Behavior of the Meson-Nucleon Cross Section at High Energies*

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(Received June 27, 1957)

It is shown that within the context of a local causal field theory the total π-nucleon cross section $\sigma(\omega)$ has the following behavior at high energies: $\lim_{\omega\to\infty}\omega\sigma(\omega)>0$.

I. INTRODUCTION

In the past few years, various derivations have been given for the dispersion relations for π-nucleon scattering.[1] These relations follow as a consequence of several general requirements invoked on the scattering amplitude such as "microscopic causality" (i.e., the condition that the commutator of two current operators taken at space-like points vanish), Lorentz covariance, etc. Also several detailed assumptions concerning the interaction such as conservation of isotopic spin and the pseudoscalar nature of the π-meson field are made. Aside from assumptions of the type listed above (which one would in general invoke on any local field theory that is to describe meson-nucleon interactions), one must also make a postulate concerning the behavior of the π-nucleon cross sections at high energies in order to obtain a unique set of dispersion relations. It is usually assumed that the cross sections approach a constant at high energies. This condition seems reasonable in view of the direct experimental measurements of the high-energy cross section.[2] One may also take the experimental verification of the π-nucleon dispersion relations in the low-energy region[3] as indirect verification of this behavior for the total cross section.

At present, then, the assumption that the total cross sections approach a constant value in the high-energy region may be viewed as a reasonable phenomenological condition on the scattering amplitude. On the other hand, it is obvious that a solution of the field equations of motion would uniquely determine the high-energy behavior of the cross section. In this note we shall show that by making greater use of the nature of the Lagrangian governing π-nucleon scattering it is possible to make a partial statement about the high-energy behavior. The theorem that we shall prove in the next section is that

$$\lim_{\omega\to\infty}\omega\sigma(\omega)>0,$$

(where $\sigma(\omega)$ is the total π-nucleon cross section and ω is the pion energy in the laboratory system) provided that the constant λ in the $\lambda\phi^4$ direct meson-meson interaction is not zero.

II. HIGH-ENERGY THEOREM

In order to prove the theorem mentioned at the end of the previous section, we shall make use of two different representations of the scattering amplitude. For definiteness let us consider the scattering of π^+ mesons and protons. If q' and p' are the initial meson and proton momenta and q and p the final momenta, the S-matrix can be written as the sum of a noninteracting contribution plus a part which gives rise to the scattering:

$$\langle pq|S|p'q'\rangle=\langle p|p'\rangle\langle q|q'\rangle+(2\pi)^4 i\delta^4(p+q-p'-q')\times[m^2/(4\omega\omega' EE')]^{\frac{1}{2}}T_+. \quad (1)$$

In the usual representation,[4] the T_+ matrix is expressed as one-nucleon matrix elements of the current operators which generate the meson field; i.e.,[5]

$$\left(\frac{m^2}{EE'}\right)T_+(pqp'q')=i\bigg[\int e^{-iqx}\theta(x_0)\langle p|[j(x),j^\dagger(0)]|p'\rangle d^4x -i\int\{e^{-iqx}\overleftrightarrow{\partial_0}\langle p|[\phi(x),j^\dagger(0)]|p'\rangle\}\delta(x_0)d^4x\bigg] -i\langle p|p'\rangle\bigg[\int e^{-iqx}\theta(x_0)\langle 0|[j(x),j^\dagger(0)]|0\rangle d^4x -\int\{e^{-iqx}\overleftrightarrow{\partial_0}\langle 0|[\phi(x),j^\dagger(0)]|0\rangle\}\delta(x_0)d^4x\bigg]. \quad (2)$$

In Eq. (2), $j(x)$ is defined by

$$j(x)=[-\Box^2+\mu^2]\phi(x),\quad j^\dagger(x)=[-\Box^2+\mu^2]\phi^\dagger(x), \quad (3)$$

* Supported in part by the U. S. Atomic Energy Commission and the National Science Foundation.

[1] M. L. Goldberger, Phys. Rev. **99**, 979 (1955); K. Symanzik, Phys. Rev. **105**, 743 (1957); Goldberger, Miyazawa, and Oehme, Phys. Rev. **99**, 986 (1955).

[2] Cool, Piccioni, and Clark, Phys. Rev. **103**, 1081 (1956).

[3] Anderson, Davidon, and Kruse, Phys. Rev. **100**, 339 (1955); Uri Haber-Schaim, Phys. Rev. **104**, 1113 (1956).

[4] By the "usual representation," we mean the one used by Goldberger in reference 1.

[5] We use italic letters p, q to represent four-vectors, while bold-face letters **p**, **q** denote three-vectors. The metric used is such that $pq=\mathbf{p}\cdot\mathbf{q}-p_0q_0$. For a meson $q_0^2\equiv\omega^2=q^2+\mu^2$ (μ = meson mass) while for a nucleon $p_0^2\equiv E^2=p^2+m^2$ (m = nucleon mass). We take $\hbar=c=1$.

where $\phi(x)$ is the meson field operator. $\theta(x_0)$ is the step function, zero for $x_0<0$, unity for $x_0>0$ and $|0\rangle$ is the vacuum state. The symbol $\leftrightarrow$ above ∂ means

$$f(x)\overleftrightarrow{\partial_0}g(x)=f(x)\frac{\partial g(x)}{\partial x_0}-\frac{\partial f(x)}{\partial x_0}g(x). \tag{4}$$

One can, however, express the T_+ matrix in an alternate fashion as one-meson matrix elements of operators $J(x)$ and $\bar{J}(x)$ which may be viewed as generating the nucleon field.[6] In this representation, one easily obtains

$$\begin{aligned}\left(\frac{1}{4\omega\omega'}\right)^{\frac{1}{2}}&T_+(pqp'q')\\ &=i\bigg[\int e^{-iqx}\theta(x_0)\bar{u}_p\langle q|\{J(x),\bar{J}(0)\}|q'\rangle u_{p'}d^4x\\ &\quad-i\int e^{-ipx}\bar{u}_p\gamma_0\langle q|\{\psi(x),\bar{J}(0)\}|q'\rangle u_{p'}\delta(x_0)d^4x\bigg]\\ &\quad-i\langle q|q'\rangle\bigg[\int e^{-ipx}\theta(x_0)\bar{u}_p\langle 0|\{J(x),\bar{J}(0)\}|0\rangle u_{p'}d^4x\\ &\quad-i\int e^{-ipx}\bar{u}_p\gamma_0\langle 0|\{\psi(x),\bar{J}(0)\}|0\rangle u_{p'}\delta(x_0)d^4x\bigg],\end{aligned} \tag{5}$$

where

$$J(x)=(-i\gamma\partial/\partial x+m)\psi(x),\quad \bar{J}(x)=J^\dagger(x)\gamma_0, \tag{6}$$

and $\psi(x)$ is the nucleon field operator. Also

$$\gamma\partial/\partial x=\gamma_i\partial/\partial x_i+\gamma_0\partial/\partial x_0,\quad \{A,B\}=AB+BA. \tag{7}$$

The $\gamma_\mu=(\gamma_i,\gamma_4=i\gamma_0)$ are anti-Hermitian matrices and $\gamma_0\equiv\beta$ is Hermitian. The u_p is a positive-energy nucleon spinor of a given spin (the spin label has been suppressed) and is normalized according to

$$\bar{u}_pu_p=u^\dagger{}_p\gamma_0u_p=1.$$

The terms in Eq. (2) proportional to $\langle p|p'\rangle$ and in Eq. (5) proportional to $\langle q|q'\rangle$ have been inserted for completeness. They affect only forward scattering in cancelling out vacuum fluctuations.[7] We shall call the representation given in Eq. (2) the "N representation" and that of Eq. (5) the "M representation."

We shall go immediately to forward scattering in both representations. In the N representation we choose the Lorentz frame which is at rest with respect to the nucleon while in the M representation we consider the frame at rest with respect to the meson. Accordingly, Eqs. (2) and (5) define two functions $T_+(\omega)$ and $T_+(E)$, respectively, where $T_+(\omega)$ is the T_+ obtained from Eq. (2) by putting $\mathbf{p}$ and $\mathbf{p}'$ to zero and $T_+(E)$ is the T_+ obtained from Eq. (5) when $\mathbf{q}$ and $\mathbf{q}'$ are set to zero. These two functions are related to each other by a simple Lorentz transformation. Since pq is an invariant one has $E/m=\omega/\mu$ where E is the nucleon energy in the meson rest frame and ω is the meson energy in the nucleon rest frame. Thus $T_+(E)=T_+(\omega)$.

We shall first prove the theorem by making use of a specific interaction Lagrangian involving only nucleon and π-meson interactions. Later we will discuss the generalizations necessary to include hyperon and heavy-meson interactions. According to pseudoscalar meson theory, the interaction Lagrangian density is given by

$$L(x)=g\bar{\psi}\gamma_5\tau_i\psi\phi_i+(\delta m)\bar{\psi}\psi+\tfrac{1}{2}(\delta\mu)^2\phi_i\phi_i-\tfrac{1}{4}\lambda(\phi_i\phi_i)^2, \tag{8}$$

where τ_i are the usual isotopic spin matrices and δm and $\delta\mu$ are the nucleon and meson mass renormalizations respectively. This leads to the current operators

$$j(x)=\sqrt{2}g\bar{\psi}\gamma_5\tau_-\psi+(\delta\mu)^2\phi-\lambda\phi(\phi_i\phi_i), \tag{9}$$

$$J(x)=g\gamma_5\tau_i\psi\phi_i+(\delta m)\psi, \tag{10}$$

where

$$\phi=\frac{1}{\sqrt{2}}(\phi_1-i\phi_2),\quad \tau_-=\tfrac{1}{2}(\tau_1-i\tau_2). \tag{11}$$

Proceeding in the manner of Goldberger,[1] we break T_+ into its dispersive and absorptive parts by writing $\theta(x_0)=\frac{1}{2}[1+\epsilon(x_0)]$ where $\epsilon(x_0)$ is the step function, i.e., $\epsilon(x_0)$ equals plus or minus one according as x_0 is greater or less than zero. Thus in the N representation one defines

$$\begin{aligned}D_+(\omega)=\tfrac{1}{2}i\bigg[&\int e^{-iqx}\epsilon(x_0)\langle m|[j(x),j^\dagger(0)]|m\rangle d^4x\\ &-2\int e^{-iqx}\langle m|[\phi(x),j^\dagger(0)]|m\rangle\delta(x_0)d^4x\bigg],\end{aligned} \tag{12}$$

and

$$A_+(\omega)=\tfrac{1}{2}\int e^{-iqx}\langle m|[j(x),j^\dagger(0)]|m\rangle d^4x, \tag{13}$$

where $|m\rangle$ is a state of one nucleon at rest and $T_+(\omega)=D_+(\omega)+iA_+(\omega)$. Similarly, in the M representation one has

$$\begin{aligned}\frac{D_+(E)}{2\mu}=\tfrac{1}{2}i\bigg[&\int e^{-ipx}\epsilon(x_0)\bar{u}_p\langle\mu|\{J(x),\bar{J}(0)\}|\mu\rangle u_pd^4x\\ &-2i\int e^{-ipx}\bar{u}_p\gamma_0\langle\mu|\{\psi(x),\bar{J}(0)\}|\mu\rangle u_p\delta(x_0)d^4x\bigg],\end{aligned} \tag{14}$$

$$\frac{A_+(E)}{2\mu}=\tfrac{1}{2}\int e^{-ipx}\bar{u}_p\langle\mu|\{J(x),\bar{J}(0)\}|\mu\rangle u_pd^4x. \tag{15}$$

Here $|\mu\rangle$ is a state of one meson at rest and again $T_+(E)=D_+(E)+iA_+(E)$. In Eqs. (12)–(15) it is understood that the required terms proportional to $\langle p|p'\rangle$ or $\langle q|q'\rangle$ are to be subtracted from the structures explicitly written down.

[6] This representation has also been used by Symanzik (reference 1) but with scalar nucleons.

[7] See footnote 4 of Goldberger's paper in reference 1.

Owing to the presence of the factor $\delta(x_0)$, the second integral in Eqs. (12) and (14) may be explicitly evaluated since only equal-time commutators (and anticommutators) appear. From Eq. (10), one sees that $\{\psi(\mathbf{r},0),\bar{J}(0)\}$ is the sum of a c-number and a term linear in ϕ_i. The c-number cancels out when one does the appropriate subtraction of the $\langle q|q'\rangle$ term, while $\langle\mu|\phi_i(0)|\mu\rangle$ is zero by Furry's theorem. Thus one has

$$\frac{D_+(E)}{2\mu}=\tfrac{1}{2}i\int e^{-ipx}\epsilon(x_0)\langle\mu|\{J(x),\bar{J}(0)\}|\mu\rangle u_p d^4x. \quad (16)$$

On the other hand, the term proportional to λ in $j(x)$ as defined in Eq. (9) gives a contribution to $[\phi(\mathbf{r},0),j^\dagger(0)]$ which is not a c-number, and leads to a term of the form

$$\lambda[\langle m|\phi_i^2(0)+2\phi(0)\phi^\dagger(0)|m\rangle -\langle 0|\phi_i^2(0)+2\phi(0)\phi^\dagger(0)|0\rangle]\delta^3(\mathbf{r}). \quad (17)$$

Thus the second integral in Eq. (12) contributes a constant to $D_+(\omega)$ proportional to λ:

$$D_+(\omega)=\tfrac{1}{2}i\int e^{-iqx}\epsilon(x_0)\langle m|[j(x),j^\dagger(0)]|m\rangle d^4x+\lambda C. \quad (18)$$

The theorem that we wish to prove is that

$$\lim_{\omega\to\infty} A_+(\omega)>0, \quad (19)$$

or

$$\lim_{E\to\infty} A_+(E)>0, \quad (20)$$

[Eqs. (19) and (20) being equivalent statements]. In order to establish the validity of Eqs. (19) and (20), we first assume the contrary, i.e., that

$$\lim_{\omega\to\infty} A_+(\omega)=0. \quad (21)$$

Under the assumption of Eq. (21), the principal-value integral

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{A_+(\omega')}{\omega'-\omega}d\omega', \quad (22)$$

exists. Using now Goldberger's approach[1] to the derivation of the more conventional dispersion relations and noting from the definition of $A_+(\omega)$ that

$$A_+(-\omega)=-A_-(\omega), \quad (23)$$

where $A_-(\omega)$ refers to π^- scattering, one easily obtains the relation

$$D_+(\omega)=\frac{1}{\pi}P\int_0^{\infty}\left[\frac{A_+(\omega')}{\omega'-\omega}+\frac{A_-(\omega')}{\omega'+\omega}\right]d\omega'+\lambda C. \quad (24)$$

On the other hand, from the assumption

$$\lim_{E\to\infty} A_+(E)=0, \quad (25)$$

the integral

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{A_+(E')}{E'-E}dE' \quad (26)$$

exists and by a similar analysis leads to the dispersion relation

$$D_+(E)=\frac{1}{\pi}P\int_0^{\infty}\left[\frac{A_+(E')}{E'-E}+\frac{A_-(E')}{E'+E}\right]dE'. \quad (27)$$

In Eq. (27), $A_-(E')$ is the absorptive part of the π^+-antiproton scattering amplitude. From charge conjugation invariance, this is equal to the π^--proton amplitude.[8]

We are here considering forward scattering, and $D(\omega)$ and $A(\omega)$ are the real and imaginary parts of $T(\omega)$. Similarly, $D(E)$ and $A(E)$ are the real and imaginary parts of $T(E)$. Further, since the Lorentz transformation relating the frames used in the N and M representation show that $T(E)$ and $T(\omega)$ are equal, we see that Eqs. (27) and (24) are inconsistent if $\lambda\neq 0$. Under this condition then, Eqs. (21) and (25) are impossible and thus Eqs. (19) and (20) must hold.[9]

From the "optical theorem" one may relate the absorptive part of the forward scattering amplitude to the cross section according to

$$\sigma_+(\omega)=A_+(\omega)/|\mathbf{q}|. \quad (28)$$

Here $\sigma_+(\omega)$ is the total (elastic plus inelastic) π^+-proton cross section. Hence Eq. (19) may be restated to read

$$\lim_{\omega\to\infty}\omega\sigma(\omega)>0. \quad (29)$$

In deriving Eq. (24), use was made of the causality condition

$$[j(x),j^\dagger(0)]=0 \quad \text{for } x^2>0, \quad (30)$$

while to obtain Eq. (27) one must assume

$$\{J(x),\bar{J}(0)\}=0 \quad \text{for } x^2>0. \quad (31)$$

Equation (31) is an extended causality condition and is certainly a reasonable one for fermion fields $J(x)$ in a local theory.

III. M REPRESENTATION

In this section we briefly discuss some other results that may be obtained by making use of the M representation of the scattering amplitude. First, it is of

[8] To establish Eq. (27) one must of course show that $A_+(-E)=-A_-(E)$. In evaluating $A_+(-E)$, E is replaced by $-E$ in Eq. (15) both in the exponential and in the spinor u_p. It should be noted that $u_p(-E)$ is a negative-energy spinor but normalized in the "wrong way," i.e., $\bar{u}_p(-E)u_p(-E)=+1$, while the negative-energy spinors v_p are normalized such that $\bar{v}_pv_p=-1$. The contribution to the integral in Eq. (24) from the region $0\leq\omega'\leq\mu$ and in Eq. (27) in the region $0\leq E'\leq m$ can be evaluated by the methods of Goldberger, Miyazawa, and Oehme.[1] Both contributions lead to the same result.

[9] It is interesting to note that neither Eq. (24) nor Eq. (27) reproduces perturbation theory. It is necessary to write down the relation which contains a second denominator under the integral in order to reproduce perturbation theory.

course possible to derive all the conventional dispersion relations in this representation. Thus, if one makes the usual assumption that the cross sections approach a constant limit in the high-energy region, one can form the integral

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{A(E')}{(E'-E)(E'-E_0)}dE', \tag{32}$$

the extra denominator being required to insure convergence. In this way one obtains the same results as Goldberger *et al.*[1] The results are now consistent with that obtained in the N representation since the extra denominator produces the structure $D(\omega)-D(\omega_0)$ and the λC term cancels out in the usual representation.

It is possible, however, in the M representation to obtain other dispersion-type relations, not between the real and the imaginary parts of the scattering amplitude, but between other quantities which are closely related. Thus, let us define a quantity, $\mathcal{T}_+$, which is the same as T_+ defined by Eq. (5) but with negative-energy spinors v substituted for u:

$$(4\omega\omega')^{-\frac{1}{2}}\mathcal{T}_+=i\int e^{-ipx}\theta(x_0)\bar{v}_p\langle q|\{J(x),\bar{J}(0)\}|q'\rangle v_p d^4x. \tag{33}$$

$\mathcal{T}_+$ is not a scattering amplitude,[10] but may be obtained from the meson-nucleon propagator. However, one can still break $\mathcal{T}_+$ into its "dispersive" and "absorptive" part $\mathfrak{D}_+$ and $\mathfrak{A}_+$ by writing $\theta(x_0)=\frac{1}{2}[1+\epsilon(x_0)]$ and derive "dispersion" relations based on causal restrictions. These relations are very similar to the usual ones between D_+ and A_+. Similarly, one can obtain relations for quantities where one of the spinors in Eq. (5) is a positive-energy spinor and one a negative-energy spinor. Thus it is possible to obtain other restrictions on quantities such as propagators which one would calculate in a Lagrangian approach.

IV. DISCUSSION AND CONCLUSIONS

Although the proof of the theorem on the high-energy behavior of $\sigma(\omega)$ was given for a specific Lagrangian, one can see that a much more general Lagrangian could have been assumed. The main point in the proof was that $D(\omega)$ as defined in Eq. (12) contained an extra term proportional to the direct meson-meson interaction constant, λ, whereas $D(E)$ as defined in Eq. (14) did not contain such a term. One could add to the Lagrangian terms containing the hyperon and heavy-meson fields without affecting the result. Thus terms such as $\bar{\psi}_N\psi_\Lambda\phi_K^\dagger$ and $\bar{\psi}_\Sigma T_i\psi_\Sigma\phi_i$ could be included where N, Λ, Σ, and K refer to the nucleon, Λ particle, Σ particle and K meson respectively and T_i are the isotopic spin-one matrices. On the other hand, a nonlinear interaction such as $(\bar{\psi}\gamma_5\tau_i\psi)^3\phi_i$ would lead to an extra term in Eq. (16), and thus certain nonlinear interactions would make the proof less clear-cut. An interaction structure of the form $\lambda'\phi_K^\dagger\phi_K\phi_i\phi_i$ would lead to an additional term in Eq. (24) of the form $\lambda'C'$, where C' is a constant. The theorem would still be valid, provided $\lambda C+\lambda'C'\neq 0$. For the usual pseudoscalar Lagrangian it is perhaps possible that $\lambda\equiv 0$. This does not seem likely, however, if we think of our theory as being simply the limit of a theory with a cutoff. In the above derivation we have made more specific use of the Lagrangian than is usually done, in that we have assumed the condition of Eq. (31) in addition to the usual causality condition (30).

Symanzik,[11] using the so-called tangent approximation, has shown that

$$\lim_{\omega\to\infty}\left(\frac{\sigma(\omega)}{\omega}\right)=0. \tag{34}$$

Thus this result and the one derived above, taken together, say that the total cross section must increase less rapidly than ω but more rapidly than ω^{-1} at high energies. A cross section which approaches a constant is of course consistent with both results.

As a corollary to our theorem it follows, of course, that in one of the dispersion relations [say, the one for $D_+(\omega)+D_-(\omega)$], at least one extra denominator must occur. This introduces the scattering amplitudes at the reference energy ω_0. That is, $D_+(\omega_0)+D_-(\omega_0)$ becomes an extra parameter in the dispersion relations. Upon choosing $\omega_0=\mu$, the dispersion relations will involve the following parameters: the coupling constant g^2, and the sum $\alpha_1+2\alpha_3$ of the zero-energy s-wave phase shifts.

In conclusion, it should be mentioned that the derivation of the cross-section theorem given here follows the methods of Goldberger,[1] and hence is subject to the same objections as to rigorous justification as Goldberger's original derivation of the more conventional dispersion relations. A rigorous derivation of the theorem using the procedures of Bogoliubov[12] seems difficult because of the fact that Bogoliubov's approach utilizes the total scattering amplitude, $T(\omega)$, while the theorem as derived here seems to depend upon breaking $T(\omega)$ into its dispersive and absorptive parts.[13]

We should like to thank Dr. R. Oppenheimer for hospitality shown us at the Institute for Advanced Study where part of this work was done.

[10] At first sight it might appear that $\mathcal{T}_+$ is the amplitude for π^+-antiproton scattering, but that quantity is proportional to $i\int e^{-ipx}\theta(x_0)\bar{v}_p\langle q|\{J(0),\bar{J}(x)\}|q'\rangle v_{p'}d^4x$.

[11] K. Symanzik, Nuovo cimento **5**, 659 (1957).

[12] Bogoliubov, Medvedev, and Polivanov, "Problems of the Theory of Dispersion Relations," lecture notes (unpublished).

[13] We should like to thank R. Karplus for pointing out this difficulty to us.