Weak Collective Effects in the Nuclear Shell Model

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Configuration mixing between the odd particle and the core is formulated to account for the weak collective quadrupole effects near closed shells. The configuration mixing sum is evaluated formally for the general case and applied to O^{17}. An equivalent formalism in which closure is done over the configuration mixing sum is developed and applied to O^{17}, giving good agreement with experiment. The closure result is found to be very insensitive to variation of parameters. The closure formalism is also applied near the lead closed shell, using a nuclear charge density distribution for the core instead of a sum over particle states. The collective effects in this region are well accounted for, and the formalism is compared with that for weak surface coupling.

I. INTRODUCTION

PERHAPS the most fertile model of the nucleus introduced in recent years is the shell model of Mayer and Jensen.[1] In this model, eigenstates are generated by considering the independent motion of nucleons in a potential well with a large spin-orbit coupling, and it is assumed that a good approximation can be obtained for the ground state and low-energy properties of the nucleus by considering only a few of these states. The assumption of independent motion is ostensibly a very surprising one in view of the very short range and great strength of nuclear forces, but the nuclear shell model has proved extremely rewarding and thus gained strong pragmatic justification. Recently the work of Brueckner *et al.*[2] has indicated the way to a sound theoretical justification as well.

If the initial generating potential is considered to be spherically symmetric, then in the region of the doubly closed shells the nuclear wave functions should be particularly simple, and for a doubly closed shell plus or minus one particle the nuclear wave function should reduce to a single antisymmetric term, i.e., for this case the $j-j$ coupling independent-particle model should reduce to the single-particle model. In gross properties this seems in fact to be the case, but in most cases where detailed information is available, there seem to be residual collective effects that cannot be easily attributed to the odd particle in the pure single-particle model. Attempts to explain these weak collective effects within the framework of the collective model of Bohr and Mottelson[3] using the so-called surface coupling approach have been only moderately successful and have not as yet been put on any fundamental footing. The subject of this paper is the treatment of these collective effects within the framework of the independent-particle model.[4]

In the nuclear shell model the generating potential, U, must arise from some sort of averaging of the two-body interactions,[5] V, between the nucleons. But even if this average has been performed in an optimum, self-consistent manner, the shell-model states will not be diagonal in the residual interaction, $V-U$. Since the shell-model states generally used are chosen for their analytic rather than self-consistent properties we should not be surprised if the residual interaction has a large perturbing effect. The mixing of zero-order shell-model states by the residual interaction is known as configuration mixing, and calculations of this mixing have considerably extended the range of success of the shell model.[6] But calculations with the residual interaction are extremely complex and have in general only been carried out among particles outside $L-S$ closed shells, and then except for the light nuclei, only near closed shells. Limited calculations in which any number of particles outside closed shells have been considered have also been done for ground-state magnetic moments,[7] ground-state electric quadrupole moments,[8] and comparative half-lives in beta decay.[9] The calculations for magnetic moments and logft values give quite good agreement with experiment, whereas the agreement for quadrupole moments is not nearly so good far from closed shells, but this can be understood in terms of the sensitivity of the quadrupole moment to shape deformations of the nucleus.[3]

Just as the residual interaction mixes different zero-order shell-model configurations of particles outside closed shells, so the interaction of an odd particle with the closed-shell core will mix in excited configurations of the core, even for a nucleus with only one particle outside the closed shell. Because of the great stability of the closed shell, the admixtures of these excited states will be small. But because there are many core protons, even very small admixtures can give fairly

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[1] M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley and Sons, Inc., New York, 1955).

[2] For references see H. A. Bethe [Phys. Rev. **103**, 1353 (1956)].

[3] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).

[4] A preliminary note on this work has appeared. R. D. Amado and R. J. Blin-Stoyle [Proc. Phys. Soc. (London) **A70**, 532 (1957)].

[5] Only two-body interactions are considered.

[6] A complete account of this and other aspects of the shell model and a complete list of references are given by J. P. Elliott and A. M. Lane, "The Nuclear Shell Model," in *Handbuch der Physik* [Springer-Verlag, Berlin (to be published)].

[7] A. Arima and H. Horie, Progr. Theoret. Phys. (Japan) **11**, 509 (1954).

[8] H. Horie and A. Arima, Phys. Rev. **99**, 778 (1955).

[9] R. J. Blin-Stoyle and C. A. Caine, Phys. Rev. **105**, 1810 (1957).

large collective contributions, particularly for electric moment operators like the quadrupole operator. From this configuration mixing point of view, then, collective effects are due to mixtures of excited states of the core caused by interaction with the odd particle.

In Sec. II the general procedure for the direct evaluation of the configuration mixing sum in the case of electric quadrupole effects is outlined and applied to the two weak collective effects in O^{17}.

In Sec. III an alternative formulation for the weak collective effects is presented in which closure is done over the configuration mixing sum. For O^{17} this procedure is exactly equivalent to the one in Sec. II, and has the advantage of being a formalism in which parameters are easily varied. The calculation gives a good account of the weak collective effects in O^{17}, and further, variations of the parameters will change the results by a factor of two, but no reasonable set of parameters will account for the old, small value for the ground-state quadrupole moment.[10] A brief discussion is also given of other calculations of the weak collective effects in O^{17}.

In the heavy nuclei direct evaluation of the configuration mixing sum for weak electric quadrupole effects becomes excessively difficult, but the closure formalism developed in Sec. III can be effectively applied. In Sec. IV the closure formalism is discussed further and applied to weak collective effects in the region of the lead doubly closed shell. It is seen that the calculation can be carried out in terms of a nuclear charge density distribution without having to specify single particle states for particles in the core. The formalism gives a consistent account of the weak collective effects in this region, and also provides a general framework for the discussion of these effects. A discussion is given of other calculations of the weak collective effects in the region of lead, and in particular the configuration mixing formalism with closure is compared with the weak surface coupling formalism of Bohr and Mottelson.[3]

II. THE CONFIGURATION MIXING SUM

For a nucleus differing by only one neutron from a doubly closed shell, there will be no zero-order shell-model contribution to electric quadrupole effects. The entire effect must then arise from excitations of the core. Since the core initially has spin zero and positive parity, and since the quadrupole operator is a one-body operator transforming under rotations like an eigenstate of angular momentum 2 with positive parity, the relevant core excitations in first order must be 2+ states of the core differing from the doubly closed shell in the state of one proton only. Since the generating potential, U, is a one-body scalar operator, it cannot produce core excitations of this type and thus for the odd-neutron case we need only consider the effect of the two-body interaction, V. The first-order collective contribution to the matrix element of the nuclear electric quadrupole operator between two states J and J_1 can then be written:

$$-\sum \frac{\langle 0J|V|(2+,J_1)J\rangle\langle(2+,J_1)J|Q|0J_1\rangle}{E_{2+}+\Delta_{JJ_1}} -\sum \frac{\langle 0J|Q|(2+,J)J_1\rangle\langle(2+,J)J_1|V|0J_1\rangle}{E_{2+}-\Delta_{JJ_1}}, \quad (1)$$

where the sum extends over all 2+ states of the core formed by excitation of a single proton, and where E_{2+} represents the energy for this excitation and Δ_{JJ_1} the difference in energy between the odd neutron states and J_1. By $(2+,J)J_1$ is meant the core in a 2+ state and the odd particle in the state J, coupled together to form a state of angular momentum J_1. Since we shall only be concerned with low-lying states J and J_1 or with $J=J_1$, Δ_{JJ_1} is always negligible compared with E_{2+} by virtue of the energy stability of the closed-shell core.

A. Evaluation of the Mixing Sum

In order to evaluate the sum in (1) we take single particle eigenstates of the form

$$\phi_{lJM}=\mathfrak{R}_{Nl}\sum_m\langle l\tfrac{1}{2}mM-m|l\tfrac{1}{2}JM\rangle Y_{l,m}\chi_{\frac{1}{2},M-m}, \quad (2)$$

where $\mathfrak{R}_{Nl}$ is the radial function, $\chi_{\frac{1}{2},m}$ a spin function, and $Y_{l,m}$ the usual spherical harmonic. The vector addition or Wigner coefficient and the spherical harmonics are chosen to agree in phase and normalization with Condon and Shortley.[11] It is convenient in (1) to consider separately the matrix elements of V and of Q. We take for V an arbitrary exchange mixture of the form

$$V(r_{12})=-B(\gamma+\epsilon P_M+\beta P_B+\delta P_B P_M)f(r_{12}), \quad (3)$$

where the P's are the exchange operators, P_M space exchange, and P_B spin exchange, as defined in Blatt and Weisskopf.[12] B is the strength of the interaction and the constants γ, ϵ, β, δ are defined so that $\gamma+\epsilon+\beta+\delta=1$. Putting this exchange mixture into the matrix elements of V in (1), one gets four terms and thus may write

$$\langle 0J|V|(2^+,J_1)J\rangle=\gamma M_W+\epsilon M_M+\beta M_B+\delta M_H. \quad (4)$$

We can now proceed to evaluate each of these terms by first expanding the interaction function, $f(r_{12})$, in

[10] Geschwind, Gunther-Mohr, and Townes, Phys. Rev. 83, 209 (1951).

[11] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1935).

[12] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (J. Wiley and Sons, Inc., New York, 1952).

spherical harmonics.

$$f(r_{12})=4\pi\sum_{k=0}^{\infty}f_k(r_1,r_2)\sum_m Y_{k,m}(\theta_1\varphi_1)Y_{k,m}^*(\theta_2,\varphi_2). \quad (5)$$

Writing in the couplings in terms of Wigner coefficients in the usual way and using the elegant formalism for manipulating holes and particles in terms of creation and annihilation operators developed by Brink and Satchler,[13] one can reduce the general matrix element of V in (4) from a matrix element between nuclear states to a sum of matrix elements of the interaction between single-particle states. These are then most easily evaluated by transforming from a $j-j$ to LS coupling scheme and using the usual techniques for manipulation of angular momenta.[14]

Considering the core 2+ state formed by exciting a proton from the core state j to the excited state j_1 and using

$$J=l\pm\tfrac{1}{2},\quad J_1=l_1\pm\tfrac{1}{2},\quad j=l'\pm\tfrac{1}{2},\quad j_1=l_1'\pm\tfrac{1}{2},$$

one gets for the four matrix elements corresponding to the four exchange possibilities

$$M_W=(-1)^{J_1+j+l_1+l_1'}BF_2(ll'l_1l_1')\langle 2l_100\,|\,2l_1l0\rangle$$
$$\times\langle 2l'00\,|\,2l'l_1'0\rangle[5(2j+1)(2J_1+1)(2j_1+1)$$
$$\times(2l_1+1)(2l'+1)]^{\frac{1}{2}}W(l_1lJ_1J;2\tfrac{1}{2})$$
$$\times W(l_1'l'j_1j;2\tfrac{1}{2}),$$

$$M_M=(-1)^{j+J_1+l_1+l_1'}B\sum_k\{(2k+1)F_k(ll'l_1'l_1)$$
$$\times\langle kl_1'00\,|\,kl_1'l0\rangle\langle kl'00\,|\,kl'l_10\rangle[5(2l_1'+1)(2l'+1)$$
$$\times(2J_1+1)(2j+1)(2j_1+1)]^{\frac{1}{2}}W(l_1lJ_1J;2\tfrac{1}{2})$$
$$\times W(l_1'l'j_1j;2\tfrac{1}{2})W(ll_1'l_1l';k2)\},$$

$$M_B=(-1)^{j_1+J+l+l'}5BF_2(ll'l_1l_1')\langle 2l_100\,|\,2l_1l0\rangle$$
$$\times\langle 2l'00\,|\,2l'l_1'0\rangle[5(2j+1)(2J_1+1)(2j_1+1)$$
$$\times(2l_1+1)(2l'+1)]^{\frac{1}{2}}\mathfrak{L}+M_W,$$

$$M_H=(-1)^{j_1-J+l+l'}B\sum_k\{(2k+1)F_k(ll'l_1'l_1)$$
$$\times\langle kl_1'00\,|\,kl_1'l0\rangle\langle kl'00\,|\,kl'l_10\rangle[5(2l_1'+1)(2l'+1)$$
$$\times(2J_1+1)(2j_1+1)(2j+1)]^{\frac{1}{2}}W(jj_1J_1J;2k)$$
$$\times W(lJl_1'j_1;\tfrac{1}{2}k)W(l_1J_1l'j;\tfrac{1}{2}k)\}.$$

$W(abcd;ef)$ is the usual Racah coefficient,[15] and the F_k's are the radial integrals given by

$$F_k(abcd)=\int_0^\infty\int_0^\infty\mathfrak{R}_a(r_1)\mathfrak{R}_b(r_2)\mathfrak{R}_c(r_1)\mathfrak{R}_d(r_2)$$
$$\times f_k(r_1,r_2)r_1^2r_2^2dr_1dr_2. \quad (6)$$

In the case of M_B, that is the Bartlett or spin-exchange case, it has not been possible to evaluate all the sums analytically. Nevertheless it is possible to reduce the expression so that only the sum $\mathfrak{L}$ need be done numerically. This is

$$\mathfrak{L}=\sum_L(2L+1)W(jj_1JJ_1;2L)W(l'l_1'll_1;2L)$$
$$\times W(l_1'j_1l_1J_1;\tfrac{1}{2}L)W(l'jlJ;\tfrac{1}{2}L).$$

The matrix elements of the quadrupole operator appearing in (1) can be evaluated in a similar manner and for the case of a core proton excited from the core state j to the state j_1, one gets

$$\langle(2+,J_1)JM\,|\,Q_\mu\,|\,0J_1M'\rangle$$
$$=(-1)^{\frac{1}{2}+l'-j}\left(\frac{(2l_1'+1)(2j+1)(2j_1+1)}{4\pi}\right)^{\frac{1}{2}}$$
$$\times W(j_1l_1'jl';\tfrac{1}{2}2)\langle J_12JM\,|\,J_12M'\mu\rangle$$
$$\times\langle l_1'2l'0\,|\,l_1'200\rangle\int_0^\infty\mathfrak{R}_{l_1'}{}^*r^4\mathfrak{R}_{l'}dr, \quad (7)$$

where we have taken the quadrupole operator to have the form

$$Q_m=r^2Y_{2,m}.$$

B. Application to O^{17}

In the shell model, O^{17} has one neutron outside the extremely stable O^{16} ($Z=8$, $N=8$) doubly closed shell. One is therefore not surprised to find that the low-lying positive parity levels[16] as well as the ground-state magnetic dipole moment[17] are very well accounted for in the extreme single-particle model. However, two electric quadrupole effects are known that, from our point of view, must be attributed to configuration mixing. The first excited state, 0.872 Mev above the ground state and assigned the single-particle configuration $2s_{\frac{1}{2}}$, decays to the ground state by $E2$ radiation with a mean lifetime of $(2.5\pm1)\times10^{-10}$ sec,[16] corresponding to a reduced matrix element[18] for the decay of 0.027 ± 0.006 barn. The ground state, which on the shell model has the odd neutron in a $1d_{\frac{5}{2}}$ state has a

[13] D. M. Brink and G. R. Satchler, Nuovo cimento **4**, 549 (1956).
[14] See A. R. Edmonds, *Angular Momentum in Quantum Mechanics*, CERN 55–26 (1955).
[15] G. Racah, Phys. Rev. **62**, 438 (1942); L. C. Biedenharn, Oak Ridge National Laboratory Report ORNL-1098, 1952 (unpublished).
[16] F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1955).
[17] R. J. Blin-Stoyle, Revs. Modern Phys. **28**, 75 (1956).
[18] We use the normalization convention for the reduced matrix element (or double-bar matrix element) of some tensor operator $T(\lambda\mu)$ between angular momentum eigenstates jm and JM of $\langle JM\,|\,T(\lambda\mu)\,|\,jm\rangle=\langle J\|T(\lambda)\|j\rangle\langle j\lambda JM\,|\,j\lambda m\mu\rangle$. This is related to the lifetime following G. R. Satchler [Proc. Phys. Soc. (London) **A67**, 1024 (1954)].

small static electric quadrupole moment of -0.0265 ± 0.003 barn.[19]

In order to evaluate the radial integrals in (6) for the case of O^{17}, we must specify the form of the interaction as well as the form for the radial functions. For light nuclei it is usual to take the single-particle functions as eigenstates in a three-dimensional isotropic harmonic-oscillator well. This gives for the radial function[20]

$$\Re_{Nl}=N_{Nl}\exp(-\rho^2/2)\rho^l L_{Nl}(\rho^2), \tag{8}$$

where N_{Nl} is a normalization constant, L_{Nl} is the associated Laguerre polynomial, and $\rho=r/b$ with $b=(\hbar/Mk)^{\frac{1}{2}}$ if the generating potential U is given by $U=\frac{1}{2}Mk^2r^2$. For ease of calculation we take $f(r_{12})$ to have a Gaussian form

$$f(r_{12})=\exp(-r_{12}^2/a^2). \tag{9}$$

The method for getting the f_k's in this case and for doing the radial integrals has been given by Swiatecki.[21] A generalization of his procedure and an expression in closed form for the integrals is given in the Appendix.

The choice of harmonic-oscillator radial functions of the form (8) severely limits the possible excitations that give nonvanishing contributions to the configuration mixing sum. The combination of angular momentum selection rules and selection rules for matrix elements of r^2 between harmonic-oscillator radial functions[22] limit the possible excitations of protons from the O^{17} core that will contribute to (7) to $1s_{\frac{1}{2}}$ to $1d_{\frac{3}{2}}$, $1s_{\frac{1}{2}}$ to $1d_{\frac{5}{2}}$, $1p_{\frac{1}{2}}$ to $2p_{\frac{3}{2}}$, $1p_{\frac{3}{2}}$ to $2p_{\frac{1}{2}}$, $1p_{\frac{3}{2}}$ to $2p_{\frac{3}{2}}$, $1p_{\frac{1}{2}}$ to $1f_{\frac{5}{2}}$, $1p_{\frac{3}{2}}$ to $1f_{\frac{5}{2}}$, and $1p_{\frac{3}{2}}$ to $1f_{\frac{7}{2}}$.

We are interested in three types of terms in the combination of the interaction and quadrupole matrix elements—those for which $J=J_1=\frac{5}{2}$, those for which $J=\frac{5}{2}$ and $J_1=\frac{1}{2}$, and those for which $J_1=\frac{5}{2}$ and $J=\frac{1}{2}$. These three correspond to diagonal mixing for the ground-state quadrupole moment, and the two cases for the $E2$, mixing into the ground state and the excited state respectively. For each of these three cases we must evaluate the contribution of each of the four types of exchange interaction, and this must be done for the eight possible excitations enumerated above. The results will depend on the exchange character of the force. We take for this a Rosenfeld mixture[23] for which $\gamma=-0.13$, $\epsilon=0.93$, $\beta=0.46$, and $\delta=-0.26$. The radial function parameter, b, can be obtained by relating the density distribution of the wave functions to the nuclear radius[21] and this gives for O^{17} $b=1.6\times10^{-13}$ cm. In the interaction we take $a=1.9\times10^{-13}$ cm.[23] The contribution of each of the excitations is given in Table I. We notice that in general the relative contribution of each excitation, or group of excitations to Q, E_D, and E_S is roughly of the same order in each case, and thus we cannot change the results drastically by simply disallowing some particular excitation, or by giving it an anomalously low or high energy.

TABLE I. The contribution is listed for each of the possible excitations in O^{17} to each of the three terms; Q for the quadrupole moment, E_D for the $E2$ reduced matrix element with admixtures into the ground state, and E_S for the $E2$ reduced matrix element with admixtures into the excited state. The contributions are given in units of $Bb^2\alpha^{\frac{3}{2}}/[E_i(\alpha+2)^{11/2}]$ where $\alpha=(a/b)^2=1.4$ and where E_i is the appropriate excitation energy for each contribution.

	$1s-1d$		$1p-2p$			$1p-1f$		
	$\frac{1}{2}-\frac{3}{2}$	$\frac{1}{2}-\frac{5}{2}$	$\frac{1}{2}-\frac{3}{2}$	$\frac{3}{2}-\frac{1}{2}$	$\frac{3}{2}-\frac{3}{2}$	$\frac{1}{2}-\frac{5}{2}$	$\frac{3}{2}-\frac{5}{2}$	$\frac{3}{2}-\frac{7}{2}$
$-Q$	22.2	33.5	3.6	3.6	3.8	52	12.8	84.4
E_S	7.1	18	12	8.7	10	8.3	4.9	17
E_D	−8.8	−6.5	2.5	2.8	2.8	19	4.7	36

Since we have no clear-cut experimental evidence on which to base a choice, we must pick the energy denominators for the excitations from the model. Neglecting spin-orbit splitting, all the excitations we are considering have the same energy. This is a consequence of the selection rules for matrix elements of r^2 between harmonic-oscillator radial functions. Since for light nuclei spin-orbit splittings are small compared with shell splittings, we expect that all the excitation energies will still be of the same order when we include spin-orbit splittings. This suggests that the configuration mixing sum in this case could be evaluated using closure with a mean excitation energy equal to twice the oscillator shell splitting. The closure formulation for the problem allows one to take into account the effect of variation of parameters, and forms a more convenient basis for comparison with experiment. As we shall see in the next section, the results are in good agreement with the experimental $E2$ and ground-state quadrupole moment in O^{17}.

III. THE CLOSURE FORMALISM

One can always treat the sums in (1) by replacing the energy denominators by some mean excitation energy, $\bar{E}$, and then doing closure over the excitations. This gives for (1)

$$-(1/\bar{E})\langle 0J|VQ+QV|0J_1\rangle. \tag{10}$$

This expression can be treated as a definition of the mean excitation energies, and has the advantage of averaging the information about the excited configurations into the mean energy so that one need not specify any but the zero-order configuration. The closure expression is particularly useful if one can assume that the mean excitation energy is a property that can be assigned to the core, independent of the odd-particle state. If this can be done then (10) can be used to give

[19] Kamper, Lea, and Lustig, Proc. Phys. Soc. (London) (to be published); see also M. J. Stevenson and C. H. Townes [Phys. Rev. **107**, 635 (1957)].
[20] I. Talmi, Helv. Phys. Acta **25**, 185 (1952).
[21] W. J. Swiatecki, Proc. Roy. Soc. (London) **A205**, 238 (1951).
[22] W. H. Shaffer, Revs. Modern Phys. **16**, 245 (1944).
[23] L. Rosenfeld, *Nuclear Forces* (North Holland Publishing Company, Amsterdam, 1948).

a one-parameter fit for the weak collective quadrupole effects in a given nucleus or for a given closed shell core.

Since we are interested in obtaining just such a fit of the weak quadrupole effects near closed shells, it is of interest to see in what cases the closure assumption, that is the assumption that the mean excitation energy is independent of the single-particle states for a given core, will be valid. Clearly the assumption will be trivially true if all the excited core states contributing to the sum have the same excitation energy. It will still remain a good approximation if the major contribution to the sum comes from a spread of excitations small compared with the mean excitation energy. As we have seen this is the case for the harmonic-oscillator shell model of O^{17} with small spin-orbit splitting. For the closure assumption to be valid, however, it is not necessary that the important contribution to the sum comes from a small energy range. The closure expression (10) defines $\bar{E}$ as the harmonic mean of the excitation energies over the configuration mixing sum. And hence the closure assumption is valid so long as the odd-particle state does not affect the harmonic mean of the excitation energies. Thus, for example, if the relative contribution of each excitation is always the same, and the only effect of the odd-particle state is to provide a scale for the contributions, then the assumption will be valid. We shall return to the closure assumption in the discussion of lead in Sec. IV.

A. Application to O^{17}

One can apply the closure form of (10) to the ground-state quadrupole moment and the $E2$ transition probability in O^{17} either by simply adding up the results of the direct configuration mixing calculation, or by a straightforward evaluation of the matrix elements appearing in (10). This latter procedure is relatively simple for the closure case since the nuclear states appearing in the matrix element are all states for one particle outside a closed shell and thus the matrix element between nuclear states is easily converted into a sum of matrix elements between particle states.[11] Since the closed shell is an $L-S$ closed shell, there is no need to couple up the angular momentum of the states of the core as is done in (2). Furthermore, for matrix elements between states differing in only one particle from a closed shell, it is easily verified that the only effect of the spin-exchange operator, P_B, in the interaction is to reduce by one the number of spin sums that are free. Since each of these sums contributes a factor of two to the matrix element, one can simply replace P_B by a factor of one half. Thus the interaction in (3) can be rewritten

$$V(r_{12})=-B[\gamma+\tfrac{1}{2}\beta+(\epsilon+\tfrac{1}{2}\delta)P_M]f(r_{12}),$$

and the expression for the quadrupole moment or the $E2$ transition matrix element splits up into a direct part and a space-exchange part. One can write

$$\langle Q\rangle=(\gamma+\tfrac{1}{2}\beta)\langle Q\rangle_{\text{direct}}+(\epsilon+\tfrac{1}{2}\delta)\langle Q\rangle_{\text{exchange}},$$

for the quadrupole moment and there is clearly a similar expression for the $E2$.

Using the Gaussian form for the interaction as before, and harmonic-oscillator radial functions, the evaluation of the matrix elements follows in a straightforward manner. The techniques described in the Appendix are used to evaluate the radial integrals. One finds

$$\langle Q\rangle_{\text{direct}}=-\frac{8Bb^2\alpha^{\frac{3}{2}}}{\bar{E}(\alpha+2)^{11/2}}(6\alpha^2+11\alpha+7),$$

and

$$\langle Q\rangle_{\text{exchange}}=-\frac{8Bb^2\alpha^{\frac{3}{2}}}{\bar{E}(\alpha+2)^{11/2}}(\alpha^2+3\alpha^2+10\alpha+7),$$

where, as before, $\alpha=(a/b)^2$. Similarly, for the reduced matrix element for the $E2$ transition,

$$\langle E2\rangle_{\text{direct}}=\frac{Bb^2\alpha^{\frac{3}{2}}}{\bar{E}(\alpha+2)^{11/2}}\left(\frac{15}{2\pi}\right)^{\frac{1}{2}}(24\alpha^2+26\alpha+19),$$

and

$$\langle E2\rangle_{\text{exchange}}=\frac{Bb^2\alpha^{\frac{3}{2}}}{2\bar{E}(\alpha+2)^{11/2}}\left(\frac{15}{2\pi}\right)^{\frac{1}{2}}(11\alpha^3+15\alpha^2+68\alpha+38).$$

Putting in the values for a, b, and α as before, one finds

$$\begin{aligned}\langle Q\rangle_{\text{direct}}&=-1.2(B/\bar{E})\times10^{-26}\text{ cm}^2;\\ \langle Q\rangle_{\text{exchange}}/\langle Q\rangle_{\text{direct}}&=0.9,\end{aligned}\tag{11}$$

for the quadrupole moment, and similarly for the $E2$:

$$\begin{aligned}\langle E2\rangle_{\text{direct}}&=0.85(B/\bar{E})\times10^{-26}\text{ cm}^2,\\ \langle E2\rangle_{\text{exchange}}/\langle E2\rangle_{\text{direct}}&=1.\end{aligned}\tag{12}$$

It is striking to note that both for the quadrupole moment and for the $E2$, for reasonable values of the parameters, the direct and exchange contributions are about the same. This means that in both cases we can write the results in the form

$$\langle Q\text{ or }E2\rangle=(\gamma+\tfrac{1}{2}\beta+\epsilon+\tfrac{1}{2}\delta)\langle Q\text{ or }E2\rangle_{\text{direct}},$$

and hence that the relative value of the quadrupole and $E2$ are independent of the choice of exchange mixture. Using the Rosenfeld mixture to evaluate the exchange factors, one finds

$$\langle Q\rangle=-1.0(B/\bar{E})\times10^{-26}\text{ cm}^2,$$

for the quadrupole moment, and for the reduced matrix element for the transition

$$\langle E2\rangle=0.78(B/\bar{E})\times10^{-26}\text{ cm}^2.$$

Comparing these results with the experimental values we find that $B/\bar{E}=3.4\pm0.8$ fits the $E2$ transition rate and $B/\bar{E}=2.65\pm0.3$ fits the quadrupole moment. There is a satisfactory region of overlap between these two values.

In order to compare these values with other results for the strength of the interaction we must pick $\bar{E}$. The most reasonable procedure seems to be to pick the excitation energy semiempirically. The first excited state of O^{16} occurs at 6 Mev, and in O^{17} there is a group of negative-parity states at about 4 Mev.[16] These must correspond to excitations of the same order as those involving a change in the oscillator principal quantum number. Taking 6 Mev as a mean value for this shell separation energy and taking $B/\bar{E}$ to be about 3 gives $B=40$ Mev. This is a reasonable value for the interaction strength and fits well with other work that has been done with configuration mixing around $A=19$.[24]

B. Variation of Parameters

It is interesting that both for the quadrupole moment and the $E2$, the direct and exchange parts are essentially the same. We can investigate how this result changes with variations in the range of the force or the shape of the nuclear wave functions by varying α. The ratio in (11) only changes by 10% as α is varied from zero to 3.5. For the $E2$ the ratio in (12) is even less sensitive to variation of α. Thus the insensitivity of the results to the exchange character of the force holds for all reasonable values of the ratio of the range to nuclear radius. The ratio of the total $E2$ matrix element to the ground-state quadrupole moment shows a similar insensitivity to variations in α. It is gratifying that the results are insensitive to variations in the exchange character or range of the force. This means that no set of experimental results greatly different from the present values could be accounted for by the theory.

For reasons of simplicity, the calculations thus far have been done with a Gaussian radial dependence for the force, but they can also be done with a Yukawa dependence for $f(r_{12})$. Evaluation of the radial integrals for this case is very much more complicated, and therefore we have only investigated the direct part of the quadrupole moment and of the $E2$. It is probably safe to assume, nevertheless, that the exchange parts will not be very different from the direct parts since in O^{17} the range of the force is of the same order as the nuclear radius, and the matrix elements of the interaction should not be very sensitive to what is essentially the fine structure of the interaction. For the Yukawa case the direct contribution to the quadrupole moment is a factor of two smaller relative to the $E2$ than is the case for the Gaussian interaction. This result is also fairly insensitive to variations of the range of the force relative to the range of the particle functions. No exact conclusions can be reached from this calculation but the experimental results do seem to indicate a slightly smaller quadrupole moment than is given by the calculations with the Gaussian interaction, and this might be accounted for by taking a Yukawa interaction.

Since it is probable that in a proper self-consistent independent-particle-model formulation for O^{17} the odd neutron states will be more loosely bound than is the case in the harmonic-oscillator model, it is of interest to investigate the effect of "pushing out" the radial functions for the $1d$ and $2s$ states relative to the core states. This can be done in a reasonable way by taking a smaller b for the odd-particle states than for the core states, that is to say, by taking the core states as eigenfunctions in a narrower well than the odd-particle states. This variation will only affect the radial integrals and as before we have only investigated the effect on the direct parts with a Gaussian dependence for the interaction. The evaluation of the radial integrals can be carried out in a generalization of the procedure of the Appendix, and one finds that even when large variations of the odd-particle radial-function range relative to the core states are considered the ratio of the direct part of the $E2$ to the quadrupole moment remains essentially unaltered.

We have seen that reasonable variations within the framework of first-order configuration mixing with closure will change the relative results for the $E2$ and the quadrupole moment by factors of order unity, but no more. Thus, these calculations are in agreement with the recent value for the ground-state quadrupole moment,[19] but could not, by any stretch, be put into agreement with the old, very much smaller, value.[10] We have also seen in Sec. II that the results for direct evaluation of the configuration mixing sum are not very sensitive to assumptions about the excitations. We have not investigated the effect of variation of parameters in the direct configuration mixing calculation, but in view of the closure calculation it seems reasonable to assume that they will be insensitive to these variations as well.

C. Other Calculations

Other attempts have been made to explain the weak collective effects in O^{17} from different points of view. Perks[25] and Horie and Arima[8] have looked at the configuration mixing without closure. In particular Horie and Arima have evaluated the first-order configuration mixing contribution to the ground-state quadrupole moment for a large number of nuclei. They make the two simplifying assumptions of zero-range forces and harmonic-oscillator radial functions. Although one would not *a priori* expect zero-range forces to be a good approximation in O^{17} we have seen that in fact the calculations are extremely insensitive to the range of the force so that their calculations should be good for oxygen. They find a ground-state moment of

[24] J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) **A229**, 536 (1955).

[25] M. A. Perks, Proc. Phys. Soc. (London) **A68**, 1083 (1955).

-0.04 barn, which is the correct order of magnitude but somewhat larger than the experimental results or the results of our calculations. They do not calculate transition probabilities, but in view of our calculation, one would guess that their method would give reasonably good agreement with the $E2$ in O^{17}.

The two weak collective effects have been investigated by Barker[26] from the point of view of weak surface coupling.[3] He finds that consistent with the $E2$ transition probability the ground-state quadrupole moment comes out about a factor of two larger than the experimental value. But we might expect surface coupling to be a bad approximation in a light nucleus like oxygen. A semiempirical calculation of the two effects has also been carried out that seems to give a consistent account of the order of magnitude of the two effects.[27]

IV. THE LEAD CLOSED SHELL

For heavy nuclei the nuclear density distribution is more "square" than in oxygen and hence harmonic-oscillator radial functions are no longer a good approximation. We are thus no longer helped by the strong selection rules imposed on the configuration mixing sum by these functions, and the evaluation of the proper first-order configuration mixing sum (1) taking all possible contributions into account would be very difficult indeed. On the other hand, application of the closure formalism (10) is relatively simple. However, since we are using closure in this case out of ignorance, we cannot begin with a justification of the closure assumption, that is the assumption that the mean excitation energy will be a property of the core independent of the odd-particle state, but as we shall see this assumption seems in fact to be corroborated.

A. Development of the Formalism

In expanding the matrix element in (10) into a sum of matrix elements between single-particle states,[11] one encounters two types of terms, two-body terms in which only the odd-particle state and one core state appears, and three-body terms in which there are two core states. In O^{17} the three-body terms contribute slightly less than 20% of the two-body terms, and one expects that because of the absence of harmonic-oscillator selection rules, the relative contribution in the case of heavy nuclei will be even less. Further it is expected that the three-body terms will have roughly the same dependence on odd-particle states as do the two-body terms, and thus from the point of view of the closure assumption we make a negligible error in neglecting these terms.

If we neglect the three-body terms, then for the odd-neutron case we can write

$$\frac{1}{\bar{E}}\langle 0J|VQ|0J_1\rangle = \frac{1}{\bar{E}}\sum_i\langle\phi_i(1)\phi_J(2)|V(12)Q(1)|\phi_i(1)\phi_{J1}(2)\rangle, \quad (13)$$

where the sum goes over all core proton states.

In lead the range of the force is much less than the nuclear radius so we can treat the interaction in a range expansion and keep only the first couple of terms. For the first term in the expansion (zero range) P_M, the space exchange operator, simply gives unity. In this case the sum over core proton states reduces to the charge density distribution, after suitable care has been taken over the spin parts of the matrix element. Thus (13) can be rewritten

$$\frac{Z\zeta}{\bar{E}}\int_0^\infty\int_0^\infty \phi_J^*(2)\phi_{J1}(2)\rho(1)V(12)Q(1)d^3r_1d^3r_2, \quad (14)$$

where ρ is the charge density normalized to unity, and ζ is a numerical factor depending on the exchange mixture, and hence on the spin matrix elements.

The introduction of a density distribution is a considerable computational simplification. Further, it reduces the physically difficult problem of having to choose single-particle eigenfunctions for the protons to the problem of choosing a density distribution, and the recent high-energy electron scattering experiments at Stanford have yielded much information about this distribution.[28] We take a simpler density than the one obtained in these experiments, but one that still retains the gross features of the Stanford distribution. We take the spherically symmetric distribution

$$\begin{aligned}
&\rho(r,t)=\rho_0 && \text{for } 0\leq r\leq R(1-t),\\
&\rho(r,t)=\rho_0[(R-r)/Rt] && \text{for } R(1-t)\leq r\leq R,\\
&\rho(r,t)=0 && \text{for } r>R.
\end{aligned}$$

The normalization is $4\pi\int_0^R\rho(r)r^2dr=1$, giving

$$\rho_0=\frac{3}{R^3(4-6t+4t^2-t^3)\pi}.$$

This density has been used by Blin-Stoyle[29] but with a slightly different normalization. If one treats $R_0=r_0\times A^{\frac{1}{3}}$ as the radius of the square density distribution that has the same mean square radius as ρ, then R_0 is related to R by

$$R^2=R_0^2\beta^2(t),$$

where

$$\beta^2(t)=\frac{3(4-6t+4t^2-t^3)}{2(6-15t+20t^2-15t^3+6t^4-t^5)}.$$

[26] F. C. Barker, Phil. Mag. 1, 329 (1956).
[27] S. Fallieros and R. A. Ferrell, Bull. Am. Phys. Soc. Ser. II, 2, 26 (1957).
[28] R. Hofstadter, Revs. Modern Phys. 28, 214 (1956).
[29] R. J. Blin-Stoyle, Phil. Mag. 46, 973 (1955).

In order to evaluate the integral in (14), we expand the integrand in a power series in the range of the force.[30] Going to the first nonvanishing term beyond zero range, and putting P_M equal to unity for this term as well, gives

$$-\frac{Z\zeta}{\bar{E}}\mathfrak{A}\left[\int_0^\infty \phi_J^*\phi_{J1}\rho Q d^3r + C'\int_0^\infty \phi_J^*\phi_{J1}\nabla^2(\rho Q)d^3r\right], \quad (15)$$

where

$$\mathfrak{A}=B4\pi\int_0^\infty f(s)s^2ds,$$

$$C'=\int_0^\infty f(s)s^4ds \Big/ 6\int_0^\infty f(s)s^2ds.$$

The integral can be evaluated in a straightforward manner by using single-particle eigenstates for the odd particle of the form (2). For the case of lead, however, we take the radial functions as solutions of the radial equation for eigenstates in an infinite square well. We assume that the radius of the generating potential is the same as the R that appears in the density distribution; this is a reasonable assumption since it makes the density and the single-particle functions go to zero at the same radius. The angular integrals appearing in (15) are easily evaluated in the usual way and the remaining radial integrals are done graphically.

The formalism presented above is for the case of an odd neutron or odd neutron hole nucleus and is easily extended to the case of two odd neutrons or neutron holes. Some change arises in considering odd protons. Firstly there is a zero-order contribution to the electric quadrupole effects so that the configuration mixing only provides an additional collective part. Further, care must be taken in the closure sum to subtract the diagonal term since to do closure one must sum over all states. This diagonal term vanishes for the odd neutron since the zero-order matrix element of the quadrupole operator vanishes. In addition the antisymmetry of the odd particle with the core will make the spin matrix element somewhat different. In the case of a diagonal matrix element of the quadrupole operator, that is, a ground-state moment, the collective contribution for the odd-proton case corresponding to (14) is

$$\frac{Z\eta}{\bar{E}}\int_0^\infty\int_0^\infty |\phi_J(2)|^2\rho(1)Q(1)V(12)d^3r_1d^3r_2$$

$$-\frac{1}{2}\frac{Z\eta}{\bar{E}}\int_0^\infty |\phi_J|^2Qd^3r\int_0^\infty\int_0^\infty V(12)\rho(1)$$

$$\times|\phi_J(2)|^2d^3r_1d^3r_2, \quad (16)$$

[30] D. M. Brink, Proc. Phys. Soc. (London) **A67**, 757 (1954).

where η is a factor depending on the exchange character of the force.

In the approximation of zero-range forces that we are using, the exchange factors, ζ and η, are easily expressed in terms of the exchange character of (3). P_M is equivalent to unity for zero range and for the odd neutron P_B gives a factor of $\frac{1}{2}$. Thus we get

$$\zeta=\gamma+\epsilon+\tfrac{1}{2}(\beta+\delta).$$

In the odd-proton case, P_B gives minus the direct part, and we get

$$\eta=\gamma+\epsilon-(\beta+\delta).$$

For a Rosenfeld mixture we find $\zeta=0.90$ and $\eta=0.60$.

For r_0 we take 1.2×10^{-13} cm,[28] which corresponds to an R_0 for lead of 7.1×10^{-13} cm. For the interaction parameters $\mathfrak{A}$ and C', we take the values given by Brink.[30] $\mathfrak{A}=1.3\times10^{-36}$ Mev cm^2, which in the usual way is independent of the precise radial form of the interaction; and $C'=0.92\times10^{-26}$ cm^2, which is an average for the Yukawa and square well interaction. The precise choice of C' is not very important since the range expansion seems to be quite good, the term in C' being only 10% of the first term.

B. Comparison with Experiment

The energy levels and spin assignments for Pb^{207}, one neutron hole in the doubly closed shell, are given by Alburger and Sunyar.[31] The $E2-M1$ mixing ratio for the 1.78-Mev transition between the $f_{7/2}$ and $f_{\frac{5}{2}}$ state is known and Satchler[32] has calculated the $E2$ reduced matrix element for the transition from this mixing ratio assuming a single-particle value for the $M1$. He gets for the reduced matrix element corresponding to the $E2$ part of the transition 0.15 barn. The assumption that the $M1$ part of the transition is well given by the single-particle value has been investigated and corroborated by Caine.[33] He has calculated the first-order configuration mixing contribution to matrix elements of the magnetic dipole operator and finds that the mixing reduces the $M1$ matrix element for the transition by only about 7% of the single-particle value.

The reduced matrix element for the 0.569-Mev $E2$ transition from the first excited state ($f_{\frac{5}{2}}$) to the $p_{\frac{3}{2}}$ ground state has been measured by Stelson and McGowan[34] using Coulomb excitation and they get 0.096 barn. These two $E2$ transitions in Pb^{207} must clearly be collective and we can try to fit them with the formalism developed above. If we equate the experimental results to (15) in each case, we get an expression for $\bar{E}$ as a function of the density parameter, t, needed to fit the experimental results. It is convenient to treat t as a parameter even though it is given by the electron

[31] D. E. Alburger and A. W. Sunyar, Phys. Rev. **99**, 695 (1955).
[32] G. R. Satchler (private communication).
[33] C. A. Caine (private communication).
[34] P. H. Stelson and F. K. McGowan, Phys. Rev. **99**, 112 (1955).

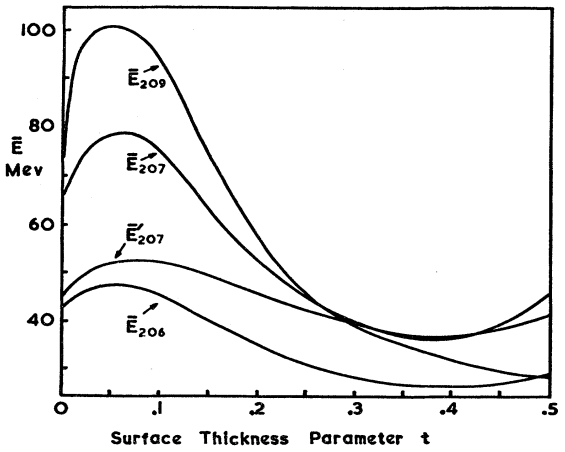

FIG. 1. The mean excitation energies, $\bar{E}$, are plotted as a function of the surface thickness parameter, t, for the ground state quadrupole moment in Bi^{209}, $\bar{E}_{209}$, the 0.569-Mev $E2$ transition to the ground state of Pb^{207}, $\bar{E}_{207}$, the 1.78-Mev $E2$ in Pb^{207}, $\bar{E}'_{207}$, and the $E2$ to the ground state of Pb^{206}, $\bar{E}_{206}$.

scattering experiments since it is interesting to see the effect of variation in density distribution. The mean excitation energies as a function of t are plotted in Fig. 1.

In the two-neutron-hole nucleus, Pb^{206}, the first excited state, 2+, decays to the 0+ ground state by a 0.803-Mev $E2$ gamma ray. The reduced matrix element for the transition, measured by Stelson and McGowan,[34] is 0.158 barn. The formalism of configuration mixing with closure extended to the case of two neutrons can be applied to this case, after single-particle configurations have been chosen for the two neutrons. To do this we have taken the particle configurations calculated by Kearsley[35] taking into account configurations mixing between the two holes. Once again the $\bar{E}$ as a function of t needed to fit the experimental result is given in Fig. 1.

The only measured electric quadrupole effect in the odd-proton nucleus Bi^{209} is the ground-state quadrupole moment of -0.4 barn.[17] The contribution of the odd proton, in the $h_{9/2}$ state, to this moment is -0.28 barn calculated by using single-particle eigenstates in an infinite square well. It is better to use these functions to get the mean value of r^2 than the usual procedure involving $\frac{3}{5}$ the square of the nuclear radius[17] since the $h_{9/2}$ state is strongly "pushed out." Assuming the remaining -0.12 barn to be of collective origin, one can use the formalism of (16) and once again plot the mean excitation energy required as a function of t in Fig. 1.

In Fig. 1 we see that there is surprisingly good agreement between $\bar{E}_{207}$, $\bar{E}'_{207}$, and $\bar{E}_{209}$. This agreement occurs for $t=0.3$, a value consistent with the results of the electron scattering experiments,[28] and, it is interesting to note, not for values of t very different from 0.3. For $t=0.3$ the three mean excitation energies are about 40 Mev. $\bar{E}_{206}$ comes out some 25% smaller but we might expect this since, differing by two particles from a closed shell, Pb^{206} should have a "softer" core than those nuclei differing by only one. In addition there is an uncertainty of about 15% in all the results due to the neglect of the three-body terms. One expects the mean excitation energy to be somewhat greater than twice the shell splitting in the heavy nuclei and this is in fact the case if one takes the average shell splitting to be given by the maximum of the giant dipole photoelectric cross section.[36]

The question remains of why there should be a unique mean excitation energy at all, that is, why we should expect the closure assumption to be valid. To give a complete justification for this assumption, apart from the empirical justification, would, of course, entail being able to do the configurations mixing sum exactly, or, at least, knowing a good deal about the contribution of each term in that sum. Since it is precisely the lack of this information that led us to the closure formalism, we cannot hope to give the results anything but a certain plausibility.

We have already seen that if the relative contribution of each excitation is independent of the odd-particle state except for an over-all scale factor, then the closure assumption is fully justified. The closure assumption will still be a good approximation if the relative contribution from groups of excitations is the same, so long as we consider groups with an energy spread small compared with the mean excitation energy. This is likely to be the case if in the range of important contributions there are a large number of excitations of different types in each small energy interval. If the number in each interval is large enough and diverse enough, then we can perform an average over the contributions in each interval, and assume that the relative values of each average does not depend on the odd-particle state. This is a kind of statistical approach to the problem, and it is likely to be fairly good in heavy nuclei like lead because the large number of particles in the nucleus will make for a large number of diverse possible excitations in each small energy interval. In addition it is probable that the total width of the distribution of excitations giving the major contribution is small compared with the mean excitation energy since the combined effect of the large energy denominator, the selection of states by the quadrupole operator, and the potential coupling will all contribute to cutting off the configuration-mixing sum quickly.

C. Comparison with the Weak Surface Coupling Method

It is of interest to compare the formalism for weak collective effects using configuration mixing and closure with the quasi-hydrodynamical method of Bohr and

[35] M. Kearsley (to be published).

[36] S. Rand, Phys. Rev. 99, 1620 (1955).

Mottelson.[3] They assume that near closed shells the nucleus may be represented by a spherical core with small compressibility, in which case the collective degrees of freedom which have the lowest energy will be shape deformations with approximate preservation of volume. The normal coordinates of these oscillations will be the expansion parameters, α_μ, of the nuclear surface given by

$$R'=R(1+\textstyle\sum_\mu \alpha_\mu Y_{2,\mu}),$$

where R is the equilibrium radius, and where we have restricted ourselves to deformations of the quadrupole type. Bohr[37] and Bohr and Mottelson[3] have developed an elegant quantum field-theoretical formalism for treating these deformations. They quantize the surface oscillations, or phonons, and find that α_μ essentially contains the creation and annihilation operators for these phonons. In the hydrodynamical approximation the phonon energy is easily expressed in terms of the surface tension and mass transport parameters for the nuclear "fluid" of the core, and, as one would expect from the quadrupole nature of the deformation, the phonons have angular momentum 2 and positive parity. The zero-order shell-model core state is then considered to be a state with no phonons present, and the effect of the interaction between the odd particle and the core is to mix in states with one phonon (in lowest order). In such a formalism the only mechanism for the interaction is the model generating potential U.

Following the field-theoretical formalism, one can write both the interaction and the quadrupole operator in terms of the creation and annihilation operators, and thus one gets for the weak surface coupling contribution to the ground-state quadrupole moment of a nucleus with one particle in a state J outside the closed-shell core

$$\langle Q_B\rangle=\frac{3}{(5\pi)^{\frac{1}{2}}}\mathcal{Y}_{JJ}\frac{k}{C}ZR^2, \tag{17}$$

where $\mathcal{Y}_{JJ}$ is an angular matrix element of $Y_{2,0}$ between the angular and spin parts of the single particle functions, C is the surface-tension energy, and k is essentially a measure of the interaction strength between the odd particle and the core, and is related to the radial derivative of the generating potential. In the case that U is a square well, then the derivative is a delta function at the surface and k can be expressed in terms of the value of the odd-particle radial function evaluated at the surface. But in general this specialization is not made and k is treated as a constant. The surface-tension energy, C, is normally treated as a fitting parameter, and like $\bar{E}$ is expected to be a constant for a given core.

It is instructive to compare (17) with the corresponding expression in the configuration mixing formalism with closure. The comparison is most easily made by considering the case of $t=0$ and taking only the first term in the range expansion. We then get for the quadrupole moment of some odd-neutron nucleus

$$\langle Q\rangle=\frac{6\mathfrak{A}Z}{(5\pi)^{\frac{1}{2}}R^3\bar{E}}\mathcal{Y}_{JJ}\int_0^R|\mathcal{R}_J|^2r^4dr, \tag{18}$$

where $\mathcal{R}_J$ is the odd-particle radial function.

It is interesting to note that this is just the expression we would get if we assigned to the neutron an effective charge of magnitude $3\mathfrak{A}Z/2\pi R^3\bar{E}$ so long as the single-particle functions vanish at the core radius. It seems then that the effect of the perturbation has been to add to the uniform charge density of the zero-order core, a density just proportional to the odd-particle density. One of the essential differences between this and the weak-coupling formalism being that we no longer consider the core as incompressible and hence the odd particle can affect the core throughout rather than only at the surface. The surface part is still present as can be seen by doing a partial integration in (18).

$$\langle Q\rangle=\frac{6\mathfrak{A}ZR^2}{5(5\pi)^{\frac{1}{2}}\bar{E}}\mathcal{Y}_{JJ}|\mathcal{R}_J(R)|^2-\frac{6\mathfrak{A}Z}{5(5\pi)^{\frac{1}{2}}R^3\bar{E}}\mathcal{Y}_{JJ}\int_0^R r^5\frac{d}{dr}[|\mathcal{R}_J(r)|^2]dr.$$

The first term is a surface term of the type one would have for a square-well generating potential in the weak surface coupling formalism. The second term, the bulk term, is, in fact, larger than the first for all the cases we have considered.

The expression for the quadrupole moment would in the formalism of configuration mixing with closure seem to depend in a more natural way on the odd-particle function than does the weak surface coupling result, particularly since the former allows one to take into account the odd-particle function overlap with the core density. It is this overlap that gives the curves in Fig. 1 their characteristic shape. For example, in Bi^{209} the single-particle function is strongly "pushed out" and thus the collective contribution for small t is very large since the core density in the region of the peak in the particle function is large. As we increase t the charge density in the region of the odd particle goes down and so does the collective contribution to the quadrupole moment. Similarly the rise in the collective contribution in Pb^{207} for large t can be attributed to a node in the radial function.

The weak surface coupling formalism is easily generalized to $E2$ transitions as well as ground-state moments and has been applied by True[38] to the transitions to the ground state in Pb^{207} and Pb^{206} and to the ground-state moment of Bi^{209}. He finds a surface-tension parameter, C, of 1100 Mev for the transition in Pb^{207} but only 520 Mev for the transition in Pb^{206}. Using the wave

[37] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26,** No. 14 (1952).

[38] W. W. True, Phys. Rev. **101.** 1342 (1956).

functions with configuration mixing between the holes for Pb^{206} this can be raised to 715 Mev, but this is still too large a discrepancy to be explained on the surface coupling formalism. The better fit obtained with the configuration mixing and closure approach can be explained in that it allows one to take into account the particle overlap in Pb^{206}, and further since in the configuration mixing formalism one is dealing with a more familiar mechanism than surface coupling, the remaining discrepancies can be more easily accounted for. Calculations on the 1.78-Mev transition in Pb^{207} indicate that very different parameters would be necessary to fit the transition on the surface coupling model from those used by True for the transition to the ground state. Once more this is largely due to radial integrals.

For the ground-state quadrupole moment of Bi^{209}, True finds a collective contribution of -0.25 barn using $C=1100$ Mev. This is much larger than one expects if one estimates the single-particle part of the moment taking into account the odd-proton wave function. In this case an important contribution to the configuration mixing approach is the introduction of anti-symmetry between the odd proton and the core. This cannot be conveniently done in the weak surface coupling formalism.

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APPENDIX

If we expand the radial dependence of the interaction as in (5), then

$$f_k(r_1,r_2)=\tfrac{1}{2}\int_0^\pi f(r_{12})P_k(\cos\theta_{12})\,\sin\theta_{12}d\theta_{12},$$

where P_k is the usual Legendre polynomial.

For a Gaussian form as in (9), one gets for the general f_k

$$f_k(r_1,r_2)=\tfrac{1}{2}\exp\left(-\frac{r_1^2+r_2^2}{a^2}\right)\sum_{r=0}^{\frac{1}{2}k \text{ or } \frac{1}{2}(k-1)}\frac{(2k-2r)!(-1)^r}{2^k r!(k-r)!}$$
$$\times\sum_{s=0}^{k-2r}\frac{1}{(k-2r-s)!}\left(\frac{(-1)^s e^x-(-1)^k e^{-x}}{x^{s+1}}\right), \quad \text{(A1)}$$

where $x=2r_1r_2/a^2$.

Evaluation of radial integrals with harmonic oscillator functions of the form (8) always reduces to the evaluation of radial integrals of the form

$${}_kI_{mn}=\int_0^\infty\int_0^\infty f_k(r_1,r_2)r_1^m r_2^n \times\exp\left(-\frac{r_1^2+r_2^2}{b^2}\right)dr_1dr_2, \quad \text{(A2)}$$

where from parity considerations if k is even, m and n are even, and if k is odd, m and n are odd. In view of this the general integrals can be generated from the integral with $m=n$ by repeated differentiation with respect to $1/b^2$.

Putting (A1) into (A2) with $m=n$, and following a generalization of the procedure used by Swiatecki[21] for the evaluation of these integrals, we get

$${}_kI_{mm}=\frac{\pi a^{2m+2}}{2^{2m+k+1}[\alpha(\alpha+2)]^{\frac{1}{2}}}\sum_{r=0}^{\frac{1}{2}k \text{ or } \frac{1}{2}(k-1)}\sum_{s=0}^{k-2r}\sum_{p=0}^{m-s-1}\left[\frac{(2k-2r)!(m-s-1)!(2m-2s-2p-3)!!(2p-1)!!(-1)^{p+r+s}2^s}{r!(k-r)!(k-2r-s)!p!(m-s-p-1)!\alpha^{m-s-p-1}(\alpha+2)^p}\right],$$

where $\alpha=(a/b)^2$ and $(-1)!!=1$.