New Dispersion Relations for Pion-Nucleon Scattering

WALTER GILBERT*
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts

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New relations expressing the imaginary part of the scattering amplitude as a coupling constant term and an integral over the physical spectrum of the real part of the amplitude are developed. These relations are applied to compute the meson-nucleon coupling constant from the experimental phase shifts, yielding the value $f^2=0.084$, and are used to compute the s-wave scattering lengths from the p-wave data.

INTRODUCTION

We shall develop a new, more convergent form for the dispersion relations for meson-nucleon scattering. These relationships will express the imaginary part of the amplitude in terms of a coupling constant term and an integral of the real part of the amplitude in the physical region. These new relations differ from the conjugate Hilbert transforms that connect the imaginary part of an amplitude to an integral of the real part in that the integrals are restricted to the experimentally accessible region above the threshold. These expressions have an additional convergence factor, the reciprocal of the laboratory momentum, and so provide relationships between the phase shifts less dependent upon the high-energy behavior of the theory or upon additional constant terms than the usual dispersion relations. We shall make two applications of these new relations: a coupling constant determination and a computation of the s-wave scattering lengths. In this paper we shall restrict ourselves to the dispersion relations for scattering into the forward direction, although the argument could be applied to the more general relations off the forward direction.

DISPERSION RELATIONS

Many authors have given the dispersion relations applicable to meson-nucleon scattering.[1–5] We shall collect in this section the relations and the formulas that we shall need.[6] The invariant T matrix for the pion-nucleon scattering process, considered as a two-by-two matrix in the nucleon isotopic spin space, we divide into two parts, even and odd, in the meson isotopic indexes,

$$T=\delta_{\alpha\beta}T^e+\tfrac{1}{2}[\tau_\alpha,\tau_\beta]T^o. \quad (1)$$

These even and odd amplitudes are related to the isotopic spin decomposition of the T matrix by

$$T^e=\tfrac{1}{3}(T^1+2T^3),\qquad T^o=\tfrac{1}{3}(T^1-T^3), \quad (2)$$

where the superscripts "1" and "3" refer to the $T=\frac{1}{2}$ and $T=\frac{3}{2}$ states, respectively. We shall separate out the nucleon spin dependence of the T matrix as

$$T(p,p',q)=\bar{u}(p)u(p')f(pp',pq)+\bar{u}(p)\gamma q u(p')g(pp',pq) \quad (3)$$

on the energy shell. The two invariant amplitudes, f and g, separately obey dispersion relations. The g amplitude is, to the order of μ^2/κ^2, the spin-flip amplitude, and since the π meson is pseudoscalar, the bound-state terms involving the coupling constant occur only in the dispersion relations for the g amplitude. We need only the relations in the forward direction, which we take to be, using z for $pq/\kappa\mu$, the laboratory energy of the meson divided by the meson's mass:

$$\operatorname{Re} f^o(\bar{z})=\frac{2}{\pi}\bar{z}P\int_1^\infty \frac{dz}{z^2-\bar{z}^2}\operatorname{Im} f^o(z), \quad (4)$$

$$\operatorname{Re} g^o(\bar{z})=\frac{bG^2/\kappa\mu}{\bar{z}^2-b^2}+\frac{2}{\pi}P\int_1^\infty \frac{zdz}{z^2-\bar{z}^2}\operatorname{Im} g^o(z), \quad (5)$$

$$\operatorname{Re} f^e(\bar{z})-\operatorname{Re} f^e(z')=\frac{2}{\pi}(\bar{z}^2-z'^2)P\int_1^\infty \frac{zdz\operatorname{Im} f^e(z)}{(z^2-\bar{z}^2)(z^2-z'^2)}, \quad (6)$$

$$\operatorname{Re} g^e(\bar{z})=-\frac{\bar{z}G^2/\kappa\mu}{\bar{z}^2-b^2}+\frac{2}{\pi}\bar{z}P\int_1^\infty \frac{dz}{z^2-\bar{z}^2}\operatorname{Im} g^e(z). \quad (7)$$

Here $b=\mu/2\kappa$, the position of the bound state singularity, and G is the renormalized, unrationalized pseudoscalar coupling constant. We shall also use the rationalized pseudovector coupling constant

$$f^2=b^2g^2=b^2G^2/4\pi. \quad (8)$$

We need to know the relation between this invariant decomposition of the scattering amplitude and the usual phase-shift expansion. We shall introduce direct and spin-flip amplitudes as, in the center-of-mass

* National Science Foundation Predoctoral Fellow.
[1] M. L. Goldberger, Phys. Rev. **99**, 979 (1955).
[2] Goldberger, Miyazawa, and Oehme, Phys. Rev. **99**, 986 (1955).
[3] A. Salam, Nuovo cimento **3**, 424 (1956), and A. Salam and W. Gilbert, Nuovo cimento **3**, 607 (1956).
[4] R. Oehme, Phys. Rev. **100**, 1503 (1955) and **102**, 1174 (1956).
[5] R. H. Capps and G. Takeda, Phys. Rev. **103**, 1877 (1956).
[6] We take $\hbar=c=1$ and use a timelike metric. pp' is the invariant product of two four-vectors. We are considering the scattering of a nucleon of mass κ, four-momentum p', by a meson of mass μ, four-momentum q', and isotopic spin β. The final state is characterized by p, q, and α. The nucleons are described by Dirac spinors $u(p)$ obeying the equation $(\gamma p-\kappa)u(p)=0$, $\bar{u}u=2\kappa$. In calculations we take $\kappa=6.7\mu$.

system,

$$T=D+i\boldsymbol{\sigma}\cdot\hat{n}\sin\theta\left(\frac{\mu^2\eta^2E}{\kappa}\right)S, \tag{9}$$

where θ is the angle of scattering in the center-of-mass system, $\hat{n}$ is a unit vector perpendicular to the plane of scattering, η is the center-of-mass momentum divided by the meson's mass, and E is the total center-of-mass energy. We shall also use the symbol $\beta=\rho_0+\kappa$, the sum of the nucleon energy and mass in the center-of-mass system, and $\alpha=E+\kappa$. Then we have the algebraic relations

$$\begin{aligned} E&=(\kappa^2+\mu^2+2\kappa\mu z)^{\frac{1}{2}},\\ \beta E&=\kappa(\alpha+\mu z),\\ (z^2-1)^{\frac{1}{2}}\kappa&=\eta E.\end{aligned} \tag{10}$$

If the spinors in (3) are specialized to the center-of-mass system, the invariant functions $f(z)$ and $g(z)$ can be related to the direct and spin-flip amplitudes:

$$f(z)=\frac{\alpha}{2\beta E}D(z)-\mu zS(z), \tag{11}$$

$$g(z)=[D(z)/2\beta E]+S(z). \tag{12}$$

The phase-shift expansions for the direct and spin-flip amplitudes may be determined from the unitary condition on the invariant T matrix. In the forward direction the expansions are, with our normalization,

$$D(z)=\frac{8\pi E}{\mu\eta}\sum_{l=0}^{\infty}[la_{l-}+(l+1)a_{l+}], \tag{13}$$

$$S(z)=\frac{8\pi\kappa}{\mu^3\eta^3}\sum_{l=1}^{\infty}(a_{l-}-a_{l+})\frac{l(l+1)}{2}. \tag{14}$$

Here $a_{l\pm}$ is $e^{i\delta}\sin\delta$ for the state $j=l\pm\frac{1}{2}$ if the scattering is elastic. We shall normally cut off these expansions after the p-wave terms and assume that the scattering is elastic at all energies of interest. We use the usual notation for the s and p waves, a_1 and a_3 referring to $e^{i\delta}\sin\delta$ for the $T=\frac{1}{2}$ and $\frac{3}{2}$ s waves, $a_{2T,2J}$ for the p waves. We shall use a superscript "0" to denote the scattering lengths, taking the low-energy behavior of the phases to be $\delta\sim\delta^0\eta$ for the s-waves and $\delta\sim\delta^0\eta^3$ for the p waves.

The optical theorem is, with this normalization,

$$\operatorname{Im} D=2\mu\eta E\sigma, \tag{15}$$

or for the even and odd isotopic index amplitudes:

$$\operatorname{Im} D^e=\mu\eta E(\sigma^++\sigma^-), \tag{16}$$

$$\operatorname{Im} D^o=\mu\eta E(\sigma^--\sigma^+), \tag{17}$$

σ^+ and σ^- being the total cross sections for the scattering of positive and negative pions off protons. The relations (4) and (5), (6) and (7) may be combined using (10), (11), and (12) to yield Goldberger's[1,2] relations for the direct amplitude:

$$\operatorname{Re} D^o(\bar{z})=\frac{2\bar{z}bG^2}{\bar{z}^2-b^2}+\frac{2}{\pi}\bar{z}P\int_1^{\infty}\frac{dz\operatorname{Im} D^o(z)}{z^2-\bar{z}^2}, \tag{18}$$

$$\operatorname{Re} D^e(\bar{z})-\operatorname{Re} D^e(1)=(\bar{z}^2-1)\left\{\frac{2b^2G^2}{(\bar{z}^2-b^2)(1-b^2)}+\frac{2}{\pi}P\int_1^{\infty}\frac{zdz\operatorname{Im} D^e(z)}{(z^2-\bar{z}^2)(z^2-1)}\right\}. \tag{19}$$

The dispersion relations that we have written are consistent with the assumption that the cross sections σ^+ and σ^- become constant and equal at high energies and, further, that the spin-flip amplitude vanishes at high energies. That the cross sections become constant and the spin-flip amplitude vanish would follow from the existence of a finite range for the interaction and the interaction becoming completely inelastic at high energies.

NEW RELATIONS

Now we shall develop a more convergent form of the dispersion relations for each of the amplitudes introduced in the previous section, basing our arguments on the form of the usual dispersion relations for these amplitudes. We shall only discuss the forward direction, but a similar development could be made for the general case. Consider, as an example, the function $g^e(z)$. We have previously written relation (7) for this function which corresponds to the following properties:

(1) $g^e(z)$ is analytic in the upper half of the complex z plane,

(2) $g^e(z)$ is less than z at infinity in the upper half of the complex plane,

(3) the real part of $g^e(z)$ is an odd function on the real axis, the imaginary part of $q^e(z)$ is an even function on the real axis, and

(4) $\operatorname{Im} g^e(z)$ is zero on the real axis for $-1<z<1$ with the exception of a delta-function contribution

$$\tfrac{1}{2}\pi(G^2/\kappa\mu)[\delta(z-b)+\delta(z+b)].$$

We consider the function $g^e(z)/(z^2-1)^{\frac{1}{2}}$. By $(z^2-1)^{\frac{1}{2}}$ we mean that branch of the function that is analytic in the upper half plane, positive for z real, $z>1$, negative for z real, $z<-1$, and positive imaginary for z real, $-1<z<1$. The imaginary part of this function is positive in the upper half-plane, and the function has no zeros above the real axis.[7] This function is just the dimensionless laboratory momentum of the incoming

[7] We limit our discussion of the behavior of these functions to the upper half-plane as a matter of convenience; all of these functions are analytic in the entire plane with cut lines along the real axis from ± 1 to $\pm\infty$. The behavior of these functions on the real axis is always understood to be the behavior as a limit from the upper half-plane.

meson. Now the quotient function, $g^e(z)/(z^2-1)^{\frac{1}{2}}$, will have these properties:

(1) it is analytic in the upper half-plane,
(2) it goes to zero at infinity,
(3) the limit of the function onto the real axis from above has an even real part and an odd imaginary part, and
(4) the real part of this function is zero between $z=-1$ and $z=1$ on the real axis with the exception of two delta-function singularities, the real part of this function being $\mathrm{Im}\,[g^e(z)]/(1-z)^{\frac{1}{2}}$ in this region.

From the boundedness and symmetry properties we can write, using Cauchy's theorem,

$$\mathrm{Im}\left[\frac{g^e(\bar{z})}{(\bar{z}^2-1)^{\frac{1}{2}}}\right]=-\frac{2}{\pi}\bar{z}P\int_0^\infty \frac{dz}{z^2-\bar{z}^2}\,\mathrm{Re}\left[\frac{g^e(z)}{(z^2-1)^{\frac{1}{2}}}\right], \quad (20)$$

in which Im [ ] and Re [ ] mean the imaginary and the real parts of the limit on the real axis from above of the analytic function inside the brackets. No trouble arises when the contour of integration is brought to the real axis. The integration is to be thought of, at first, as including semicircles above the points $z=\pm1$. The contribution of these semicircles vanishes as the radius shrinks to zero, since the singularity in

$$\mathrm{Re}\,[g^e(z)/(z^2-1)^{\frac{1}{2}}]$$

at $z=\pm1$ is integrable. Using the relation between the real part of $g^e(z)/(z^2-1)^{\frac{1}{2}}$ and the imaginary part of $g^e(z)$ in the nonphysical region, we have:

$$\mathrm{Im}\left[\frac{g^e(\bar{z})}{(\bar{z}^2-1)^{\frac{1}{2}}}\right]=-\frac{2}{\pi}\bar{z}P\int_1^\infty \frac{dz}{z^2-\bar{z}^2}\,\frac{\mathrm{Re}\,g^e(z)}{(z^2-1)^{\frac{1}{2}}}+\frac{\bar{z}}{\bar{z}^2-b^2}\,\frac{G^2}{\kappa\mu(1-b^2)^{\frac{1}{2}}}. \quad (21)$$

The left-hand side of this relation is

$$\mathrm{Im}\,[g^e(\bar{z})]/(\bar{z}^2-1)^{\frac{1}{2}} \quad \text{for } \bar{z}>1,$$

and

$$-\mathrm{Re}\,[g^e(\bar{z})]/(1-\bar{z}^2)^{\frac{1}{2}} \quad \text{for } 0<\bar{z}<1.$$

There is an infinite discontinuity in this relation as the threshold is approached from below, since $\mathrm{Im}\,g^e(z)$ goes to zero as η at threshold while $\mathrm{Re}\,g^e(z)$ is a constant in the neighborhood of the threshold.

Similar relations can be written for the other amplitudes. We introduce the symbol $\xi=(z^2-1)^{\frac{1}{2}}$ and drop the factor $(1-b^2)^{\frac{1}{2}}$ that occurs in the bound state terms. Then some of the other possible relations are

$$\mathrm{Im}\left[\frac{g^o(\bar{z})}{\bar{\xi}}\right]=-\frac{2}{\pi}P\int_1^\infty \frac{zdz}{z^2-\bar{z}^2}\,\frac{\mathrm{Re}\,g^o(z)}{\xi}-\frac{bG^2/\kappa\mu}{\bar{z}^2-b^2}, \quad (22)$$

and

$$\mathrm{Im}\left[\frac{f^e(\bar{z})}{\bar{\xi}}\right]=-\frac{2}{\pi}P\int_1^\infty \frac{zdz}{z^2-\bar{z}^2}\,\frac{\mathrm{Re}f^e(z)}{\xi}, \quad (23)$$

where we do not have to perform a subtraction in order to obtain convergence provided that the $f^e(z)$ amplitude is less singular than z at infinity. The $f^o(z)$ amplitude obeys relation (21) without the bound-state term. For the direct amplitudes we have:

$$\mathrm{Im}\left[\frac{D^o(\bar{z})}{\bar{\xi}}\right]=-\frac{2}{\pi}\bar{z}P\int_1^\infty \frac{dz}{z^2-\bar{z}^2}\,\frac{\mathrm{Re}\,D^o(z)}{\xi}-\frac{\bar{z}}{\bar{z}^2-b^2}2bG^2, \quad (24)$$

$$\mathrm{Im}\left[\frac{D^e(\bar{z})}{\bar{\xi}}\right]-\mathrm{Im}\left[\frac{D^e(z')}{\xi'}\right]=(\bar{z}^2-z'^2)\left\{-\frac{2b^2G^2}{(\bar{z}^2-b^2)(z'^2-b^2)}-\frac{2}{\pi}P\int_1^\infty \frac{zdz}{(z^2-\bar{z}^2)(z^2-z'^2)}\,\frac{\mathrm{Re}\,D^e(z)}{\xi}\right\}, \quad (25)$$

or, in terms of cross sections, using (16) and (17), $\bar{z}>1$,

$$\sigma^+-\sigma^-=\frac{bG^2}{\kappa\mu\bar{z}}+\frac{2}{\pi}\frac{\bar{z}}{\kappa\mu}P\int_1^\infty \frac{dz}{z^2-\bar{z}^2}\,\frac{\mathrm{Re}\,D^o}{\xi}, \quad (26)$$

$$(\sigma^++\sigma^-)_{\bar{z}}-(\sigma^++\sigma^-)_{z'}=\frac{(\bar{z}^2-z'^2)}{\kappa\mu}\left\{-\frac{2b^2G^2}{\bar{z}^2z'^2}-\frac{2}{\pi}P\int_1^\infty \frac{zdz}{z^2-\bar{z}^2}\,\frac{\mathrm{Re}\,D^e}{\xi}\,\frac{1}{z^2-z'^2}\right\}. \quad (27)$$

The integrals over the real parts of the amplitudes would be difficult to handle near threshold, since the integrand is so singular there. The integrands can be modified, however, to remove either the principal-part singularity or the singularity at the threshold. Since ξ has the analytic properties mentioned earlier, the following integral is an immediate consequence of Cauchy's theorem:

$$\frac{2}{\pi}P\int_1^\infty \frac{zdz}{z^2-\bar{z}^2}\,\frac{1}{\xi}=\begin{cases}0 & \text{for } \bar{z}>1\\ 1/(1-\bar{z}^2)^{\frac{1}{2}} & \text{for } \bar{z}<1.\end{cases} \quad (28)$$

We shall use this and similar integrals to isolate the dominant contributions to these singular integrals when we apply these new relations.

COUPLING CONSTANT DETERMINATION

The new relations are more rapidly convergent than the usual ones and therefore are better adapted for the analysis of the experimental data since the high-energy effects can be neglected. Of all the relations, the new relation (21) for the even-isotopic-index spin-flip amplitude is best suited for the determination of the coupling constant. The relation is rapidly convergent. The coupling constant is weighted with a large factor with respect to the phase shifts, being given essentially additively rather than as a difference of experimental quantities. We shall use relation (21) at threshold. In this limit, the dominant contribution from the integral arises from the neighborhood of the singularity. We use the fact that

$$\lim_{\bar{z}\to 1+} \frac{2}{\pi} P\int_1^\infty \frac{dz}{(z^2-\bar{z}^2)}\frac{1}{z\xi} = -1, \tag{29}$$

which follows from (28), to reduce the singularity. Thus (21) becomes

$$\frac{G^2}{\kappa\mu}+\operatorname{Re} g^e(1) = \lim_{\bar{z}\to 1}\frac{\operatorname{Im} g^e(\bar{z})}{\bar{\xi}} + \frac{2}{\pi}\int_1^\infty \frac{dz}{\xi^3}\left[\operatorname{Re} g^e(z) - \frac{1}{z}\operatorname{Re} g^e(1)\right]. \tag{30}$$

No difficulty arises from the limit of the principal-part function since the integral, after the subtraction, has an integrable singularity at threshold. The term arising from the imaginary part of the amplitude is related to the square of the s-wave scattering lengths. It is

$$\lim_{z\to 1}\frac{\operatorname{Im} g^e(z)}{\xi} = \frac{2\pi}{3\mu(\kappa+\mu)}[(\delta_1{}^0)^2 + 2(\delta_3{}^0)^2],$$

which is about 0.1% of the coupling constant term, if one uses Orear's values for the s-wave lengths.[8] Since the contribution to $g^e(z)$ from the direct amplitude is about one percent, we shall ignore it, although it could be taken into account using (12). The integral is rapidly convergent; dropping all higher waves in $g^e(z)$, we rewrite (30) as

$$3f^2 = \delta_{33}{}^0 - \delta_{31}{}^0 + \tfrac{1}{2}(\delta_{13}{}^0 - \delta_{11}{}^0) - [\lambda_{33} - \lambda_{31} + \tfrac{1}{2}(\lambda_{13} - \lambda_{11})], \tag{31}$$

in which

$$\lambda = \frac{1}{\pi}\int_1^\infty \frac{dz}{\xi^3}\left(\frac{\sin 2\delta}{\eta^3} - \frac{2\delta^0}{z}\right). \tag{32}$$

We shall evaluate the coupling constant using the Anderson fitted phase shifts.[9] See Table I. The contribution from the scattering lengths is

$$3f^2 \sim 0.301,$$

TABLE I. The scattering lengths, δ^0, and the integrals, λ, defined by (32) based upon the p-wave phases given by Anderson.[a]

State	δ^0	λ
33	0.248	0.0404
31	−0.042	−0.0060
13	0.0048	0.0029
11	−0.0175	−0.0033

[a] See reference 9.

and the correction from the integral is −0.050, resulting in a value for the coupling constant,

$$f^2 = 0.084.$$

In this determination, the 33 state contributes 83% and the 31 state, 14%. An alternative, but less convergent, way to calculate the coupling is to use the usual dispersion relation for $g^e(z)$, (7), which we write analogously to (30)

$$\frac{G^2}{\kappa\mu}+\operatorname{Re} g^e(1) = \frac{2}{\pi}\int_1^\infty \frac{dz}{\xi^2}\operatorname{Im} g^e(z).$$

Only the 33-state contribution to the integral is important. This yields

$$3f^2 = 0.301 - 0.052 \quad \text{or} \quad f^2 = 0.083.$$

Both of these values for the coupling constant are in good agreement with the value found by Haber-Schaim,[10] $f^2=0.082$. Our result is less than the value $f^2=0.10$ found by Davidon and Goldberger[11] using the same phases and a combination of relations (5) and (7). The relation we have used, however, is more convergent than theirs by a factor of z^2 and gives the coupling constant essentially additively in terms of the experimental data.

Another way to compute the coupling constant would be to modify Eq. (21) so that it could be used to represent the experimental data as a straight line whose intercept would be the coupling. This would also provide a check on the consistency of the data. Equation (21) can easily be brought to the form:

$$\frac{G^2}{\kappa\mu}+\bar{z}^2\left[\operatorname{Re} g^e(1) - \frac{2}{\pi}\int_1^\infty \frac{dz}{\xi^3}\chi(z)\right] = \frac{\bar{z}}{\bar{\xi}}\operatorname{Im} g^e(\bar{z}) + \frac{2}{\pi}\bar{z}^2\bar{\xi}^2\int_1^\infty \frac{dz}{z^2-\bar{z}^2}\frac{1}{\xi^3}\left[\chi(z) - \frac{\bar{z}}{z}\chi(\bar{z})\right], \tag{33}$$

where

$$\chi(z) = \operatorname{Re} g^e(z) - \frac{1}{z}\operatorname{Re} g^e(1).$$

[8] J. Orear, Phys. Rev. **101**, 288 (1956).
[9] H. L. Anderson, *Sixth Annual Rochester Conference on High-Energy Physics*, 1956 (Interscience Publishers, Inc., New York, 1956).
[10] U. Haber-Schaim, Phys. Rev. **104**, 1113 (1956).
[11] W. C. Davidon and M. L. Goldberger, Phys. Rev. **104**, 1119 (1956).

If the right-hand side of (33) were plotted against z^2, the data should fit a straight line. Alternatively, we could use the fact that the threshold value of $\bar{z}/\bar{\xi}\,\mathrm{Im}\,g^e(\bar{z})$ is insignificant compared to the coupling constant term and plot

$$\frac{G^2}{\kappa\mu}=-\frac{\bar{z}}{\bar{\xi}^3}\,\mathrm{Im}\,g^e(\bar{z})-\frac{2}{\pi}\bar{z}^2\int_1^\infty\frac{dz}{z^2-\bar{z}^2}\frac{1}{\xi^3}\left[\chi(z)-\frac{\bar{z}}{z}\chi(\bar{z})\right],\quad(34)$$

which should be a constant.

S-WAVE SCATTERING LENGTHS

We shall now use these new relations to calculate the s-wave scattering lengths from the p-wave phase shifts. More exactly, given that the s waves are small, we shall show that they can be approximately calculated from the p waves alone, but that if the experimental data about the second maximum in the pion-nucleon scattering cross section, the $T=\frac{1}{2}$ maximum, is used, the scattering lengths can be calculated quite accurately. We shall not have to make any very extreme assumptions about the high-energy behavior of the theory; we need only the assertion that the cross sections for the scattering of positive and negative mesons by protons approach the same value at high energies. We use the two relations for the odd-isotopic-index direct scattering amplitude specialized to threshold. Relation (18) yields

$$\delta_1{}^0-\delta_3{}^0=6\frac{\kappa}{\kappa+\mu}f^2+\frac{3}{4\pi^2}\frac{\mu}{\kappa+\mu}\int_1^\infty\frac{dz}{\xi^2}\,\mathrm{Im}\,D^o,\quad(35)$$

and the new relation (24) yields

$$(\delta_1{}^0)^2-(\delta_3{}^0)^2=-6f^2-\frac{3}{4\pi^2}\frac{\mu}{\kappa}\lim_{\bar{z}\to1+}P\int_1^\infty\frac{dz}{z^2-\bar{z}^2}\frac{\mathrm{Re}\,D^o}{\xi}.\quad(36)$$

Expression (35) can be used to compute the difference of the s-wave scattering lengths, and this difference along with the experimental p-wave phase shifts, can be inserted into (36) to compute the sum of the scattering lengths.

Equation (35) was used by Goldberger, Miyazawa, and Oehme[2] to calculate this difference of scattering lengths from the experimental cross sections. Since the s waves are small, they and the small p waves may be neglected under the integral in (35). Exhibiting the important contributions to the difference, we have

$$\delta_1{}^0-\delta_3{}^0=6\frac{\kappa}{\kappa+\mu}f^2-\frac{4}{\pi}\frac{\kappa}{\kappa+\mu}\int_1^\infty\frac{dz}{z^2-1}\frac{E}{\kappa}\frac{\sin^2\delta_{33}}{\eta}+\frac{3}{4\pi^2}\frac{\mu^2\kappa}{\kappa+\mu}\int\frac{dz}{\xi}(\sigma^--\sigma^+),\quad(37)$$

where the integral is to be extended over the second maximum in the pion-nucleon cross section. The first term in the Born approximation, in pseudoscalar theory, for the difference of the scattering lengths. For a coupling constant $f^2=0.084$, it is 0.44. The p-wave integral is the dominant correction to the Born approximation. If this is integrated using Anderson's[9] values for the 33-phase shift along with the experimental values of Mukhin and Pontecorvo[12] above the resonance, the result is $\delta_1{}^0-\delta_3{}^0=0.20$ The contribution from the $T=\frac{1}{2}$ maximum brings this value up into agreement with the experimental value[8] $\delta_1{}^0-\delta_3{}^0=0.27$. Turning to the second relation, we see that a large part of the integral comes from the neighborhood of the singularity. We modify the integrand to eliminate this singular behavior. Since

$$\lim_{\bar{z}\to1+}P\int_1^\infty\frac{dz}{z^2-\bar{z}^2}\frac{1}{\xi}=-1,$$

we can subtract the threshold value of $\mathrm{Re}\,D^o$ from the integrand:

$$(\delta_1{}^0)^2-(\delta_3{}^0)^2=-6f^2+\frac{2}{\pi}\frac{\kappa+\mu}{\kappa}(\delta_1{}^0-\delta_3{}^0)-\frac{3}{4\pi^2}\frac{\mu}{\kappa}\int_1^\infty\frac{dz}{\xi^3}[\mathrm{Re}\,D^o(z)-\mathrm{Re}\,D^o(1)].\quad(38)$$

To the extent that the s waves are small and linear in the momentum, this subtraction takes into account, within 10%, the s-wave contribution to the integral. The rest of the integral may be approximated by the p-wave terms. The sum of the s-wave scattering lengths is then

$$\delta_1{}^0+\delta_3{}^0=\frac{2}{\pi}\frac{\kappa+\mu}{\kappa}+\frac{\Delta-6f^2}{\delta_1{}^0-\delta_3{}^0},\quad(39)$$

where

$$\Delta=\frac{1}{\pi}\int_1^\infty\frac{dz}{\xi^2\eta^2}[2\sin2\delta_{33}+\sin2\delta_{31}-2\sin2\delta_{13}-\sin2\delta_{11}].\quad(40)$$

Using the Anderson phases,[9] Δ may be evaluated numerically, yielding $\Delta=0.315$. Various values for the scattering lengths are collected in Table II. The first

TABLE II. The s-wave scattering lengths computed from values of the coupling constant, an integral, Δ, over the p-wave phases, and the difference of scattering lengths using Eq. (39).

	f^2	Δ	$\delta_1{}^0-\delta_3{}^0$	$\delta_1{}^0$	$\delta_3{}^0$
(1)	0.084	0.315	0.27	0.15	−0.12
(2)	0.082	0.315	0.27	0.17	−0.10
(3)	0.084	0.315	0.20	−0.01	−0.21
(4)	0.084	0.350	0.20	0.10	−0.10

[12] A. I. Mukhin and B. M. Pontecorvo, J. Exptl. Theoret. Phys. (U.S.S.R.) **31**, 550 (1956) [translation: Soviet Phys. (JETP) **4**, 373 (1957).

and second lines are in good agreement with Orear's values,[8] but the scattering lengths are sensitive to the coupling constant. The third line of Table II uses the p-wave approximation for the difference of the lengths, producing rough agreement. The small p waves do not influence the qualitative picture. In line four, only the 33-phase shift is assumed to be different from zero, $\Delta=0.350$, and the scattering lengths are in qualitative agreement with experiment.

We should point out that for the first relation to hold, $\sigma^+ - \sigma^-$ must vanish at high energies. For the new relation (35) to hold, all that is required is that the cross sections become constant at high energies or, more weakly, that neither the real nor the imaginary part of $D^o(z)$ increases as fast as z^2. The significance of this calculation is that a large part of the s-wave scattering may be treated as a relativistic effect induced by the p-wave interaction.

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Meson-Meson Interaction in the Bethe-Salpeter Approximation

A. N. MITRA AND R. P. SAXENA
Department of Physics, Muslim University, Aligarh, U. P., India
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The π-π interaction is studied in the Bethe-Salpeter approximation by first obtaining the "effective" interaction Hamiltonian with the help of the lowest order S matrix for π-π scattering. This Hamiltonian is then used to set up a Goldberger-type integral equation for π-π scattering which is solved for the cases of total isotopic spins $I=0, 2$, assuming the mesons to be in s states with respect to each other. It is found that the interaction in the state $I=0$ is too strongly attractive to give a resonance. On the other hand, a resonance is obtained for the case $I=2$, at a momentum (in the center-of-mass system) $k\sim 93$ Mev/c for each meson, taking $G^2/4\pi=15.5$.

1. INTRODUCTION

THE concept of meson-meson interaction (which can be pictured as taking place through virtual nucleon pairs) is almost as old as that of the more fundamental meson-nucleon interaction. However, unlike the latter which can be realized directly by means of a meson-nucleon scattering experiment, the former finds only indirect verification by the effect it produces on a system consisting of two or more mesons, since it is not possible to conduct any direct experiment for scattering a meson beam by another. One of the most important processes in which π-π interaction plays an indirect role is one involving scattering of mesons by nucleons, if it is remembered that a nucleon is always "dressed" with its own meson field which the incident meson beam has to encounter. Strong resonant interactions of the meson beam with the meson-cloud of the nucleon were in fact suggested phenomenologically by Dyson[1] and Takeda[2] as possible interpretations for the second maximum in π^--p scattering near 1 Bev. Earlier, Mitra and Dyson[3] had suggested an attractive π-π interaction as a possible explanation for the anomalously small (in magnitude) and negative s-phase shift in the $T=\frac{3}{2}$ state of the π-p system. Strong π-π interactions have also been suggested by a number of investigators in various other connections.

Though the importance of the π-π interaction in influencing various physical phenomena has been generally recognized, a field-theoretical treatment of this interaction has hardly received any attention so far. It is at least clear that a perturbation treatment is hopelessly inadequate for the purpose. Among nonperturbation methods, the moderate amount of success of the Tamm-Dancoff approximation in its application to meson-nucleon scattering (at least in the "low" energy region, 0–200 Mev) has partially encouraged the belief that this approximation is perhaps fairly reasonable in the low-energy region. It is therefore of interest that the π-π interaction be also investigated in this approximation, at least for small relative momenta of the pions. However, a Tamm-Dancoff treatment for this problem has the disadvantage that a large number of intermediate states are involved in the equation connecting the various Tamm-Dancoff amplitudes, *viz.*, one must consider all the states which can be reached from the (0,2) state by *two steps* of the interaction Hamiltonian, unlike the situation in the pion-nucleon scattering[4] where only *one step* of the interaction Hamiltonian was necessary. A second disadvantage of the Tamm-Dancoff treatment for π-π interaction is that its noncovariant nature prevents a clear-cut separation of the well known primitive divergence associated with the "contact" meson-meson interaction (the so called meson-meson divergence in the pseudoscalar theory). To avoid these difficulties it has been

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