Many-Body Problem in Quantum Field Theory

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A formalism for the covariant description of scattering processes involving composite particles is derived from first principles. Its basic ingredients are the Green's functions of quantum field theory combined with a physically sound formulation of the adiabatic hypothesis. Explicit expressions are presented for various examples of nucleon-nucleus and boson-nucleus collisions. It is shown that the impulse approximation can be formulated for such problems. A general method of obtaining electromagnetic moments of nuclei, based on the scattering formalism, is also derived. The normalization condition for covariant amplitudes is discussed and its application to bound-state problems reviewed. In particular, a method of carrying out perturbation theory for the discrete spectrum is suggested.

I. INTRODUCTION

MOST recent studies of local field theories have employed the formulation in the Heisenberg representation. Almost all aspects of the problem have received some attention in the literature. For example, the energies of bound states should in principle be obtainable from the Bethe-Salpeter equation and suitable generalizations,[1] and indeed at least one idealized case has been worked out in full detail.[2] Again, the definition of the S-matrix in the Heisenberg representation for processes involving only elementary particles has been studied extensively.[3] Perhaps the least amount of attention has been accorded the problems of scattering involving composite particles, though even here a number of aspects have been treated.[4] There remain, nevertheless, several issues outstanding in connection with the problem of composite particles which require clarification, particularly with regard to the formulation of the adiabatic hypothesis for scattering problems, the orthonormalization properties of bound states and the proper description of charge-current distributions in bound states. In addition to a thorough discussion of these matters, the account which follows contains several novel elements such as the introduction of the impulse approximation[5] for covariant problems, a method for obtaining the electromagnetic moments of nuclei, and a suggested form of perturbation theory for bound states when the unperturbed interaction is noninstantaneous.[6] The original intention of the authors in undertaking the investigation of the many-body problem was to achieve a set of starting formulas for certain two-nucleon reaction processes (such as pion-deuteron scattering and photodisintegration of the deuteron) which have hitherto been treated only by noncovariant techniques. It is intended to develop these and other applications in a subsequent publication.

For the sake of definiteness we shall consider, unless otherwise specified, two interacting renormalized fields[7]; the nucleon (fermion) field will be represented by a Heisenberg operator[8] $\psi(x)$, the meson (boson) field by a corresponding operator $\phi(\xi)$. We commence our study with a survey of the premises underlying the development.

A. The physical significance of the field operators is provided by their algebraic properties *vis a vis* the constants of the motion, of which there are a full complement besides the generators of the space-time transformations, such as charge and nucleon number. The latter, for instance, is defined by the normal product[4] of operators,

$$N=\int :\ \bar{\psi}(x)\gamma_\mu\psi(x):\ d\sigma_\mu, \tag{1}$$

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[2] G. C. Wick, Phys. Rev. **96**, 1124 (1954); R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954); F. L. Scarf, Phys. Rev. **100**, 912 (1953); D. A. Geffen and F. L. Scarf, Phys. Rev. **101**, 1829 (1956); R. E. Cutkosky and G. C. Wick, Phys. Rev. **101**, 1830 (1956); see also J. S. Goldstein, Phys. Rev. **91**, 1516 (1953); H. S. Green, Phys. Rev. **97**, 540 (1955).

[3] C. N. Yank and D. Feldman, Phys. Rev. **79**, 972 (1950); G. Källen, Arkiv. Fysik **2**, 187 (1951); W. Glaser and W. Zimmermann, Z. Physik **134**, 346 (1953); W. Zimmermann, Z. Physik **135**, 473 (1953); E. Freese, Z. Naturforsch. **8a**, 776 (1953); Lehmann, Symanzik, and Zimmermann, Nuovo cimento **1**, 205 (1955); R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 12 (1955); F. E. Low, Phys. Rev. **97**, 1392 (1955); M. L. Goldberger, Phys. Rev. **97**, 508 (1955); Y. Nambu, Phys. Rev. **98**, 803 (1955); S. S. Schweber, Nuovo cimento **2**, 397 (1955); A. Klein, Phys. Rev. **102**, 913 (1956); H. Ekstein, Nuovo cimento **4**, 1017 (1956).

[4] K. Nishijima, Progr. Theoret. Phys. (Japan) **10**, 549 (1953); **12**, 279 (1954); **14**, 175 (1955); S. Mandelstam, Proc. Roy. Soc. (London) **233**, 248 (1955); A. Klein, Progr. Theoret. Phys. (Japan) **14**, 580 (1956); see also R. J. Eden, Proc. Roy. Soc. (London) **219**, 109 (1953). Many of the germinal ideas of this paper are contained in the papers of Nishijima and Mandelstam.

[5] G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952).

[6] For an instantaneous unperturbed interaction, perturbation theory was first formulated by E. E. Salpeter, Phys. Rev. **87**, 328 (1952).

[7] We shall be dealing throughout with closed equations, which can presumably be renormalized by a scale transformation. We assume that this operation has been carried out on all quantities in the theory, i.e., that we are dealing from the start with a finite theory.

[8] Isotopic and spinor indices as well as charge and spin variables will be suppressed throughout, the space-time, or momentum label also including these variables.

where $\bar{\psi}(x)=\psi^\dagger(x)\gamma_0=\psi^\dagger(x)\beta$, and

$$[\psi(x),N]=\psi(x),\quad [\psi^\dagger(x),N]=-\psi^\dagger(x). \qquad (2)$$

B. There exist two alternative complete sets of states $|\alpha^{(+)}\rangle$ and $|\alpha^{(-)}\rangle$ which are eigenstates of total energy-momentum and of a complete compatible set of constants of the motion. The two sets are differentiated in the usual manner—the superscripts distinguishing the outgoing-wave boundary condition (+) from the incoming one (−). Excepting only the vacuum, the states of both categories possess energies, momenta, and perhaps other quantum numbers lying in a continuum. Wave packets constructed by a superposition of neighboring outgoing-wave (ingoing-wave) states represent, for times in the remote past (remote future) assemblies of localized isolated particles and thus correspond to experimentally realizable situations. The probability amplitude that an assembly α of particles localized in the remote past will be observed as a new assembly β in the remote future is then obtained by applying the packeting operation to the matrix element $S_{\beta\alpha}$ of the scattering matrix,

$$S_{\beta\alpha}=\langle\beta^{(-)}|\alpha^{(+)}\rangle. \qquad (3)$$

In this description, the notion of "particle" applies equally well to single quanta and to stable bound states. The observation that in scattering problems of physical interest, the interparticle interactions are ineffective at remote times motivates the so-called "adiabatic hypothesis"; specifically, that no error is introduced into the treatment of a physically realizable scattering process by a formulation of the theory in which the interactions among the particles of interest are "turned off" at remote times—provided, of course, that the turning-off procedure is sufficiently gradual that it does not, of itself, create disturbances.

The presence of a decay mechanism for excited bound states replaces the discrete level structure defined by the binding interaction by a continuum and true energy eigenstates for such situations include components possessing one or more radiation quanta. Here, a distinction between outgoing-wave and ingoing-wave states is again of importance. A wave packet of outgoing-wave states describes the decay, in the course of time, of a state which initially consisted of a bound excited nucleus only. In contradistinction to the scattering case, where the energy widths of the packets may be made arbitrarily small at the expense of localization, the packeted bound state has an irreducible width associated with the strength of the decay mechanism. If the latter is weak compared to the binding, the density of states will be appreciable only in the neighborhood of the discrete energy levels and only at these levels may narrow packets be constructed at all.

Finally we may note that S is to be so normalized as to leave the one-particle states, $|\alpha_1\rangle$, (as well as the vacuum) invariant,

$$S|\alpha_1\rangle=|\alpha_1\rangle. \qquad (4)$$

C. The theory will be developed with the aid of the renormalized Green's functions[9]

$$\begin{aligned}G[n,\nu]&=G(x_1,\cdots x_n;\,x_1',\cdots x_n';\,\xi_1,\cdots\xi_\nu)\\&=\langle 1\cdots n|G[n;\,\xi_1,\cdots\xi_\nu]|1'\cdots n'\rangle\\&=i^n i^{[\frac{1}{2}\nu]}\langle 0|T(\psi(x_1)\cdots\psi(x_n)\bar{\psi}(x_n')\cdots\psi(x_1')\\&\qquad\times\phi(\xi_1)\cdots\phi(\xi_\nu))|0\rangle, \qquad (5)\end{aligned}$$

where $[\frac{1}{2}\nu]$ signifies the enclosed integer if ν is even, the nearest (smaller) integer if ν is odd, and T is the chronological ordering symbol used by Wick.[10]

For $\nu=0$, we are dealing with the n-nucleon propagation function, $G[n]$, which satisfies a differentio-integral equation of the symbolic form[11]

$$\{G[n-1]^{-1}G_n^{-1}-1[n]I([n-1],\,n)\}G[n]=1[n]. \qquad (6)$$

Here G_n is the single-nucleon Green's function for the nth particle, $I([n-1],\,n)$ the interaction between the nth particle and the $n-1$ others, and $1[n]$ the antisymmetric δ function in n variables, i.e., the determinantal form

$$1[n]=\sum\epsilon(x_{s_1}'\cdots x_{s_n})\delta(x_1-x_{s_1}')\cdots\delta(x_n-x_{s_n}'), \qquad (7)$$

with ϵ the alternating symbol in n variables. The form of Eq. (6) is obtained by induction from the two-nucleon case,

$$\{G_1^{-1}G_2^{-1}-1[2]I_{12}\}G[2]=1[2], \qquad (8)$$

and

$$\begin{aligned}I([n-1],\,n)&=\sum_{j=1}^{n-1}(I_{nj}G_j)G_1^{-1}\cdots G_{n-1}^{-1}\\&\quad+\sum_{i\neq j=1}^{n-1}(I_{nji}G_iG_j)G_1^{-1}\cdots G_{n-1}^{-1}+\cdots+I_{12\cdots n}, \qquad (9)\end{aligned}$$

comprising interactions of at least two particles and up to n particles at a time.

An equivalent relation employing the single-nucleon Green's function is

$$\{G_1^{-1}\cdots G_n^{-1}-1[n]I[n]\}G[n]=1[n], \qquad (10)$$

where $I[n]$ is the total interaction among all groups of nucleons:

$$\begin{aligned}I[n]&=\sum_{i\neq j}(I_{ij}G_iG_j)G_1^{-1}\cdots G_n^{-1}\\&\quad+\sum_{i\neq j\neq k}(I_{ijk}G_iG_jG_k)G_1^{-1}\cdots G_n^{-1}+\cdots+I_{12\cdots n}. \qquad (11)\end{aligned}$$

When the interaction of the nth particle with the other particles is neglected, the n-nucleon Green's

[9] R. P. Feynman, Phys. Rev. **80**, 440 (1950); J. Schwinger, reference 1 and unpublished lectures; E. Freese, Nuovo cimento **11**, 312 (1954); P. T. Matthews and A. Salam, Proc. Roy. Soc. (London) **221**, 128 (1953); F. Coester, Phys. Rev. **95**, 1318 (1954); H. Umezawa and A. Visconti, Nuovo cimento **1**, 1079 (1955); Y. Nambu, Phys. Rev. **100**, 294 (1955); **101**, 459 (1956); J. M. Jauch, Helv. Phys. Acta. **29**, 287 (1956).

[10] G. C. Wick, Phys. Rev. **80**, 268 (1950).

[11] Equation (6) has been derived in unpublished lectures by J. Schwinger. See also Schwinger, reference 1 and Umezawa and Visconti, reference 9.

function assumes the form

$$G(x_1,\cdots x_n;x_1',\cdots x_n')\to G[n-1]G_n1[n]$$
$$=\sum\epsilon(x_{s1}',\cdots x_{sn}')G(x_1,\cdots x_{n-1},x_{s1}',\cdots x_{sn-1}')\times G(x_n,x_{sn}'),\quad(12)$$

whereas, if all internucleon interactions are neglected, $G[n]$ reduces to $G[n]^{(0)}$,

$$G[n]^{(0)}=\sum\epsilon(x_{s1}',\cdots x_{sn}')G(x_1,x_{s1}')\cdots G(x_n,x_{sn}').\quad(13)$$

Equation (10) may also be written in the integral form

$$G[n]=G[n]^{(0)}+G[n]^{(0)}I[n]G[n].\quad(14)$$

In the construction (6)–(13), the detailed form of both the single-nucleon Green's functions as well as of the various interaction operators is assumed known, but need hardly concern us until we confront the problem of applications. We merely require the existence of equations of the given structure. The generalization to include bosons will be made at the apposite juncture.

To accompany the propagators $G[n]$, we require the associated amplitudes

$$\chi_\alpha^{(\pm)}[n_1,n_2]=\langle0|T(\psi(x_1)\cdots\psi(x_{n_1})\bar\psi(y_1)\cdots\bar\psi(y_{n_2}))|\alpha^{(\pm)}\rangle,$$
$$\bar\chi_\alpha^{(\pm)}[n_1,n_2]=\langle\alpha^{(\pm)}|T(\psi(y_{n_2})\cdots\psi(y_1)\bar\psi(x_{n_1})\cdots\bar\psi(x_1))|0\rangle,\quad(15)$$

which may be interpreted as probability amplitudes for the presence, in $|\alpha^{(\pm)}\rangle$, of n_1 nucleons and n_2 antinucleons. For simplicity, we shall confine ourselves to the treatment of (positive-energy) nucleon systems. With this restriction, we need only consider the nucleon amplitudes

$$\chi_\alpha^{(\pm)}[n]=\chi_\alpha^{(\pm)}(1\cdots n)=\langle0|T(\psi(1)\cdots\psi(n))|\alpha^{(\pm)}\rangle,$$
$$\bar\chi_\alpha^{(\pm)}[n]=\bar\chi_\alpha^{(\pm)}(1\cdots n)=\langle\alpha^{\pm}|T(\bar\psi(n)\cdots\bar\psi(1))|0\rangle.\quad(16)$$

For $x_{10},\cdots,x_{n0}>x_{10}',\cdots,x_{n0}'$, the inequality holding for every member of the unprimed set relative to the primed set, we can write, after introduction of either complete set $|\alpha\rangle$,

$$G(1\cdots n;1'\cdots n')=(i)^n\sum_\alpha\chi_\alpha(1\cdots n)\bar\chi_\alpha(1'\cdots n'),\quad(17)$$

the sum extending only over those states permitted by the conservation theorems. By means of the limiting procedure of Gell-Mann and Low,[1] the essential aspects of which are reviewed in Sec. II, one can then demonstrate that the $\chi_\alpha[n]$ satisfy the homogeneous equations

$$\{G[n-1]^{-1}G_n^{-1}-1[n]I([n-1],n)\}\chi_\alpha[n]=0,$$
$$\bar\chi_\alpha[n]\{G[n-1]^{-1}G_n^{-1}-I([n-1],n)1[n]\}=0.\quad(18)$$

Finally, we shall require a notation for the T symbol in the limiting instance when two or more times are equated. We shall write

$$\lim_{x_{i0}=t}T(\psi(1)\cdots\psi(n))=[\psi(1)\cdots\psi(n)]\quad(19)$$

to indicate the average over all possible modes of approach to equality.

D. The adiabatic hypothesis is adopted, but only in a restricted form which doesn't gainsay our physical experience.[12] Consider, for example, the matrix element $\chi_\alpha^{(+)}(1\cdots n)$. Let $|\alpha^{(+)}\rangle=|(A,a)^{(+)}\rangle$ signify a state which when properly packeted consists, in the remote past, of a localized stable nucleus of atomic number $n-1$, state $|A\rangle$ and a localized single nucleon, state $|a\rangle$. Anticipating application to a process in which the nucleon-nucleus interaction will be ineffective at very early times, we expect that a representation of $\chi_\alpha^{(+)}$ in terms of χ_A and χ_a, the corresponding steady-state amplitudes for $|A\rangle$ and $|a\rangle$, obtained by disregarding the nucleon-nucleus interaction will be adequate for such times. In this approximation, by choosing $x_{10},\cdots,x_{n0}>x_{10}',\cdots x_{n0}'$ and rewriting Eq. (12) with the aid of Eq. (17), we obtain for the partial sum over states of the above-specified character

$$\sum_\beta\chi_\beta(1\cdots n)\bar\chi_\beta(1'\cdots n')=\sum\epsilon(x_{s1}',\cdots x_{sn}')\times\sum_{B,b}\chi_B(1\cdots n-1)\bar\chi_B(s_1\cdots s_{n-1})\chi_b(x_n)\bar\chi_b(x_{sn})$$
$$\equiv\sum_{B,b}\|\chi_B(1\cdots n-1)\chi_b(n)\|\|\bar\chi_B(1\cdots n-1)\bar\chi_b(n)\|,\quad(20)$$

where, for example,

$$\|\chi_B(1\cdots n-1)\chi_b(n)\|=n^{-\frac12}[\chi_B(1\cdots n-1)\chi_b(n)-\chi_B(1\cdots n-2,n)\chi_b(n-1)+\cdots+(-1)^{n-1}\chi_B(2\cdots n)\chi_b(1)].\quad(21)$$

Thus, in terms of the operation

$$[\psi(1)\cdots\psi(n)]^{\text{in}}=\lim_{t\to-\infty}[\psi(1)\cdots\psi(n)],\quad(22)$$

and the definition

$$[\chi_\alpha^{(+)}(1\cdots n)]^{\text{in}}=\langle0|[\psi(1)\cdots\psi(n)]^{\text{in}}|\alpha^{(+)}\rangle,\quad(23)$$

we infer from (20)

$$[\chi_\alpha^{(+)}(1\cdots n)]^{\text{in}}=\|\chi_A(1\cdots n-1)\chi_a(n)\|.\quad(24)$$

At the other extreme, if $|\alpha^{(+)}\rangle=|(a_1\cdots a_n)^{(+)}\rangle$, that is, consists asymptotically of n single nucleons, then the right-hand side of Eq. (24) would be replaced by the usual determinantal form in $\chi_{a_i}(j)$,

$$[\chi_\alpha^{(+)}(1\cdots n)]^{\text{in}}=(n!)^{-\frac12}\sum\epsilon(j_1\cdots j_n)\chi_{a_1}(j_1)\cdots\chi_{a_n}(j_n)\equiv\|\chi_{a_1}(j_1)\cdots\chi_{a_n}(j_n)\|.\quad(25)$$

Equations (24) and (25) are illustrative of the assumption that we may effectively turn off the interaction between particles, but not the self-interaction or the interactions which produce bound states. In Appendix A, it is shown how our adiabatic hypothesis

[12] See the paper of Haag, reference 3 and those of Nishijima and Klein, reference 4, for further discussion of the underlying ideas.

follows directly from an asymptotic condition on the field operators themselves.

Similarly with respect to the states $|\alpha^{(-)}\rangle$, we can define a limiting procedure "out" exemplified by

$$[\psi(1)\cdots\psi(n)]^{\text{out}}=\lim_{t\to\infty}[\psi(1)\cdots\psi(n)], \qquad (26)$$

with its accompanying form of the adiabatic hypothesis.

By means of assumptions A–D, we shall proceed in the next two sections to the construction of formulas for the S-matrix, for most cases of interest. In Sec. II we deal with nucleon-nucleus collisions, and in Sec. III with boson-nucleus collisions. It is shown in Sec. IV that the impulse approximation can be formulated simply for the various transition amplitudes under discussion. In Sec. V the normalization condition for the covariant amplitudes which enter these expressions is derived, and the method for obtaining matrix elements of an arbitrary observable briefly discussed. Section VI takes up the question of finding the bound states of the systems considered. Here, the orthonormalization problem, the application of perturbation theory, and the relation to the contents of Sec. V are given consideration. Finally, Appendix A discusses the foundations of the adiabatic hypothesis, Appendix B establishes the equivalence of alternative forms for certain amplitudes for inelastic processes, and Appendix C contains further discussion of orthonormalization conditions.

II. NUCLEON-NUCLEUS COLLISIONS

We proceed upon the basic observation that the known structure of the propagation function $G[n,\nu]$ permits the identification of a certain subset of the matrix elements of S. Considering, for instance, the Green's function $G[n]$, we first define the operation $L(t,t')$ as follows:

$$L(t,t')G[n]=i^n\langle 0|[\psi(1)\cdots\psi(n)]^{\text{out}}\times[\bar\psi(n')\cdots\bar\psi(1')]^{\text{in}}|0\rangle. \qquad (27)$$

Upon introduction of the complete sets of states $|\alpha^{(+)}\rangle$ and $\langle\beta^{(-)}|$ and subsequent utilization of the relation

$$S_{\beta\alpha}=\langle\beta^{(-)}|\alpha^{(+)}\rangle, \qquad (28)$$

we have immediately

$$L(t,t')G[n]=i^n\sum_{\beta,\alpha}\langle 0|[\psi(1)\cdots\psi(n)]^{\text{out}}|\beta^{(-)}\rangle \times S_{\beta\alpha}\langle\alpha^{(+)}|[\bar\psi(n')\cdots\bar\psi(1')]^{\text{in}}|0\rangle. \qquad (29)$$

As will be seen in detail below, $S_{\beta\alpha}$ is then recognized from the form of $G[n]$ when the asymptotic relations satisfied by the amplitudes of Eq. (29) are noted.

In the work to follow, we shall frequently encounter equations with the structure

$$\sum_{\beta,\alpha}\chi_\beta(1\cdots n)A_{\beta\alpha}\bar\chi_\alpha(1',\cdots n') =\sum_{\beta,\alpha}\chi_\beta(1\cdots n)B_{\beta\alpha}\bar\chi_\alpha(1'\cdots n'), \qquad (30)$$

which hold when the common value t of the time components x_{i0} lies in the remote future and the common value t' of the components x_{i0}' lies in the remote past. From (2.3), we desire to infer that

$$A_{\beta\alpha}=B_{\beta\alpha},$$

for each pair (β,α).

If the amplitudes are antisymmetrized products of one-nucleon amplitudes, one need only apply the orthogonality relations

$$\int d\mathbf{x}_1\cdots d\mathbf{x}_n\bar\chi_\beta(1\cdots n)\beta^{(1)}\cdots\beta^{(n)}\chi_{\beta'}(1\cdots n)=\delta_{\beta,\beta'}. \qquad (31)$$

When (31) is not applicable, one may nevertheless write (30) in the form

$$\sum e^{-iE_\beta t}\chi_\beta(\mathbf{x}_1\cdots\mathbf{x}_n)A_{\beta\alpha}\bar\chi_\alpha(\mathbf{x}_1'\cdots\mathbf{x}_n')e^{iE_\alpha t'} =\sum e^{-iE_\beta t}\chi_\beta(\mathbf{x}_1\cdots\mathbf{x}_n)B_{\beta\alpha}\bar\chi_\alpha(\mathbf{x}_1'\cdots\mathbf{x}_n')e^{iE_\alpha t'}. \qquad (32)$$

Exploiting the validity of (32) throughout the intervals $\infty>t>|T|$ and $-|T'|>t'>-\infty$ for sufficiently large $|T|$, $|T'|$, we may conclude that

$$\sum{}'\chi_\beta A_{\beta\alpha}\bar\chi_\alpha=\sum{}'\chi_\beta B_{\beta\alpha}\bar\chi_\alpha, \qquad (33)$$

where the restricted summation $\sum'$ extends over amplitudes of states $|\beta^{(-)}\rangle$ and $\langle\alpha^{(+)}|$ with common energies E_β and E_α respectively. Two states $|\beta_1^{(-)}\rangle$ and $|\beta_2^{(-)}\rangle$ with a common energy are now distinguished by different values of some constant of the motion. A consideration of the transformation generated by this constant of the motion suffices to demonstrate the orthogonality of the associated amplitudes in the sense of (31).

As a ready illustration, let us first consider the well-known case of nucleon-nucleon scattering. Here we make use of Eq. (8), or rather of the equivalent integral equation

$$G(1\,2;1'\,2')=G(1\,1')G(2\,2')-G(1\,2')G(2\,1') +\int[G(1\,1')G(2\,2')-G(1\,2'')G(2\,1'')] \times I(1''\,2'';1'''\,2''')G(1'''\,2''';1'\,2'). \qquad (34)$$

According to the definition in Eq. (27) and the adiabatic hypothesis,

$$\begin{aligned}L(t,t')&[G(1\,1')G(2\,2')-G(1\,2')G(2\,1')\\ &=i^2\sum_{p,p'}\chi_p(1)\chi_{p'}(2)[\bar\chi_p(1')\bar\chi_{p'}(2')-\bar\chi_p(2')\bar\chi_{p'}(1')]\\ &=i^2\sum_{p,p'}\|\chi_p(1)\chi_{p'}(2)\|\|\bar\chi_p(1')\bar\chi_{p'}(2')\|\\ &=i^2\sum_{p,p',p'',p'''}\langle 0|[\psi(1)\psi(2)]^{\text{out}}|(p,p')^{(-)}\rangle\delta(\mathbf{p}-\mathbf{p}'')\\ &\quad\times\delta(\mathbf{p}'-\mathbf{p}''')\langle(p'',p''')^{(+)}|[\bar\psi(2')\bar\psi(1')]^{\text{in}}|0\rangle, \qquad (35)\end{aligned}$$

exhibiting the unit term of the S-matrix. Applying the same procedure to the remainder of Eq. (34), we find

$$\langle pp'|S|p''p'''\rangle$$
$$=\tfrac{1}{2}[\delta(\mathbf{p}-\mathbf{p}'')\delta(\mathbf{p}'-\mathbf{p}''')-\delta(\mathbf{p}-\mathbf{p}''')\delta(\mathbf{p}'-\mathbf{p}'')]$$
$$+i^2\int \|\bar{\chi}_p(1)\bar{\chi}_{p'}(2)\|I(1\,2;1'\,2')\chi_{p''p'''}{}^{(+)}(1'\,2'). \quad (36)$$

An alternative version of Eq. (36) that will be useful for our later considerations can be obtained if we replace Eq. (34) by its symbolic algebraic solution

$$G[2]=[1-G_1G_2I_{12}]^{-1}G_1G_2$$
$$=G_1G_2+G_1G_2I_{12}[1-G_1G_2I_{12}]^{-1}G_1G_2, \quad (37)$$

where the antisymmetrization has been suppressed. From Eq. (37) we then conclude that

$$\langle pp'|(S-1)|p''p'''\rangle=i^2\int \|\bar{\chi}_p(1)\bar{\chi}_{p'}(2)\|$$
$$\times\langle 1\,2|I_{12}[1-G_1G_2I_{12}]^{-1}|1'\,2'\rangle$$
$$\times\|\chi_{p''}(1')\chi_{p'''}(2')\|. \quad (38)$$

As a second example we take up the inelastic scattering of a nucleon by a two-nucleon bound state which we call the deuteron. We simply record the integral equation for $G[3]$ in the form

$$G[3]=G_1G_2G_3+G_1G_2G_3I[3]G[3], \quad (39)$$

with

$$I[3]=I_{12}G_3^{-1}+I_{13}G_2^{-1}+G_{23}G_1^{-1}+I_{123}. \quad (40)$$

The matrix element in question can then be read off essentially from Eq. (39),

$$\langle k_1k_2k_3|S|kp\rangle=i^3\int \|\bar{\chi}^{k_1}(1)\bar{\chi}^{k_2}(2)\bar{\chi}^{k_3}(3)\|$$
$$\times\langle 1\,2\,3|I[3]|1'\,2'\,3'\rangle\chi^{kp(+)}(1'\,2'\,3'), \quad (41)$$

where the k_i designate the outgoing nucleons and k, p the incoming nucleon and deuteron respectively. The δ-function term is missing, of course, for the inelastic process. An alternative version of (41) would result from inserting for $\chi_{kp}{}^{(+)}(1\,2\,3)$ the symbolic solution of the inhomogeneous integral equation which it satisfies [see Eq. (47) below].

The reader should be cautioned against concluding, from an examination of Eqs. (40) and (41), that the latter is patent nonsense. One is tempted to this conclusion by the observation that the G_i^{-1} acting on the free-particle wave functions on the left apparently annihilate them and thus the contribution to the amplitude of two-body interactions is missing. In truth, however, the differentio-integral operators G_i^{-1} act to the *right* in (41), and since we are dealing with scattering states, an integration by parts is hardly justified. Alternative versions of the scattering matrix elements are available which eschew this pitfall. These have not been preferred for the purely formal study of this paper on the grounds that they do not lend themselves as readily to such conciseness and uniformity of expression as found in (41) and corresponding equations to follow.

For the elastic scattering we require a different version of (39). Noting, for example, that

$$G_1^{-1}G_2^{-1}-1_{12}I_{12}=G[2]^{-1}, \quad (42)$$

or indeed starting most simply from the differential equation (6), specialized to $n=3$, we have

$$G[3]=G[2]G_3+G[2]G_3I([2],3)G[3]. \quad (43)$$

From Eq. (43), we may first of all obtain a different version of Eq. (41):

$$\langle k_1k_2k_3|S|kp\rangle=i^3\int \|\bar{\chi}_{k_1k_2}{}^{(-)}(1\,2)\bar{\chi}^{k_3}(3)\|$$
$$\times\langle 1\,2\,3|I([2],3)|1'\,2'\,3'\rangle\chi_{kp}{}^{(+)}(1'\,2'\,3'), \quad (44)$$

whose equivalence to the former is established in Appendix B. For the elastic scattering, on the other hand, we find

$$\langle kp|S-1|k'p'\rangle=i^3\int \|\bar{\chi}_p(1\,2)\bar{\chi}_k(3)\|$$
$$\times\langle 1\,2\,3|I([2],3)|1'\,2'\,3'\rangle\chi_{kp}{}^{(+)}(1'\,2'\,3')$$
$$=i^3\int \|\bar{\chi}_p(1\,2)\bar{\chi}_k(3)\|$$
$$\times\langle 1\,2\,3|I([2],3)[1-G[2]G_3I([2],3)]^{-1}|1'\,2'\,3'\rangle$$
$$\times\|\chi_{p'}(1'\,2')\chi_{k'}(3')\|, \quad (45)$$

the second form following from the utilization of the symbolic solution of (43) [compare Eq. (37)],

$$G[3]=G[2]G_3+G[2]G_3$$
$$\times\{1-G[2]G_3I([2],3)\}^{-1}G[2]G_3, \quad (46)$$

or by inserting directly into the first form of (45) the corresponding solution for $\chi_{kp}{}^{(+)}(1.23)$

$$\chi_{kp}{}^{(+)}=\{1-G[2]G_3I([2],3)\}^{-1}\|\chi_p\chi_k\|. \quad (47)$$

The procedures that we have exposed for the simplest cases can be generalized in a straightforward manner to the situation involving an arbitrary number of nucleons. The elastic scattering of a nucleon by a nucleus of $n-1$ nucleons is described, for example, by the formulas

$$\langle kp|S-1|k'p'\rangle=i^n\int \|\bar{\chi}_p(1\cdots n-1)\chi_k(n)\|$$
$$\times\langle 1\cdots n|I([n-1],n)|1'\cdots n'\rangle\chi_{kp}{}^{(+)}(1'\cdots n')$$
$$=i^n\int \|\bar{\chi}_p(1\cdots n-1)\bar{\chi}_k(n)\|\langle 1\cdots n|I([n-1],n)$$
$$\times[1-G[n-1]G_nI([n-1],n)]^{-1}|1'\cdots n'\rangle$$
$$\times\|\chi_{p'}(1'\cdots n-1')\chi_{k'}(n')\|. \quad (48)$$

On the other hand if we were interested in a reaction in which the final state consisted of three fragments, a nucleon (k) and two nuclear fragments (in their ground states) of atomic weight p and $n-1-p$ (p and q), then in Eq. (48) we would merely make the replacement

$$\|\bar{\chi}_p(1\cdots n-1)\bar{\chi}_k(n)\| \to \|\bar{\chi}_{pq}^{(-)}(1\cdots n-1)\bar{\chi}_k(n)\|. \quad (49)$$

The forms (48) and (49) follow directly if one invokes Eq. (6) for the propagator $G[n]$. As in the the three-body case, different versions can be obtained for inelastic processes.

At least a word is in order about the possible utilization of these results. What is required is some knowledge of the covariant amplitudes of many-particle systems for scattering states, a subject on which the literature has been notoriously silent.[13] A possible approach which has proved useful in the nonrelativistic limit is the impulse approximation, the application of which necessitates knowledge of two-particle scattering matrices and amplitudes for bound states, requirements which one should be able to meet at least in approximate terms. The formulation is given in Sec. IV.

III. BOSON-NUCLEUS COLLISIONS

The treatment of such problems requires nothing novel in principle if we postulate suitable forms for the relevant propagators. Let us study the case of single-meson production in nucleon-nucleon collisions. By analogy with Eq. (29), we recognize the appropriate matrix element from the form

$$\begin{aligned} L(t,t')G(1\,2;1'\,2';\xi) &= i^2 \sum_{k,p,p',p'',p'''} \langle 0|[\psi(1)\psi(2)\phi(\xi)]^{\text{out}}|(k,p,p')^{(-)}\rangle \\ &\times\langle kpp'|S|p''p'''\rangle \\ &\times\langle (p'',p''')^{(+)}|[\bar{\psi}(2')\bar{\psi}(1')]^{\text{in}}|0\rangle, \end{aligned} \quad (50)$$

with

$$\langle 0|[\psi(1)\psi(2)\phi(\xi)]^{\text{out}}|(k,p,p')^{(-)}\rangle = \langle 0|[\psi(1)\psi(2)]^{\text{out}}|(p,p')^{(-)}\rangle\langle 0|\phi(\xi)|k\rangle, \quad (51)$$

and

$$\langle 0|\phi(\xi)|k\rangle = \chi_k(\xi). \quad (52)$$

For the application of (50)–(52), we suppose that $G(1\,2;1'\,2';\xi)$ can be reordered in the form

$$G(1\,2;1'\,2';\xi) = G[2;\xi] = \int G_1G_2\Delta(\xi,\xi')R([2],\xi')G_1G_2, \quad (53)$$

where

$$\Delta(\xi,\xi') = i\langle 0|T(\phi(\xi)\phi(\xi'))|0\rangle. \quad (54)$$

It then follows from (50)–(52) that

$$\langle pp'k|S|p''p'''\rangle = i^3\int \|\bar{\chi}_p(1)\bar{\chi}_{p'}(2)\bar{\chi}_k(\xi)\| \times\langle 1\,2|R([2],\xi)|1'\,2'\rangle\|\chi_{p''}(1')\chi_{p'''}(2')\|. \quad (55)$$

The method of obtaining $R([2],\xi)$ will be given below subsequent to our treatment of one additional example, that of boson-deuteron scattering. We take the Green's function in question to have the form

$$G[2;\xi,\xi'] = G[2]\Delta(\xi,\xi') + \int G[2]\Delta(\xi,\xi'')R([2];\xi''\xi''')\Delta(\xi''',\xi')G[2], \quad (56)$$

which readily yields the formula (elastic scattering)

$$\langle kp|(S-1)|k'p'\rangle = i^3\int \bar{\chi}_p(1\,2)\bar{\chi}_k(\xi)\langle 1\,2|R([2];\xi\xi')|1'\,2'\rangle \times\chi_{p'}(1'\,2')\chi_{k'}(\xi'), \quad (57)$$

with a corresponding expression for the inelastic situation.

The R matrix in Eq. (56), for instance, may be considered achieved by the solution of the equation

$$G[2;\xi\xi'] = G[2]\Delta(\xi\xi') + \int G[2]\Delta(\xi\xi'')I([2];\xi''\xi''')G[2,\xi'''\xi'], \quad (58)$$

where

$$I([2];\xi,\xi') = I_1(\xi,\xi') + I_2(\xi\xi') + I_{12}(\xi\xi') \quad (59)$$

is a sum of terms in which the meson interacts with one nucleon at a time (and is thus characteristic of scattering by a single nucleon) or with both in a nondecomposable manner.[14] Comparison of (56) and (58) demonstrates that

$$R = I\{1-G[2]\Delta I\}^{-1} \quad (60)$$

in analogy with the nucleon case.

A more general approach to the R functions is contained in the idea that the Green's functions $G[n,\nu]$ are all determined for a closed physical system once the $G[n]$ are given in the presence of an arbitrary external source.[15] The following formulas provide all the requisite tools. We suppose the external field to be characterized by a source function $J(\xi)$ coupled linearly to the boson field, described by an addition to the Lagrange density[16] of the form

$$\mathcal{L}'(x) = J(x)\phi(x). \quad (61)$$

[13] See however, N. Kemmer and A. Salam, Proc. Roy. Soc. (London) **230**, 266 (1955); S. Mandelstam, Proc. Roy. Soc. (London) **237**, 496 (1956).

[14] S. Deser and P. C. Martin, Phys. Rev. **90**, 1075 (1953) have studied the Green's function for the single nucleon-meson system.

[15] Compare the work of the authors mentioned in reference 11.

[16] Introduced into the work for the first time.

In the presence of $J(x)$ (assumed to vanish in the remote past and future, at which times one can define vacuum states), the vacuum matrix element of any operator Q is defined by the expression

$$\langle Q[J]\rangle = \frac{\langle 0(+\infty)|Q|0(-\infty)\rangle}{\langle 0(+\infty)|0(-\infty)\rangle} \quad \text{which} \to \langle 0|Q|0\rangle \tag{62}$$

as $J\to 0$. We have the basic formula[15]

$$\langle 0|T(Q\phi(\xi_1)\cdots\phi(\xi_\nu))|0\rangle = \prod_{i=1}^{\nu}[\langle\phi(\xi_i)-i\delta/\delta J(\xi_i)]\langle Q[J]\rangle|_{J=0}. \tag{63}$$

As a special case we recall that

$$\Delta(\xi,\xi') = \delta\langle\phi(\xi)\rangle/\delta J(\xi')|_{J=0}. \tag{64}$$

Applied to the propagators, Eq. (63) yields

$$G[n,\nu] = i^{[\frac{1}{2}\nu]}\prod_{i=1}^{\nu}[\langle\phi(\xi_i)\rangle - i\delta/\delta J(\xi_i)]G[n;J]|_{J=0}, \tag{65}$$

and thereupon, we have for our first special case (with $\langle 0|\phi(\xi)|0\rangle_{J=0}=0$)

$$\begin{aligned} G[2;\xi] &= -i(\delta/\delta J(\xi))G[2] \\ &= -i\Delta(\xi,\xi')G_1G_2 \\ &\quad\times[G_1^{-1}G_2^{-1}(\delta G[2]/\delta\langle\phi(\xi')\rangle)G_1^{-1}G_2^{-1}]G_1G_2, \end{aligned} \tag{66}$$

the limit $J=0$ being henceforth understood. Upon comparison with Eq. (53), we can thus write

$$R([2];\xi) = -iG_1^{-1}G_2^{-1}(\delta G[2]/\delta\langle\phi(\xi)\rangle)G_1^{-1}G_2^{-1}, \tag{67}$$

whereas a corresponding procedure yields for the case of boson-deuteron scattering,

$$R([2];\xi\xi') = G[2]^{-1}\{\delta^2G[2]/\delta\langle\phi(\xi)\rangle\delta\langle\phi(\xi')\rangle\}G[2]^{-1}. \tag{68}$$

Of course, we could have based the entire discussion of this section on Eq. (65).

IV. IMPULSE APPROXIMATION

The general formulation of this approximation for the noncovariant treatment of scattering is well known.[5] In this section we show that the same type of treatment is applicable to the relativistic case, a single instance, such as Eq. (36) for elastic nucleon-nucleus scattering, sufficing to demonstrate the generality of the procedure.

Let us define

$$R[n] = I([n-1],n)\times\{1-G[n-1]G_nI([n-1],n)\}^{-1}, \tag{69}$$

and

$$R_{12} = I_{12}[1-G_1G_2I_{12}]^{-1}, \tag{70}$$

the corresponding matrix for two nucleon scattering. Assuming that it is permissible to neglect many-body interactions, we may write

$$\begin{aligned} I([n-1],n) &= \sum_{i=1}^{n-1}(G_1\cdots G_n)^{-1}G_iG_nI_{in} \\ &= \sum_{i=1}^{n-1}G^{(0)}[n]^{-1}G_iG_nI_{in}. \end{aligned} \tag{71}$$

We also recall that $G[n-1]$ satisfies the integral equation

$$\begin{aligned} G[n-1]-G^{(0)}[n-1] &= G^{(0)}[n-1]I[n-1]G[n-1] \\ &= G[n-1]I[n-1]G^{(0)}[n-1]. \end{aligned} \tag{72}$$

If the algebraic identity

$$D^{-1} = D_0^{-1}+D_0^{-1}(D_0-D)D^{-1} \tag{73}$$

is applied with the choices

$$\begin{aligned} D &= 1-G[n-1]G_nI([n-1],n), \\ D_0 &= 1-G^{(0)}[n]I([n-1],n), \end{aligned} \tag{74}$$

we ascertain by means of (72) that

$$\begin{aligned} &[1-G[n-1]G_nI([n-1],n)]^{-1} \\ &\quad= [1-G^{(0)}[n]I([n-1],n)]^{-1} \\ &\quad\quad+[1-G^{(0)}[n]I([n-1],n)]^{-1} \\ &\quad\quad\times[G[n-1]I[n-1]G^{(0)}[n-1]I([n-1],n)] \\ &\quad\quad\times[1-G[n-1]G_nI([n-1],n)]^{-1} \\ &\quad= [1-\sum_{i=1}^{n-1}G_iG_nI_{in}]^{-1}+[1-\sum_{i=1}^{n-1}G_iG_nI_{in}]^{-1} \\ &\quad\quad\times[G[n-1]I[n-1]G^{(0)}[n-1]I([n-1],n)] \\ &\quad\quad\times[1-G[n-1]G_nI([n-1],n)]^{-1}, \end{aligned} \tag{75}$$

the second form resulting from the application of Eq. (71). By ignoring temporarily the second term of (75), which represents the correction for nuclear binding, we have approximately

$$R[n] \cong \sum_{i=1}^{n-1}G^{(0)}[n]^{-1}G_iG_nI_{in}[1-\sum_{j=1}^{n-1}G_jG_nI_{jn}]^{-1}. \tag{76}$$

Equation (76) thus represents the solution of the problem of the scattering of a nucleon by $n-1$ others in which the interaction among the latter is neglected.

By repeated application of the identity (73) Eq. (76) can be exhibited as a multiple scattering series. For this purpose we fix the index i and select for D the reciprocal operator in (76), whereas for D_0 we choose

$$D_0=1-G_iG_nI_{in}. \quad (77)$$

We then find, remembering Eq. (70), that

$$G^{(0)}[n]R[n]=\sum_{\alpha_1=1}^{n-1} G_{\alpha 1}G_nR_{\alpha 1 n}+\sum_{\alpha_1\neq\alpha_2} G_{\alpha 1}G_nR_{\alpha 1 n}G_{\alpha 2}G_nR_{\alpha 2 n}+\sum_{\alpha_1\neq\alpha_2\neq\alpha_3} G_{\alpha 1}G_nR_{\alpha 1 n}G_{\alpha 2}G_nR_{\alpha 2 n}G_{\alpha 3}G_nR_{\alpha 3 n}+\cdots \quad (78)$$

the sum of single, double$\cdots$scattering terms.

We may now return to Eq. (63) to see that the previously neglected second term is subject to a similar analysis. If we denote the right-hand side of Eq. (78) by M and the factor in (75), $G[n-1]I[n-1]$, by B, inspection shows that $G^{(0)}[n]R[n]$ can be represented by the symbolic double series

$$G^{(0)}[n]R[n]=M+MBM+MBMBM+\cdots =M\sum_{\nu=0}^{\infty}(BM)^\nu=M[1-BM]^{-1}. \quad (79)$$

The conventional impulse approximation consists in retaining only the first term of M, Eq. (78). This yields for Eq. (48)

$$\langle kp|S-1|k'p'\rangle\cong i^n\int \|\bar{\chi}_p(1\cdots n-1)\chi_k(n)\| \times\langle 1\cdots n|G^{(0)}[n]^{-1}\sum_{\alpha=1}^{n-1}R_{\alpha n}|1'\cdots n'\rangle \times\|\chi_{p'}(1'\cdots n-1')\chi_{k'}(n')\|. \quad (80)$$

It need hardly be added that in view of Eq. (60), similar considerations may be applied to the case of boson-nucleus scattering.

V. NORMALIZATION OF AMPLITUDES AND OTHER OBSERVABLES

An essential requirement for the utilization of the results of the previous sections is a knowledge of the normalization of the covariant amplitudes. A convenient method uses either the conservation of charge or of nucleon number. To see how to employ the latter, we introduce into the Lagrange density the term

$$\mathcal{L}'(x)=:\bar{\psi}(x)\gamma_\mu\psi(x):V_\mu(x), \quad (81)$$

which describes the coupling of an external vector field $V_\mu(x)$ to the nucleon "particle current." If we consider the equation for $G[n]$, in the version

$$\{G_1^{-1}\cdots G_n^{-1}-1[n]I[n]\}G[n]=1[n], \quad (82)$$

its general form will be unaffected by the introduction of (81), except that the terms are now all functionals of $V_\mu(x)$. In the formula

$$\delta G[n]/\delta V_\mu(\xi)=-G[n]\{\delta G[n]^{-1}/\delta V_\mu(\xi)\}G[n], \quad (83)$$

we may then utilize the bracket in (82) for $G[n]^{-1}$. With the definition

$$\Gamma_\mu^{(i)}(\xi)=-\delta G_i^{-1}/\delta V_\mu(\xi), \quad (84)$$

we thus obtain

$$\delta G[n]/\delta V_\mu(\xi)=G[n]\{\sum_i \Gamma_\mu^{(i)}(\xi)G_iG^{(0)}[n]^{-1}+1[n]\delta I[n]/\delta V_\mu(\xi)\}G[n]. \quad (85)$$

On the other hand it follows from (81)[15] that

$$\delta G[n]/\delta V_\mu(\xi)=i^{n+1}\langle 0|T(\psi(1)\cdots\psi(n):\bar{\psi}(\xi)\gamma_\mu\psi(\xi): \times\bar{\psi}(n')\cdots\bar{\psi}(1')|0\rangle. \quad (86)$$

Applying the same limiting procedure to (84) and (85), $L(t,t')$, as in the case of scattering (keeping ξ in the finite domain), we thus deduce the formula

$$i\langle\beta^{(-)}|:\bar{\psi}(\xi)\gamma_\mu\psi(\xi):|\alpha^{(+)}\rangle=i^n\int\bar{\chi}_{\beta}{}^{(-)}(1\cdots n) \times\langle 1\cdots n|[\sum_i\Gamma_\mu^{(i)}(\xi)G_iG^{(0)}[n]^{-1} +\delta I[n]/\delta V_\mu(\xi)]|1'\cdots n'\rangle\chi_\alpha{}^{(+)}(1'\cdots n'). \quad (87)$$

As the only trivial instance available, let us apply Eq. (87) to a one-particle state of momentum p. We have

$$\langle p|N|p\rangle=\lim(\mathbf{p}\to\mathbf{p}')\delta(\mathbf{p}-\mathbf{p}')=V(2\pi)^{-3}, \quad (88)$$

where V is the volume of the system. On the other hand for a renormalized theory (see Appendix A)

$$\chi_p(x)=(2\pi)^{-\frac{3}{2}}\mu(p)e^{ipx}, \quad (89)$$

where

$$(\gamma p+m)\mu(p)=0,\quad \bar{\mu}(p)\gamma_0\mu(p)=1. \quad (90)$$

Taking cognizance of (88) and (89) and (87), the latter reduces to the condition

$$1=V^{-1}\int dxdx'd\sigma_\mu(\xi)\bar{\mu}(p)e^{-ipx}\Gamma_\mu(x-\xi,\,\xi-x')e^{ipx'}\mu(p) =\bar{\mu}(p)\Gamma_0(p,p)\mu(p). \quad (91)$$

Comparison of (90) and (91) informs us that $\Gamma_0(p,p)=\gamma_0$, as we require for a renormalized theory.

It is characteristic of the covariant theory that expectation values and matrix elements involve integrations over both space and time; when the causal nature of the interactions is not disregarded, a knowledge of the amplitudes for all (relative) times is required

As a further illustrative example, let us apply Eq. (87) to the normalization of the deuteron covariant amplitude

$$\chi_D(x_1,x_2)=\langle 0|T(\psi(x_1)\psi(x_2))|D\rangle.$$

By (1), the normalization condition may be written

$$\langle D|N|D\rangle = 2, \tag{92}$$

where, in view of (87),

$$\langle D|N|D\rangle = i\int \bar{\chi}_D(x_1x_2)\{\Gamma_0^{(1)}(x_1,x_1'\xi)G_2^{(0)-1}(x_2,x_2') + \Gamma_0^{(2)}(x_2,x_2',\xi)G_1^{(0)-1}(x_1,x_1') + \delta I(x_1,x_2;x_1'x_2')/\delta V_0(\xi)\} \times \chi_D(x_1',x_2')dx_1\cdots dx_2'd\xi. \tag{93}$$

In order to exhibit the relationship between (93) and the conventional quantum-mechanical normalization conditions, we consider the former in lowest order. Then $\delta I/\delta V$ is disregarded and

$$G_{1,2}^{(0)-1} = (\gamma p+m)^{(1),(2)}, \quad \Gamma_0^{(1),(2)} = \gamma_0^{(1),(2)}. \tag{94}$$

Then in momentum space, (93) is the sum of two terms

$$\langle D|N|D\rangle = N_1 + N_2,$$

where

$$N_1 = i\int dp_1dp_2\bar{\chi}_p(p_1,p_2)\gamma_0^{(1)}(\gamma p+m)^{(2)}\chi_D(p_1,p_2), \tag{95}$$

and a similar expression obtains for N_2. For a deuteron at rest of total energy E, Eq. (95) expressed in terms of the relative momentum p reads

$$N_1 = i\int dp\Phi^*(p)[H_2 - \tfrac{1}{2}E + p_0]\Phi(p), \tag{96}$$

where $\Phi^*(p) = \bar{\Phi}(p)\gamma_0^{(1)}\gamma_0^{(2)}$ is the relative momentum-energy wave function. As Salpeter[6] has shown, the amplitude $\Phi(p)$ may be related to the "equal times" wave function $\phi(\mathbf{p})$,

$$\phi(\mathbf{p}) = \int \Phi(p)dp_0, \tag{97}$$

when the interaction I is instantaneous, by the equation

$$\Phi(p) = [H_1 - \tfrac{1}{2}E - p_0]^{-1}[H_2 - \tfrac{1}{2}E + p_0]^{-1} \times \int I(\mathbf{p},\mathbf{q})\phi(\mathbf{q})d\mathbf{q} \tag{98}$$

and $\phi(\mathbf{p})$ satisfies

$$\phi(\mathbf{p}) = (2\pi i)(H_1+H_2-E)^{-1}(P^+ - P^-) \times \int I(\mathbf{p},\mathbf{q})\phi(\mathbf{q})d\mathbf{q}, \tag{99}$$

where P^+ is a projection operator to the positive-energy components of both nucleons, whereas P^- is a projection operator to their negative components. But if (98), and its adjoint, are inserted into (96) and the p_0 integration carried out, we find

$$N_1 = (2\pi i)^2\int d\mathbf{p}d\mathbf{q}_1d\mathbf{q}_2\phi^*(\mathbf{q}_1)I(\mathbf{q}_1,\mathbf{p}) \times (H_1+H_2-E)^2(P^+-P^-)I(\mathbf{p},\mathbf{q}_2)\phi(\mathbf{q}_2) = \int d\mathbf{p}\phi^*(\mathbf{p})(P^+-P^-)\phi(\mathbf{p}). \tag{100}$$

The result for N_2 is identical. Thus, in the limit considered, the covariant normalization condition differs from the more familiar expression in terms of equal-times wave functions only in that negative-energy components contribute negatively to the normalization.[6]

Delaying further study of Eq. (87) until Sec. VI, we turn briefly to the consideration of observables other than the S-matrix. The general problem may be viewed as that of obtaining matrix elements of various T products (or normal products) between states other than the vacuum. In all cases these may be handled by a suitable generalization of the methods described for the S-matrix and for the normalization integral. We shall not consider any hypothetical cases, since in practice the most important quantity is the energy of a bound state, to be studied in detail in the ensuing section. We merely complete the considerations of this section by a special technique available for the computation of the electromagnetic moments of a nucleus.

Toward this end, it is convenient to study the scattering of the nucleus by a weak external electromagnetic field $A_\mu(\xi)$, which vanishes in the remote past and future. The Green's function $G[n]$ will now also be a functional of A_μ. To first order in the latter we have

$$G[n,A] = G[n] + \int d\xi\{\delta G[n]/\delta A_\mu(\xi)\}_{A=0}A_\mu(\xi). \tag{101}$$

It is clear that the operator $L(t,t')$ can be applied to the right-hand side of (101), thus yielding for the scattering of the nucleus (in its ground state) from total momentum p' to total momentum p, the result

$$\langle p|(S-1)|p'\rangle = -i^n\int d\xi\bar{\chi}_p(1\cdots n) \times \langle 1\cdots n|\delta G[n]^{-1}/\delta A_\mu(\xi)|1'\cdots n'\rangle \times \chi_{p'}(1'\cdots n')A_\mu(\xi). \tag{102}$$

The various moments can be extracted from (102) by special choice of $A_\mu(\xi)$.

VI. BOND-STATE PROBLEMS

We turn to the bound-state solutions of the equation

$$\{G^{(0)}[n]^{-1} - 1[n]I[n]\}\chi(1\cdots n) = 0. \tag{103}$$

Now the workers who have grappled with actual cases[2] all overlooked Eq. (87), as the proper normalization condition to be imposed on permissible solutions.[17] At the same time, one must remember that the same equation does not provide a useful orthogonality theorem. To see this, let us apply the condition to the "ladder approximation," i.e., to an equation of the form

$$(\gamma p+m)^{(1)}(\gamma p+m)^{(2)}\chi(x_1,x_2) = \int dx_1'dx_2' I(x_1,x_2;x_1',x_2')\chi(x_1',x_2'), \quad (104)$$

with I specialized to

$$I(x_1,x_2;x_1',x_2')=\mathcal{I}(x,x')\delta(X,X'), \quad (105)$$

and X, x total and relative coordinates, respectively. Under these circumstances, we may consider $\mathcal{I}(x,x')$ to be independent of $V_\mu(\xi)$ when that external field is turned on. Applied to states of total momentum zero and energies E, E', the integrated form of Eq. (87) becomes

$$2(2\pi)^{-3}V\delta(E-E')=\int dx_1\cdots dx_2'd\sigma_\mu(\xi)\bar{\chi}_E(x_1,x_2) \times\langle 1\,2|\{\gamma_\mu^{(1)}(\xi)(\gamma p+m)^{(2)}+(\gamma p+m)^{(1)}\gamma_\mu^{(2)}(\xi)\}|1'\,2'\rangle \times\chi_{E'}(x_1',x_2'). \quad (106)$$

Recalling that

$$\langle 1|\gamma_\mu(\xi)|2\rangle=\gamma_\mu\delta(1-2)\delta(1-\xi), \quad (107)$$

choosing a surface $\xi_0=$ constant, inserting

$$\chi_{E'}(x_1x_2)=e^{-iE'T}\phi_E(x),\quad \bar{\chi}_E(x_1,x_2)=e^{iET}\bar{\phi}_E(x), \quad (108)$$

and performing the integration with respect to the total coordinates, the right-hand side of (106) reduces to

$$2\pi V\delta(E-E')\int dx\bar{\phi}_E(x) \times[\gamma_0^{(1)}(\gamma p+m)^{(2)}+(\gamma p+m)^{(1)}\gamma_0^{(2)})\phi_{E'}(x). \quad (109)$$

In the limit as $E\rightarrow E'$, we can infer that

$$\tfrac{1}{2}(2\pi)^4\int dx\bar{\phi}_E(x) \times[\gamma_0^{(1)}(\gamma p+m)^{(2)}+(\gamma p+m)^{(1)}\gamma_0^{(2)}]\phi_E(x)=1. \quad (110)$$

On the other hand, for $E\neq E'$, it is inviting to conclude the the corresponding integral is at least finite. Under that supposition, it is then amusing to remark that one can prove directly from (104) that it vanishes, in fact.

[17] The relevance of this condition was noted afterwards by Nishijima and Mandelstam (reference 4 and see below).

To see this we write

$$(\gamma p+m)^{(1)}=\gamma_0^{(1)}[\boldsymbol{\alpha}^{(1)}\cdot\mathbf{p}+\gamma_0^{(1)}m-p_0-\tfrac{1}{2}E] \equiv\gamma_0^{(1)}[\mathcal{H}_1-\tfrac{1}{2}E],$$
$$(\gamma p+m)^{(2)}=\gamma_0^{(2)}[-\boldsymbol{\alpha}^{(2)}\cdot\mathbf{p}+\gamma_0^{(2)}m+p_0-\tfrac{1}{2}E] \equiv\gamma_0^{(2)}[\mathcal{H}_2-\tfrac{1}{2}E]. \quad (111)$$

In relative coordinates we then contemplate the two equations

$$\gamma_0^{(1)}\gamma_0^{(2)}[\mathcal{H}_1-\tfrac{1}{2}E][\mathcal{H}_2-\tfrac{1}{2}E]\phi_E(x) -\int\mathcal{I}(x,x')\phi_E(x)=0, \quad (112)$$

$$\bar{\phi}_{E'}(x)\gamma_0^{(1)}\gamma_0^{(2)}[\mathcal{H}_1-\tfrac{1}{2}E'][\mathcal{H}_2-\tfrac{1}{2}E'] -\int\bar{\phi}_{E'}(x)\mathcal{I}(x',x)=0, \quad (113)$$

where as usual $\bar{\phi}p=-p\bar{\phi}$. Following a standard procedure, together they imply that

$$\tfrac{1}{4}(E'-E)\int\phi_{E'}{}^\dagger(x)[(\mathcal{H}_1-\tfrac{1}{2}E)+(\mathcal{H}_2-\tfrac{1}{2}E) +(\mathcal{H}_1-\tfrac{1}{2}E')+(\mathcal{H}_2-\tfrac{1}{2}E')]\phi_E(x)=0, \quad (114)$$

and if $E\neq E'$, we may conclude the vanishing of the integral. In deriving (114), we have presupposed the validity of various integrations by parts, i.e., the Hermiticity of the operators involved. The integrals in (114) and (109) are the same.

The appearance of the last result suggests a matrix form of writing. Indeed, if we introduce the vector

$$\Phi_E(x)=\begin{pmatrix}\phi_E(x)\\(\mathcal{H}_1-\tfrac{1}{2}E)\phi_E(x)\\(\mathcal{H}_2-\tfrac{1}{2}E)\phi_E(x)\end{pmatrix}, \quad (115)$$

and select a suitable normalization, we may write

$$\int\Phi_{E'}{}^\dagger(x)\mathfrak{M}\Phi_E(x)=\delta_{EE'}, \quad (116)$$

where $\mathfrak{M}$ is the Hermitian matrix[18]

$$\mathfrak{M}=\tfrac{1}{4}\begin{pmatrix}0&1&1\\1&0&0\\1&0&0\end{pmatrix}. \quad (117)$$

In Appendix C, the generalization to an arbitrary number of particles is considered. The establishment of (116), together with the plausible assumption of an expansion theorem, will permit us to develop below a perturbation theory for the many-particle system.

[18] $\mathfrak{M}$ is in fact a singular matrix and for this case we could equally well have introduced a two-dimensional vector space with a nonsingular metric. This will affect none of the ensuing results, however.

Prior to that, however, let us review briefly the bearing of (116) on previous work.

Mandelstam[4] has argued that the imposition of (116) or its equivalent together with its associated Hermiticity requirements rule out the existence of a discrete spectrum for the parameters considered by Goldstein,[2] in striking analogy with the situation occurring for the one-particle Dirac equation in a Coulomb field for $Z>137$.[19] On the other hand, the same equation could well yield a discrete spectrum for sufficiently weak coupling.

For the other example in the literature, Nishijima[4] has applied a corresponding normalization condition to the solutions of Wick and Cutkosky[2] and found them perfectly satisfactory except in the extreme situations of zero total energy and zero binding energy.[20] It is our opinion that the work of the latter authors represents an example with no remaining ambiguities.

We turn then to the formulation of perturbation theory, restricting ourselves to the two-body problem. In the perturbed equation, we admit an energy-dependent interaction $\mathcal{G}_W(x,x')$. Let E_j be the energies of the unperturbed system. From the equations

$$\gamma_0^{(1)}\gamma_0^{(2)}(\mathcal{H}_1-\tfrac{1}{2}W)(\mathcal{H}_2-\tfrac{1}{2}W)\psi_W(x) - \int \mathcal{G}_W(x,x')\psi_W(x')=0, \quad (118)$$

$$\bar{\phi}_j(x)\gamma_0^{(1)}\gamma_0^{(2)}(\mathcal{H}_1-\tfrac{1}{2}E_j)(\mathcal{H}_2-\tfrac{1}{2}E_j) - \int \bar{\phi}_j(x')\mathcal{G}(x;x)=0, \quad (119)$$

following the same standard procedure as for Eqs. (112) and (113), we derive the result

$$(W-E_j)\int \Phi_j{}^\dagger \mathfrak{M}\Psi_W = \int \phi_j{}^* \mathcal{V}_W\psi_W, \quad (120)$$

where Ψ_W represents a vector constructed as in Eq. (115) and

$$\mathcal{V}_W=\gamma_0^{(1)}\gamma_0^{(2)}[\mathcal{G}_W-\mathcal{G}]. \quad (121)$$

Equation (120) is reminiscent of the fundamental formula of Brillouin-Wigner perturbation theory.

To carry the treatment forward, however, we must assume an expansion theorem. With an appropriate choice of normalization of ϕ_E we suppose the validity of the representation

$$\psi_W=\phi_E+\sum_{j\neq E} a_j\phi_j. \quad (122)$$

To determine the a_j, we construct from ψ_W the vector Ψ_W, that operation defining simultaneously the right-hand side of the equation

$$\Psi_W=\Phi_{W,E}+\sum a_j\Phi_{W,j} = \Phi_E+(W-E)X_E+\textstyle\sum_j a_j[\Phi_j+(W-E_j)X_j], \quad (123)$$

where

$$X_j=\begin{pmatrix} 0 \\ -\tfrac{1}{2}\phi_j \\ -\tfrac{1}{2}\phi_j \end{pmatrix}. \quad (124)$$

From Eq. (123) there follows, by means of Eqs. (116) and (120) the equations

$$(W-E_j)^{-1}\int \phi_j{}^\dagger\mathcal{V}_W\psi_W=(W-E)\int \Phi_j{}^\dagger\mathfrak{M}X_E + a_j+\sum_k (W-E_k)\int \Phi_j{}^\dagger\mathfrak{M}X_k,$$

$$\int \Phi_E{}^\dagger\mathfrak{M}\Psi_W=1+(W-E)\int \Phi_E\mathfrak{M}X_E + \sum_k a_k(W-E_k)\int \Phi_E{}^\dagger\mathfrak{M}X_k, \quad (125)$$

the first of which serves for the determination of the a_j, the second defining the normalization integral on the left-hand side. Toegther with Eq. (122), they determine the energy, represented according to Eq. (120) by the expression

$$W=E+\left[\int \phi_E{}^*\mathcal{V}_W\psi_W \Big/ \int \Phi^\dagger\mathfrak{M}\Psi_W\right]. \quad (126)$$

APPENDIX A

The treatment of the S-matrix in the Heisenberg representation in the absence of bound states is usually based on the *in* and *out* operators,[3] defined loosely by the equations (we consider for example the nucleon operators)

$$\lim_{t\to-\infty}[\psi(\mathbf{r},t)-\psi_{\text{in}}(\mathbf{r},t)]=0, \quad (A.1)$$

$$\lim_{t\to-\infty}[\psi(\mathbf{r},t)-\psi_{\text{out}}(\mathbf{r},t)]=0. \quad (A.2)$$

ψ_{in} and ψ_{out} then satisfy free-field equations, and the total Hamiltonian when expressed in terms of either set of operators has the aspect of the Hamiltonian for uncoupled fields. The associated creation and annihilation operators in conjunction with the *physical* vacuum state are then used to construct the complete sets of states $|\alpha^{(+)}\rangle$ (constructed by means of the ψ_{in}) and $|\alpha^{(-)}\rangle$ (obtained by means of the ψ_{out}). The physical significance of these operators (i.e., of the states erected by their use) is that they describe isolated real particles.

In the event that bound states occur, we extend Eqs.

[19] K. M. Case, Phys. Rev. **80**, 797 (1950).

[20] Though we are in agreement with Nishijima's conclusions, we do not confirm the precise form of the normalization condition that he imposes.

(A.1) and (A.2) to the statements

$$\lim_{t\to-\infty}[\psi(\mathbf{r},t)-\psi_{\text{in}}(\mathbf{r},t)-\psi_B(\mathbf{r},t)]=0, \quad (A.3)$$

$$\lim_{t\to+\infty}[\psi(\mathbf{r},t)-\psi_{\text{out}}(\mathbf{r},t)-\psi_B(\mathbf{r},t)]=0, \quad (A.4)$$

where of the ψ_B (the bound state operators) it is only assumed that they are dynamically independent of the *in* and *out* operators. Thus they commute (anticommute) with the latter, which form a reducible representation of the commutation (anticommutation) relations of block form with respect to the subspaces of given bound state character.

To explain what is meant by the last statement, we construct the states $|\alpha^{(+)}\rangle$. We cannot give an explicit construction for those states which in the remote past contain only composite particles but we merely assume that there exist an appropriate set $|B^{(+)}\rangle$, including the vacuum state. Let $O_{\text{in},\alpha}{}^\dagger$ be the complex of "in" operators which defines the state $|\alpha^{(+)}\rangle$ with incoming free particles only. Then the complete Hilbert space is spanned by the vectors

$$|(\alpha,B)^{(+)}\rangle=O_{\text{in},\alpha}{}^\dagger|B^{(+)}\rangle, \quad (A.5)$$

and, if Q_{in} is any "in" operator

$$\langle(\alpha',B')^{(+)}|Q_{\text{in}}|(\alpha,B^{(+)})\rangle=\langle\alpha'^{(+)}|Q_{\text{in}}|\alpha^{(+)}\rangle\delta_{B'B}, \quad (A.6)$$

i.e., between states of given bound state character, it has the same matrix elements as if the bound state character were ignored and replaced by the vacuum, whereas between states of different bound state character, the matrix element vanishes. Of course, there is a corresponding set of states $|\alpha^{(-)}\rangle$ constructed from the "out" operators.

We turn to the study of the consequences of Eqs. (A.3) and (A.4) and subsequent statements in regard to covariant amplitudes. For a one-particle state $|p\rangle$ we consider the amplitude

$$\chi_p(x)=\langle 0|\psi(x)|p\rangle. \quad (A.7)$$

Since we suppose ourselves to be dealing with a prerenormalized theory, $\psi(p)$ satisfies

$$(\gamma p+m)\chi_p(x)=0, \quad (A.8)$$

and indeed, since we are dealing with a single elementary particle

$$\langle 0|\psi(x)|p\rangle=\langle 0|\psi_{\text{in}}(x)|p\rangle=\langle 0|\psi_{\text{out}}(x)|p\rangle = (2\pi)^{-\frac{3}{2}}\mu(p)e^{ipx}, \quad (A.9)$$

where $\mu(p)$ is a free-particle spinor normalized to $\mu^\dagger\mu=1$.

For states of nucleon number two we study the amplitude

$$\chi_\alpha{}^{(+)}(x_1,x_2)=\langle 0|T(\psi(x_1)\psi(x_2))|\alpha^{(+)}\rangle, \quad (A.10)$$

to which we apply the operation $[\,]^{\text{in}}$, as defined in Sec. I, Eq. (23),

$$[\chi_\alpha{}^{(+)}(x_1,x_2)]^{\text{in}}=\langle 0|[\psi(x_1)\psi(x_2)]^{\text{in}}|\alpha^{(+)}\rangle. \quad (A.11)$$

If now $|\alpha^{(+)}\rangle$ is a bound state $=|p(2)\rangle$, we have

$$[\chi_p(x_1,x_2)]^{\text{in}}=\langle 0|[\psi(x_1)\psi(x_2)]^{\text{in}}|p(2)\rangle =\langle 0|[\psi_B(x_1)\psi_B(x_2)]^{\text{in}}|p(2)\rangle, \quad (A.12)$$

whereas if $|\alpha^{(+)}\rangle$ is a continuum state $|(p_1,p_2)^{(+)}\rangle$,

$$\begin{aligned}[\chi_{p_1p_2}{}^{(+)}(x_1x_2)]^{\text{in}} &=\langle 0|[\psi(x_1)\psi(x_2)]^{\text{in}}|(p_1,p_2)^{(+)}\rangle\\ &=\langle 0|[\psi_{\text{in}}(x_1)\psi_{\text{in}}(x_2)]^{\text{in}}|(p_1,p_2)^{(+)}\rangle\\ &=(2!)^{-\frac{1}{2}}[\chi_{p_1}(x_1)\chi_{p_2}(x_2)-\chi_{p_1}(x_2)\chi_{p_2}(x_1)]^{\text{in}}, \quad (A.13)\end{aligned}$$

as follows directly from the representations available both for the *in* operators and for the corresponding states.

As a final example we study the asymptotic form of a mixed three-particle state $|(k,p(2))^{(+)}\rangle$, with

$$\begin{aligned}[\chi_{k,p}{}^{(+)}(x_1x_2x_3)]^{\text{in}}&=\langle 0|[\psi(x_1)\psi(x_2)\psi(x_3)]^{\text{in}}|(k,p(2))^{(+)}\rangle\\ &=\langle 0|[\psi_B(x_1)\psi_B(x_2)\psi_{\text{in}}(x_3)+\psi_B(x_1)\psi_{\text{in}}(x_2)\psi_B(x_3)\\ &\quad+\psi_{\text{in}}(x_1)\psi_B(x_2)\psi_B(x_3)]^{\text{in}}|(k,p(2))^{(+)}\rangle\\ &=3^{-\frac{1}{2}}[\chi_p(x_1x_2)\chi_k(x_3)-\chi_p(x_1x_3)\chi_k(x_2)\\ &\quad+\chi_p(x_2x_3)\chi_k(x_1)]^{\text{in}}. \quad (A.14)\end{aligned}$$

The normalization factor has been explained in the introduction.

It is clear that by continuing this procedure we can obtain the general formulation of the asymptotic condition described in the introductory section.

APPENDIX B

We seek to establish the equivalence of Eqs. (41) and (44) of the text. We limit ourselves to the case that particle 3 is considered distinct from particles 1 and 2. We must then establish the equality

$$\begin{aligned}\int \|\bar{\phi}_{k_1}(1)\bar{\chi}_{k_2}(2)\|\bar{\chi}_{k_3}(3)\langle 123|I[3]|1'2'3'\rangle\chi_{kp}{}^{(+)}(1'2'3')\\ =\int\bar{\chi}_{k_1k_2}{}^{(-)}(12)\bar{\chi}_{k_3}(3)\langle 123|I([2],3)|1'2'3'\rangle\\ \times\chi_{kp}{}^{(+)}(1'2'3'). \quad (B.1)\end{aligned}$$

We recall that

$$I[3]-I([2],3)=I_{12}G_3{}^{-1}. \quad (B.2)$$

Now $\bar{\chi}_{12}{}^{(-)}$ satisfies the equation

$$\begin{aligned}\bar{\chi}_{12}{}^{(-)}&=\|\bar{\chi}_1\bar{\chi}_2\|+\bar{\chi}_{12}{}^{(-)}I_{12}G_1G_2\\ &=\|\bar{\chi}_1\bar{\chi}_2\|[1-I_{12}G_1G_2]^{-1}\\ &=\|\bar{\chi}_1\bar{\chi}_2\|+\|\bar{\chi}_1\bar{\chi}_2\|I_{12}G_1G_2[1-I_{12}G_1G_2]^{-1}\\ &=\|\bar{\chi}_1\bar{\chi}_2\|+\|\bar{\chi}_1\bar{\chi}_2\|I_{12}G_{12}. \quad (B.3)\end{aligned}$$

Inserting the last form of (B.3) into the right-hand side of (B.1) and comparing the result with the left-hand side, we are left to prove that

$$G_3{}^{-1}\chi_{123}{}^{(+)}=G_{12}I([2],3)\chi_{123}{}^{(+)}. \quad (B.4)$$

But the latter equation follows immediately from one form of the defining equation for $\chi_{123}^{(+)}$,

$$\chi_{123}^{(+)}=\chi_{12}^{(+)}\chi_3+G_{12}G_3I([2],3)\chi_{123}^{(+)}, \quad (B.5)$$

since $G_3^{-1}\chi_3=0$.

APPENDIX C

We demonstrate that the orthogonality theorem, Eq. (95), can be extended to the case of any number n of particles. If $\phi_E[n]$ is the relative-coordinate amplitude, we study the integral ($\epsilon=E/n$)

$$\int \phi_{E'}^*[n][\prod_{i=1}^{n}(\mathcal{H}_i-\epsilon)-\prod_{i=1}^{n}(\mathcal{H}_i-\epsilon')]\phi_E[n] + \int \phi_{E'}[n][\mathcal{G}_{E'}[n]-\mathcal{G}_E[n]]\phi_E[n], \quad (C.1)$$

which vanishes in virtue of the fundamental equation

$$[\prod_{i=1}^{n}\gamma_0^{(i)}(\mathcal{H}_i-E)-\mathcal{G}_E[n]]\phi_E[n]=0, \quad (C.2)$$

and its adjoint.

Ignoring for the moment the energy dependence of the interaction, we transform the first term by means of the identity (verification left to the reader)

$$\prod_{i=1}^{n}(\mathcal{H}_i-\epsilon)-\prod_{i=1}^{n}(\mathcal{H}_i-\epsilon')$$
$$=n^{-2}(E'-E)\{\sum_{j=1}^{n}\prod_{i=1}^{n}(\mathcal{H}_i-\epsilon)(\mathcal{H}_j-\epsilon)^{-1}$$
$$+(n-1)^{-1}\sum_{j_1\neq j_2}\prod_{i}(\mathcal{H}_i-\epsilon)(\mathcal{H}_j-\epsilon)^{-1}(\mathcal{H}_{j_2}-\epsilon)^{-1}$$
$$\times(\mathcal{H}_{j_1}-\epsilon')+2![(n-1)(n-2)]^{-1}\sum_{\substack{j_1\neq j_2\neq j_3,\\ j_1<j_2}}\sum_{i}$$
$$\times(\mathcal{H}_i-\epsilon)(\mathcal{H}_{j_1}-\epsilon)^{-1}(\mathcal{H}_{j_2}-\epsilon)^{-1}(\mathcal{H}_{j_3}-\epsilon)^{-1}$$
$$\times(\mathcal{H}_{j_1}-\epsilon')(\mathcal{H}_{j_2}-\epsilon')+\cdots+(n-1)^{-1}\sum_{j_1\neq j_2}\prod_{i}(\mathcal{H}_1-\epsilon')$$
$$\times(\mathcal{H}_{j_1}-\epsilon')(\mathcal{H}_{j_2}-\epsilon')(\mathcal{H}_{j_1}-\epsilon)$$
$$+\sum_{j}\prod_{i}(\mathcal{H}_i-\epsilon')(\mathcal{H}_j-\epsilon')^{-1}\}, \quad (C.3)$$

the curly bracket being symmetric in E and E'. By means of (C.3) the first term of (C.1) can be written as

$$(E'-E)\int \Phi_{E'}^{\dagger}[n]\mathfrak{M}[n]\Phi_E[n], \quad (C.4)$$

if we define the quantities involved as follows: $\Phi_E[n]$ is a column vector of 2^n-1 components comprising the sequence $\phi_E[n]$, $(\mathcal{H}_1-\epsilon)\phi_E[n]$, $\cdots$, $(\mathcal{H}_n-\epsilon)\phi_E$, $(\mathcal{H}_1-\epsilon)\times(\mathcal{H}_2-\epsilon)\phi_E$, $\cdots$, $(\mathcal{H}_2-\epsilon)\cdots(\mathcal{H}_n-\epsilon)\phi_E$; $\Phi_E^{\dagger}$, is the adjoint vector; $\mathfrak{M}[n]$ is a square matrix with rows and columns labeled by the indices $0, 1, \cdots n, 12, 13, \cdots 23\cdots n$, whose nonvanishing matrix elements $\mathfrak{M}_{\alpha,\beta}$ can be read off from (C.3): Let $\rho(\alpha)$ be the number of integers (1 to $n-1$) other than zero specifying the row α, $\rho(\beta)$ the corresponding number for the column β. Then $\mathfrak{M}_{\alpha,\beta}$ vanishes unless $\rho(\alpha)+\rho(\beta)=n-1$. If the latter condition is satisfied and $\rho(\alpha)=s$, $\rho(\beta)=n-1-s$, then

$$\mathfrak{M}_{\alpha\beta}(s, n-1-s)=[n^2(n-1)!]s!(n-1-s)!. \quad (C.5)$$

For an energy-independent interaction and a suitable normalization, we would then have the orthogonality theorem

$$\int \Phi_{E'}^{\dagger}[n]\mathfrak{M}[n]\Phi_E[n]=\delta_{E'E}. \quad (C.6)$$

However, in the general case, even under the simplest assumptions, there is an unavoidable energy dependence of the interaction. Thus if we assume only two-particle forces, we have

$$\mathcal{G}_E[n]=\sum_{i\neq j}\gamma_0^{(i)}\gamma_0^{(j)}\prod_{k=1}^{n}\gamma_0^{(k)}(\mathcal{H}_k-\epsilon)\times(\mathcal{H}_i-\epsilon)^{-1}(\mathcal{H}_j-\epsilon)^{-1}I_{ij}, \quad (C.7)$$

where we suppose I_{ij} itself to be independent of energy. It is clear, though, that the integral containing the difference $(\mathcal{G}_{E'}-\mathcal{G}_E)$ can then be written in a form proportional to $(E'-E)$ times a suitably defined matrix product in a space of $2^{n-2}-1$ dimensions. Corresponding remarks will obtain if three and more-body interactions are involved. In general then the orthogonality theorem will be more complex than (C.6). Further details will be left to the reader.