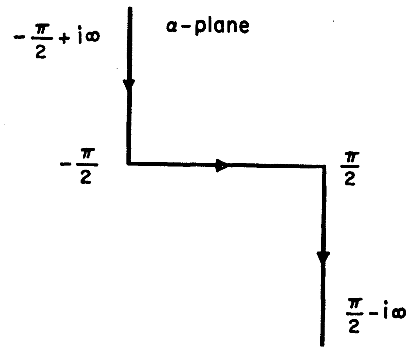

FIG. 2. Contour for α integration.

Similarly, $\psi_{\kappa,L}^{(2)}(v)$ is the orthonormal continuum solution of Eq. (11b). We may obtain this most readily by letting $a\to\infty$ in the Coulomb wave function of Eq. (12a), yielding

$$\psi_{\kappa,L}^{(2)}(v)=J_{L+\frac{1}{2}}(\kappa v)(\kappa/2v)^{\frac{1}{2}}. \tag{12b}$$

With the above definitions of the functions appearing in Eq. (10), and verification of the fact that the $g_{l,L}$ defined by this equation does actually satisfy the differential equation, Eq. (9), the contours must be specified to satisfy the outgoing flux condition. Consider, first, the integration over κ in Eq. (10), with $\psi^{(2)}$ given by Eq. (12b). The integrand has simple poles at $\kappa=\pm(k^2-K^2)^{\frac{1}{2}}$, and a path such as indicated in Fig. 1 must be defined, such that it passes below the positive pole and above the negative pole. Carrying out this integral, we find

$$\frac{1}{(vv')^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{J_{L+\frac{1}{2}}(\kappa v)J_{L+\frac{1}{2}}(\kappa v')\kappa d\kappa}{k^2-\kappa^2-K^2}=\begin{cases}[-\pi i/(vv')^{\frac{1}{2}}]J_{L+\frac{1}{2}}[v'(k^2-K^2)^{\frac{1}{2}}]H_{L+\frac{1}{2}}^{(1)}[v(k^2-K^2)^{\frac{1}{2}}] & \text{if } v>v'\\ [-\pi i/(vv')^{\frac{1}{2}}]J_{L+\frac{1}{2}}[v(k^2-K^2)^{\frac{1}{2}}]H_{L+\frac{1}{2}}^{(1)}[v'(k^2-K^2)^{\frac{1}{2}}] & \text{if } v<v'.\end{cases}$$

The remaining integral over K is somewhat more complicated. We first substitute the above result into Eq. (10), along with $\psi^{(1)}$ from Eq. (12a). We shall need to consider only the case $v>v'$, for which Eq. (10), becomes

$$g_{l,L}=\frac{(-i)}{2[(2l+1)!]^2}\int_{-\infty}^{\infty}dK\,K^2\left|\Gamma\left(1+l+\frac{1}{iKa}\right)\right|^2\exp\left[-iK(u-u')+\frac{\pi}{|k|a}\right](2Ku)^l\,{}_1F_1\left(1+l-\frac{i}{iKa},\,2l+2,\,2iKu\right)$$
$$\times(2Ku')^l\,{}_1F_1\left(1+l+\frac{1}{iKa},\,2l+2,\,-2iKu'\right)\frac{H_{L+\frac{1}{2}}^{(1)}[v(k^2-K^2)^{\frac{1}{2}}]}{v^{\frac{1}{2}}}\frac{J_{L+\frac{1}{2}}[v'(k^2-K^2)^{\frac{1}{2}}]}{v'^{\frac{1}{2}}}. \tag{13}$$

Just as in the free-space case $(a\to\infty)$,[10] determination of the effect of the radiation condition on the contour in the region u, v, $\gg u'$, v' is most readily accomplished by introducing a new variable α, and the remaining two-particle hyperspherical coordinates β, R defined by

$$\alpha=\tan^{-1}(k^2/K^2-1)^{-\frac{1}{2}},\quad \beta=\tan^{-1}(u/v),\quad\text{and}\quad R=(u^2+v^2)^{\frac{1}{2}}, \tag{13a}$$

so that $K=k\sin\alpha$ and $(k^2-K^2)^{\frac{1}{2}}=k\cos\alpha$. Equation (13) then becomes

$$g_{l,k}=\frac{-ik^3}{2[(2l+1)!]^2}(RR'\cos\beta\cos\beta')^{-\frac{1}{2}}\int_{-i\infty+\frac{1}{2}\pi}^{i\infty-\frac{1}{2}\pi}d\alpha\cos\alpha\sin^2\alpha\left|\Gamma\left(1+l+\frac{1}{ika\sin\alpha}\right)\right|^2$$
$$\times\exp\left[-ik\sin\alpha(R\sin\beta-R'\sin\beta')+\frac{\pi}{|k|a\sin\alpha}\right]\Bigg\{(2kR\sin\alpha\sin\beta)^l(2kR'\sin\alpha\sin\beta')^l$$
$$\times{}_1F_1\left(1+l-\frac{1}{ika\sin\alpha},\,2l+2,\,2ikR\sin\alpha\sin\beta\right)\Bigg\}\Bigg\{{}_1F_1\left(1+l+\frac{1}{ika\sin\alpha},\,2l+2,\,-2ikR'\sin\alpha\sin\beta'\right)$$
$$\times J_{L+\frac{1}{2}}(kR'\cos\alpha\cos\beta')H_{L+\frac{1}{2}}^{(1)}(kR\cos\alpha\cos\beta)\Bigg\}, \tag{14}$$

where on the path, as indicated in Fig. 2, the argument of the Hankel function has a positive imaginary part as $K\to\pm\infty$. The Green's function is now obtained as a single quadrature by substitution of Eq. (14) into Eq. (8).

[10] P. M. Morse and H. Feshbach, reference 5, p. 823.

IV. ASYMPTOTIC EXPANSION OF THE WAVE FUNCTION

We can now utilize the fact that for the experimental situations we are considering, the cross section involves the wave function for only such large particle separations and particle coordinates, that even for small k the phase shifts at the observation point (i.e., kR) must remain large. For this reason the two limiting processes (i.e., $kR\to\infty$, $k\to 0$) must be carried out sequentially, first letting the phase shifts associated with the observation point ($kR\sin\alpha\sin\beta$ and $kR\cos\alpha\cos\beta$) become arbitrarily large, and subsequently allowing k to become small. We consider Eq. (14), which is greatly simplified in the limit of large kR, where[11]

$$\frac{k\sin\alpha}{(2l+1)!}(2kR\sin\alpha\sin\beta)^l\left|\Gamma\left(1+l+\frac{1}{ika\sin\alpha}\right)\right|\exp\left(-ikR\sin\alpha\sin\beta+\frac{\pi}{2a|k\sin\alpha|}\right)$$
$$\times{}_1F_1\left(1+l-\frac{1}{ika\sin\alpha},\,2l+2,\,2ikR\sin\alpha\sin\beta\right)\approx\frac{\sin[kR\sin\alpha\sin\beta-\frac{1}{2}l\pi+\delta(\alpha)+\ln(2R\sin\beta|k\sin\alpha|)/ka\sin\alpha]}{R\sin\beta},\quad(15)$$

with

$$\delta(\alpha)=\arg\left[\Gamma\left(1+l+\frac{1}{ika\sin\alpha}\right)\right],$$

and

$$H_{L+\frac{1}{2}}^{(1)}(kR\cos\alpha\cos\beta)\approx\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\exp[ikR\cos\alpha\cos\beta-\frac{1}{2}iL\pi-\frac{1}{2}i\pi]}{(kR\cos\alpha\cos\beta)^{\frac{1}{2}}}.\quad(16)$$

In this limit, the integrand of Eq. (14) has points of stationary phase on the real part of the path where, above threshold, the wave function does not damp. Expressing

$$\sin[kR\sin\alpha\sin\beta-\tfrac{1}{2}l\pi+\delta(\alpha)+\ln(2R\sin\beta|k\sin\alpha|)/ka\sin\alpha]$$

in the above equations as the difference of two exponentials, the integrand containing the positive exponential has a stationary point at $\alpha=\beta$, and the other, at $\alpha=-\beta$. Using the method of stationary phase, we obtain:

$$g_{l,L}\approx\frac{-2e^{-i\pi/4}}{(2l+1)!}k\frac{e^{ikR}}{R^{\frac{3}{2}}}\exp\left\{-i\left[\frac{\pi}{2}(L+l+1)-\delta(\beta)-\frac{1}{ka\sin\beta}\ln(2|k|R\sin\beta)\right]+\frac{\pi}{2|k|a\sin\beta}\right\}\left|\Gamma\left(1+l+\frac{1}{ika\sin\beta}\right)\right|$$
$$\times\exp(ikR'\sin\beta\sin\beta')(2kR'\sin\beta\sin\beta')^l(R'\cos\beta\cos\beta')^{-\frac{1}{2}}$$
$$\times{}_1F_1\left(1+l+\frac{1}{ika\sin\beta},\,2l+2,\,-2ikR'\sin\beta\sin\beta'\right)J_{L+\frac{1}{2}}(kR'\cos\beta\cos\beta').\quad(17)$$

Substitution of this result into Eq. (8) yields the required asymptotic form of the Green's function, which in turn yields the asymptotic expression for the wave function from Eq. (5).

When these substitutions are made, we obtain the following asymptotic expression for the partial waves, valid in the limit $kR\to\infty$:

$$\Psi_{l,L,m,M}(R,\beta,\theta,\varphi,\Theta,\Phi)\approx k^{\frac{3}{2}}R^{-\frac{5}{2}}\exp\left\{i\left[kR+\frac{1}{ka\sin\beta}\ln(2|k|R\sin\beta)+\delta(\beta)\right]+\frac{\pi}{2|k|a\sin\beta}\right\}$$
$$\times A_{l,L}\left|\Gamma\left(1+l+\frac{1}{ika\sin\beta}\right)\right|(ka\sin\beta)^lI_{l,L,m,M}(k,\beta)\gamma_{l,L,m,M}(\theta,\varphi,\Theta,\Phi),\quad(18)$$

where

$$I_{l,L,m,M}(k,\beta)=\int d\tau' S_E(\tau')\gamma_{l,L,m,M}{}^*(\theta',\varphi',\Theta',\Phi')\,{}_1F_1\left(1+l+\frac{1}{ika\sin\beta},\,2l+2,\,-2ikR'\sin\beta\sin\beta'\right)$$
$$\times\exp(ikR'\sin\beta\sin\beta')\left(\frac{R'}{a}\sin\beta'\right)^l\frac{J_{L+\frac{1}{2}}(kR'\cos\beta\cos\beta')}{(kR'\cos\beta\cos\beta')^{\frac{1}{2}}},\quad(19)$$

[11] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, p. 278.

and

$$A_{l,L}=\frac{2^{l-1}}{(2l+1)!}\exp[-\tfrac{1}{2}i\pi(l+L-\tfrac{1}{2})]. \tag{19a}$$

We shall now consider the limiting energy dependence of $I_{l,L,m,M}(k,\beta)$. Since the integral has finite limits (see Sec. II and Appendix), we can obtain the limiting form of the wave function as $k\to 0$ by expanding the integrand in powers of k, and retaining only the leading term. From this point on, however, it will no longer be possible to check the development by letting the charges vanish. The reason is that the energy appears in some terms in the combination ka, which is infinite if the charges vanish, but which vanishes at threshold, otherwise. When we neglect higher order terms in ka and kR', the confluent hypergeometric function reduces to[12]:

$$\left\{\exp(ikR'\sin\alpha\sin\beta')\,{}_1F_1\left(1+l+\frac{1}{ika\sin\alpha},\,2l+2,\,-2ikR'\sin\alpha\sin\beta'\right)\right\}$$
$$\approx(2l+1)!\left(\frac{2R'\sin\beta'}{a}\right)^{-l-\frac{1}{2}}J_{2l+1}\left[2\left(\frac{2R'\sin\beta'}{a}\right)^{\frac{1}{2}}\right], \tag{20}$$

and neglecting terms of higher order in $(\frac{1}{2}kR'\cos\beta)$ in the expansion of the Bessel function appearing in Eq. (19), we obtain for the integral of Eq. (19):

$$\lim_{k\to 0}I_{l,L,m,M}(k,\beta)\equiv(ka\cos\beta)^L I_{l,L,m,M}=(ka\cos\beta)^L\int d\tau' S_{E=0}(\tau')\gamma_{l,L,m,M}^*(\theta',\varphi',\Theta',\Phi')(2l+1)!2^{-L-l-1}$$
$$\times\left(\frac{R'}{a}\sin\beta'\right)^{-\frac{1}{2}}\left(\frac{R'}{a}\cos\beta'\right)^{L}J_{2l+1}\left(2\left(2\frac{R'}{a}\sin\beta'\right)^{\frac{1}{2}}\right)[\Gamma(L+\tfrac{3}{2})]^{-1}, \tag{21}$$

where $I_{l,L,m,M}$ is finite and independent of energy and β for well-behaved source distributions and finite reaction zones of arbitrary shape and size.

In the limit $kR\to\infty$, small but finite k, and Z_2, $Z_3\neq 0$, therefore, the partial waves of Eq. (18) are given by[13]

$$\Psi_{l,L,m,M}(R,\beta,\theta,\varphi,\Theta,\Phi)\approx(2\pi/|ka|)^{\frac{1}{2}}A_{l,L}I_{l,L,m,M}\gamma_{l,L,m,M}(\theta,\varphi,\Theta,\Phi)k^{L+\frac{3}{2}}R^{-\frac{5}{2}}(\sin\beta)^{-\frac{1}{2}}(\cos\beta)^L$$
$$\times\exp\left\{i\left[kR+\frac{1}{ka\sin\beta}\ln(2|k|R\sin\beta)+\delta(\beta)\right]+\frac{\pi(a^{-1}-|a|^{-1})}{2|k|\sin\beta}\right\}, \tag{22}$$

except for $\sin\beta$ or $\cos\beta\equiv 0$. These are the asymptotic three-particle partial waves from which the threshold energy dependence of the cross section can be evaluated. They are strictly valid only in the limit $k\to 0$, but are also a valid approximation for energies sufficiently close to threshold to insure that $|kR'|$, $|ka|\ll 1$,[12] where, from the coordinate transformations of Eq. (2), and the fact that the origin is at the center of mass, R' is related to the actual (source) coordinates by

$$R'=\left[\frac{m_1}{M_t}(\mathbf{r}_1')^2+\frac{m_2}{M_t}(\mathbf{r}_2')^2+\frac{m_3}{M_t}(\mathbf{r}_3')^2\right]^{\frac{1}{2}}$$
$$=\left[\frac{(m_2\mathbf{r}_2'+m_3\mathbf{r}_3')^2}{m_1M_t}+\frac{m_2}{M_t}(\mathbf{r}_2'^2)+\frac{m_3}{M_t}(\mathbf{r}_3'^2)\right]^{\frac{1}{2}}, \tag{23}$$

with $\mathbf{r}_1'$, $\mathbf{r}_2'$, $\mathbf{r}_3'$ bounded by the "radii" of the reaction zone associated with the three particles. In configuration space, their validity is confined to the detachment region, and does not extend to a neighborhood where any of the particles "overlap" (i.e., near where $q\neq 0$), and where the wave function, in general, assumes a different asymptotic form. (These partial waves may be regarded, if we wish, to correspond to a model reaction producing point particles.)

It may be worth noting that for the case of Coulomb repulsion, these partial waves are large at threshold only when both charged particles leave the reaction zone nearly in opposite directions. This property can be demonstrated by noting that for this case $a<0$, and that for small positive k, the real negative exponent, $-\pi(k|a|\sin\beta)^{-1}$, in Eq. (22) makes the wave function arbitrarily small except in the immediate neighborhood of $\beta=\pi/2$. Values of β close to $\pi/2$ correspond to $v\ll u$ which, from the definitions of $\mathbf{u}$, $\mathbf{v}$ in Eq. (2) (and the fact that the origin is at the center of mass, i.e., $\mathbf{w}=0$) lead to

$$\frac{|\mathbf{u}|}{|\mathbf{v}|}=\left(\frac{m_1m_2m_3}{m_1+m_2+m_3}\right)^{\frac{1}{2}}\left(\frac{|\mathbf{r}_2-\mathbf{r}_3|}{|m_2\mathbf{r}_2+m_3\mathbf{r}_3|}\right)\gg 1.$$

[12] Reference 11, p. 280.
[13] For the asymptotic form of the gamma function in Eq. (18), see reference 11, p. 47.

This condition is satisfied only when the charged particles are emitted in nearly opposite directions. For the case of Coulomb attraction, on the other hand ($a>0$), the partial waves change from damped to propagating at threshold without any such strongly favored particle configuration.

Finally, it may be noted that the partial waves for even l are symmetric, and those for odd l, antisymmetric, in the interchange of the coordinates of the charged particles when these have identical mass. The symmetry of the total scattered wave function is then determined by the l dependence of the coefficients of the $\gamma_{l,L,m,M}$ in Eq. (22) and, therefore, by the angular variation of the source distribution $S_E(\tau)$.

V. CROSS SECTION

The energy dependence at threshold of the cross section is now evaluated by using the asymptotic expression for the partial wave functions, Eq. (23). We are not interested in multiplicative constants which are independent of energy and consequently they will be ignored.

Since the center of mass of the three particles has been assumed at rest, the flux integral extends only over the remaining coordinates. When the total wave function is normalized to unit incident flux, the cross section is numerically equal to the three-particle flux and is therefore proportional to

$$\sigma \propto \int d^5S\cdot[\psi\nabla_6\psi^* - \psi^*\nabla_6\psi], \tag{24}$$

where $d^5S = R^5 \sin^2\beta \cos^2\beta \sin\theta \sin\Theta d\theta d\Theta d\varphi d\Phi d\beta$ is the hypersurface element in the center-of-mass system, and where the integral extends only over that part of the surface of the six-dimensional hypersphere for which all three particles are far from each other and from the center of mass (presumably corresponding to the integration of the outgoing flux performed experimentally by the detector). This restriction excludes the two-particle current such as elastic scattering which would otherwise contribute to the cross-section integral, and which appears as surface waves in this hyperspace.[14]

Since we are interested in the three-particle flux, we have obtained an appropriate asymptotic expression for the wave function which, however, does not contain the surface waves, and is not valid in these excluded regions. In view of the infinitesimal solid angle associated with these regions, and the fact that our asymptotic solution is integrable there,[15] we can without error extend the flux integral over the entire hypersphere when we use this asymptotic solution. We substitute Eq. (23) into Eq. (24), and take immediate advantage of the orthogonality of the spherical harmonics to obtain

$$\sigma_{l,L,m,M} \propto k^{2L+3}\int_0^{\pi/2} \sin\beta(\cos\beta)^{2L+2} \times\exp\left[-\frac{\pi}{k\sin\beta}\left(\frac{1}{|a|}-\frac{1}{a}\right)\right]d\beta, \quad k\geqq 0. \tag{25}$$

For the case of Coulomb attraction ($a>0$), Eq. (25) becomes

$$\sigma_{l,L,m,M} \propto k^{2L+3}\int_0^{\pi/2} \cos^{2+2L}\beta \sin\beta d\beta \propto k^{2L+3}, \quad k\geqq 0. \tag{26}$$

The above integrand can be directly related to the energy spectrum associated with the reaction products by writing it in terms of the energy (ϵ) shared by the two charged particles in their center-of-mass system. Using Eq. (13a), and recalling that the points of stationary phase in Eq. (16) occurred at $\beta=|\alpha|$, this energy can be expressed as

$$\epsilon = \frac{k^2\hbar^2}{2M_t}\sin^2\beta = E\sin^2\beta.$$

This energy spectrum can now be explicitly displayed as the integrand of an alternate form for Eq. (26),

$$\sigma_{l,L,m,M} \propto \int_0^E (E-\epsilon)^{L+\frac{1}{2}}d\epsilon, \quad \epsilon\geq 0. \tag{26a}$$

It is interesting to note that for the fictitious case of a "neutrino" rest mass of the same order as the electron's, time-dependent perturbation theory for allowed ($L=0$) β-decay yields the same energy spectrum in the limit of very small end-point energy (E).[16]

For the case of Coulomb repulsion ($a<0$), we evaluate the integral in the limit of small k by the method of steepest descents and obtain for energies above threshold

$$\sigma_{l,L,m,M} \propto k^{3L+9/2}\exp\left[-\frac{\pi}{k}\left(\frac{1}{|a|}-\frac{1}{a}\right)\right]. \tag{27}$$

Finally, noting that [from Eq. (4b)],

$$\frac{1}{ka} = -\frac{Z_2Z_3e^2(\mu_{23})^{\frac{1}{2}}}{\hbar(2E)^{\frac{1}{2}}} = \frac{-Z_2Z_3e^2\mu_{23}}{\hbar^2 k_{23}},$$

where $\mu_{23}=m_2m_3/(m_2+m_3)$ is the reduced mass of the two charged particles, and $k_{23}{}^2=2\mu_{23}E/\hbar^2$, we rewrite the above expression for the cross section so that the exponent for Coulomb repulsion has the same form as given by Wigner in the two-particle case,[4]

$$\sigma_{l,L,m,M} \propto k_{23}{}^{3L+9/2}\exp\left(-\frac{2\pi Z_2Z_3e^2\mu_{23}}{\hbar^2k_{23}}\right), \quad k_{23}\geqq 0, \tag{27a}$$

[14] P. M. Morse and H. Feshbach, reference 5, p. 1728 ff.

[15] Although the asymptotic form of the wave function has a singularity at $\beta=0$ (where $\mathbf{r}_2=\mathbf{r}_3$), it is integrable there, and gives no contribution, since the surface element d^5S contains $\sin^2\beta$ as a factor.

[16] E. Fermi, Z. Physik **88**, 161 (1934).

where it will be recalled that L is the angular momentum quantum number for the neutral particle with respect to the center of mass of the two charged particles. The energy spectrum for this case can be displayed by rewriting Eq. (25) in the form

$$\sigma_{l,L,m,M} \propto \int_0^E (E-\epsilon)^{L+\frac{1}{2}} \exp\left[-\pi Z_2 Z_3 \frac{e^2}{\hbar c}\left(\frac{2\mu_{23}c^2}{\epsilon}\right)^{\frac{1}{2}}\right] d\epsilon. \tag{27b}$$

Just as in the case of only two charged (and no uncharged) particles, the limiting energy dependence of the cross section is independent of l and m, the quantum numbers for relative angular momentum between the two charged particles. Also, just as for cases involving an arbitrary number of neutral particles, the limiting form does not involve any z-component angular momentum quantum numbers. The appearance of states of the angular momentum quantum numbers l, m, and M is determined, even at threshold, by the short-range details of the interaction. When $R'/a \ll 1$, however (as it can be for some nuclear reactions), a power series expansion of $I_{l,L,m,M}$ [see Eq. (21)] indicates that only the $l=0$ partial wave contributes at threshold.

The three-particle result (Eq. 27) differs from the corresponding one for two particles[4] in having an energy-dependent factor multiplying the same exponential term which arises when the charged particles are of like signs.[17] It vanishes in the correspondence-principle limit $\hbar \to 0$, indicating that at threshold the process would not occur classically. This can be explained physically as follows: When this detachment takes place at threshold, the two charged particles must arrive "at infinity" with an arbitrarily small energy. But outside the reaction zone, where only the repulsive Coulomb potential is effective, the charges must be regarded classically as "rolling down" this potential hill (from the reaction zone), and will therefore always arrive at infinity with a finite energy. Evidently they must "tunnel" through the repulsive Coulomb potential which acts as a barrier. In fact, the exponential part of the threshold law can be obtained by considering transmission through such a classically forbidden region in the usual way.

The range in energy over which the $L=0$ term of Eqs. (26) and (27) might be expected to describe an experimental yield curve will be mentioned here only briefly. It is clearly limited by the neglect of higher order terms containing ka and kR'. The energy dependence of S_E, including the requirement that no other new reactions have thresholds nearby, must also be considered in estimating the range of validity.

[17] This exponential should not be confused with the usual Gamow penetration factor for the *incident* particle, which is not displayed in Eq. (27a) because sufficiently close to threshold it is a constant factor not dependent on the excess energy above threshold. For a further discussion of this point, see Wigner.[4]

For electron detachment from H^-, for example, the threshold law becomes

$$\sigma \propto E^{9/4} \exp\{-16.4[E(\text{ev})]^{\frac{1}{2}}\}, \quad 0 \leq E \ll 0.4 \text{ ev}, \tag{28a}$$

for detachment by electrons, and

$$\sigma \propto E^{\frac{3}{2}}, \quad 0 \leq E \ll 0.5 \text{ ev}, \tag{28b}$$

for detachment by positive ions. In both the above cases, the condition on kR' appears to be the most stringent limitation on the range of validity for a reaction zone as large as one Bohr radius. For the order-of-magnitude estimate above, the reaction zone radius was taken as 4 Bohr radii for the electron case (a value apparently somewhat larger than that of the negative ion),[17] and 5 Bohr radii for the positive ion case.

For nuclear-scale reactions [e.g., (n,np) reactions], the reaction zone may be regarded as being of the same order of magnitude as the nuclear radius. Upon expressing the energy E now in Mev, taking the detached particle to be a proton, and denoting the mass and charge of the product nucleus by A and Z, the threshold law becomes

$$\sigma \propto E^{9/4} \exp\{-Z[E(\text{Mev})]^{-\frac{1}{2}}(1+A^{-1})^{-\frac{1}{2}}\}$$
$$\text{for } 0 < E \ll \text{the smaller of } \begin{cases} 0.02\, Z^2 \text{ Mev} \\ 7 \times A^{-\frac{2}{3}} \text{ Mev}, \end{cases} \tag{29}$$

where, in evaluating R' [see Eq. (23)], r_1' and r_3' have been taken as $1.2A^{\frac{1}{3}} \times 10^{-13}$ cm.

VI. CONCLUDING REMARKS

Having started with three particles, one is probably entitled to feel that nine quantum numbers should have appeared somewhere in the analysis, instead of only eight (the three associated with the center of mass of the system, and l, L, m, M, E). Certainly the ninth quantum number does appear in the case of three neutral particles, where it describes states in the radial separation between the particles.[7] There, it appears in an entirely natural way when the Green's function is expanded in terms of eigenfunctions satisfying the appropriate radiation condition. But such an expansion is usually very difficult in cases such as the present one, where the Coulomb interaction term has prevented us from finding separable solutions of the Schrödinger equation which individually satisfy the desired radiation condition. Since we have not found an expansion of the radial part of the Green's function in terms of eigenfunctions satisfying this boundary condition, the associated quantum number has remained concealed. A further breakdown into reaction channels presumably exists which would display the ninth quantum number. The major point is that since the Green's function contains all detachment channels, whether or not they

[17] L. R. Henrich, Astrophys. J. **99**, 59 (1943).

are resolved, or displayed explicitly, no unresolved channel can appear less rapidly with energy than the lowest energy dependence determined directly from the "unresolved" Green's function.

This treatment has shown how the functional form of the energy dependence of the cross section near threshold is determined without a detailed knowledge of the reaction, just as for uncharged particles. It is based primarily on applying to a solution of the Schrödinger equation valid outside a reaction zone of finite extent the requirements that the wave function and its gradient remain bounded and continuous (even at threshold). In order that the concept of a finite reaction zone be a valid one, it is necessary to consider all long-range interactions explicitly in this solution of the Schrödinger equation. For this reason, although the treatment appears to be generalizable to include as many uncharged particles as desired, any extension to include additional charged particles appears to be more difficult because it is then necessary to determine properties of the wave function when the Schrödinger equation does not appear to be separable.

APPENDIX

Here, we shall illustrate the transformation of the integral

$$J=\int\int q_0(\Psi-\Psi_i)G d\mathbf{u}'d\mathbf{v}' \qquad \text{(A-1)}$$

into an integral over a finite region (of a magnitude comparable to, although somewhat larger than, the reaction zone), plus a remainder term which damps exponentially. In order to do this, we shall need only rather general properties of the bound-state wave functions, and while we shall carry out the transformation only for square-well binding potentials (q_0), the result is apparently valid for bound states in general. The transformation involves, principally, the separation of $\Psi-\Psi_i$ into outgoing and ingoing parts (in the three-dimensional space of the unbound particle) and the deformation of the path of integration (in the complex plane) along paths on which these wave functions damp. The example given here pertains only to uncharged particles, in order to illustrate the procedure in its simplest form. When two of the particles are charged, the Coulomb Green's function (and wave functions) must be used in place of the corresponding free-space functions, but the transformation is otherwise analogous for that case.

As an example, we shall consider only the channel corresponding to binding between particles 2 and 3, i.e., we define

$$q_0(u)=\begin{cases}-k_0^2 \text{ for } u\leq b \text{ (and outside the reaction zone)}\\ 0 \text{ otherwise.}\end{cases}$$

As a further simplification, let particle 1 be infinitely massive, and let $m_2=m_3=m$. In the region of integration defined by $q_0\neq 0$, which may be regarded as a ("dielectric") wave guide, $\Psi-\Psi_i$ may be expanded in the complete set of normal modes, Ψ_N, for such a guide. These satisfy the Schrödinger equation

$$\{\nabla_u^2+\nabla_v^2+k^2-q_0(u)\}\Psi_N=0, \qquad \text{(A-2)}$$

with the boundary conditions

$$\Psi_N\Big|_{u'=b}=0 \quad\text{or}\quad \frac{\partial\Psi_N}{\partial u'}\Big|_{u'=b}=0. \qquad \text{(A-3)}$$

The contribution of the damped modes need not be considered since the corresponding integrals can be cut off at several (wave guide) wavelengths with exponentially damping remainder. The coefficients of the exponentially increasing modes are required to vanish, so we are left with a (finite) number of propagating modes for which the integrals must be investigated.

The partial waves of these modes are given by

$$\frac{J_{\lambda+\frac{1}{2}}(K_\lambda u)}{u^{\frac{1}{2}}}P_\lambda^\gamma(\cos\theta)\frac{H_{\Lambda+\frac{1}{2}}^{(1)\text{ or }(2)}[v(k^2+k_0^2-K_\lambda^2)^{\frac{1}{2}}]}{v^{\frac{1}{2}}}$$

$$\times P_\Lambda^\Gamma(\cos\Theta)\begin{cases}\cos\gamma\varphi\cos\Gamma\Phi\\ \cos\gamma\varphi\sin\Gamma\Phi\\ \sin\gamma\varphi\cos\Gamma\Phi\text{'}\\ \sin\gamma\varphi\sin\Gamma\Phi\end{cases} \qquad \text{(A-4)}$$

where K_λ is a root of

$$J_{\lambda+\frac{1}{2}}(K_\lambda b)=0 \quad\text{or}\quad J_{\lambda+\frac{1}{2}}'(K_\lambda b)=0,$$

and where the argument of the Hankel function is real.

The free-space Green's function satisfies

$$(\nabla_u^2+\nabla_v^2+k^2)G(\mathbf{u},\mathbf{v};\mathbf{u}',\mathbf{v}')=\delta(\mathbf{u}-\mathbf{u}')\delta(\mathbf{v}-\mathbf{v}'), \qquad \text{(A-5)}$$

with the singularity at $(\mathbf{u}',\mathbf{v}')$ corresponding to a source, and is given, for $R>R'$, by[10]

$$G(\mathbf{R},\mathbf{R}')=\sum_{l,L,m,M,n} f_{l,L,m,M,n}(R,\theta,\Theta,\beta)\cos[m(\varphi-\varphi')]$$

$$\times\cos[M(\Phi-\Phi')]\cos^l\beta'\sin^L\beta'$$

$$\times F(-n,l+L+n+2;L+\tfrac{3}{2};\sin^2\beta')$$

$$\times P_l^m(\cos\theta')P_L^M(\cos\Theta')\cdot\frac{J_{l+L+2n+2}(kR')}{(R')^2}, \qquad \text{(A-6)}$$

where the $f_{l,L,m,M,n}$ are functions only of the field point, and are not relevant here. Substituting Eq. (A-5) and one of the modes from Eq. (A-4), say the mode with ($\cos\gamma\varphi\cos\Gamma\Phi$ and $H^{(1)}$), into Eq. (A-1), and carrying out the angle integrations, we obtain

$$J=\sum_n C_{\lambda,\Lambda,\gamma,\Gamma} f_{\lambda,\Lambda,\gamma,\Gamma,n}(R,\theta,\Theta,\beta)\cos\gamma\varphi\cos\Gamma\Phi$$

$$\times\left\{\int_{u_L}^{b}(u')^2du'\int_{v_L}^{\infty}(v')^2dv'\cos^\lambda\beta'\sin^\Lambda\beta'\right.$$

$$\times F(-n,\lambda+\Lambda+n+2;\Lambda+\tfrac{3}{2};\sin^2\beta')$$

$$\times\frac{J_{\lambda+\Lambda+2n+2}[k(u'^2+v'^2)^{\frac{1}{2}}]}{(u')^2+(v')^2}\frac{J_{\lambda+\frac{1}{2}}(K_\lambda u')}{u'^{\frac{1}{2}}}$$

$$\left.\times\frac{H_{\Lambda+\frac{1}{2}}^{(1)}[v'(k^2+k_0^2-K_\lambda^2)^{\frac{1}{2}}]}{v'^{\frac{1}{2}}}\right\}, \quad \text{(A-7)}$$

(where $u'=R'\sin\beta'$, $v'=R'\cos\beta'$, and the $C_{\lambda,\Lambda,\gamma,\Gamma}$ are coefficients arising from the angle integrations) with u_L and v_L determined by $u_L^2+v_L^2=R_0^2$, where R_0 is the reaction zone boundary.

We now show that each of the above integrals can be restricted to a region in (u',v')-space which is no greater than several wave-guide mode wavelengths from the origin, with an error that damps exponentially in the parameter "cutoff$\times(k^2+k_0^2-K_\lambda^2)^{\frac{1}{2}}$." Since the u' integration extends only to $u'=b$, we need merely show that the v' integral can be cut off at several guide mode wavelengths, say at $v'=v_0$. We now investigate the behavior of the v' integral for $v'>v_0$, where the Hankel function may be expressed by its asymptotic form and the hypergeometric function is equal to $1+O(b^2/v_0^2)$. This integral can also be expressed as an integral over a finite region with an exponentially damped remainder. This can be seen by deforming the path of integration (from v_0 to ∞ on the real axis) to a new path running from v_0 to iv_0 (say along an arc of the circle $|v'|=v_0$), then from iv_0 to $i\infty$ along the imaginary axis, and finally along an infinite arc to $v'=\infty$. (If we had taken a mode having $H^{(2)}$ rather than $H^{(1)}$, the path would have been deformed into the lower half plane, rather than the upper.) The infinite arc contributes nothing because the integrand vanishes there. The integral from iv_0 to $i\infty$ is clearly exponentially damped [i.e., it has as a factor

$$\exp[-(v_0(k^2+k_0^2-K_\lambda^2)^{\frac{1}{2}}-k(v_0^2-b^2)^{\frac{1}{2}})],$$

which damps for energies sufficiencly close to threshold that $k<(k^2+k_0^2-K_\lambda^2)^{\frac{1}{2}}/(1-b^2/v_0^2)^{\frac{1}{2}}$]. Finally, the integral from v_0 to iv_0 extends over a finite region, and can be lumped with the integral from v_L to v_0 to yield the desired transform of J into an integral over a finite region.

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Mass Distribution in Fission of U^{235} by Resonance Neutrons*

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Radiochemical analyses of samples of U^{235} irradiated with monoenergetic neutrons at 1.1, 3.1, and 9.0 ev indicated that the fission yield of Ag^{111}, Cd^{115}, and Sb^{127} relative to the yield of Sr^{89} does not change from that produced by thermal neutrons.

INVESTIGATIONS of the variation of fission yield with mass have been made under a wide variety of experimental conditions, but these studies have not included fission induced by resonance neutrons of well-defined energy. A determination of the relative probabilities of symmetric and asymmetric modes of fission at specific resonances might give further insight into the nature of the fission process and the properties of the states of the compound nucleus corresponding to the resonances. In particular, Bohr[1] has presented qualitative considerations relating the relative probabilities of symmetric and asymmetric fission modes to the spin and parity of the state of the compound nucleus.

Measurements of ν, the number of neutrons per fission, have indicated that this quantity remains essentially constant for all resonances.[2] However, it was felt that a study of the features of the curve of yield *vs* mass would provide a more sensitive measure of possible differences in fission modes at different resonances.

Samples of U^{235} metal (about 90 g each, 1 cm$\times$1 cm $\times$10 cm) were irradiated with neutrons from a crystal

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